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Semirings which are union of principal left k-radicals

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Abstract. Here we study the principal left k-radicals of a semiring with semilattice additive reduct and characterize the semirings which are disjoint union of principal left k-radicals via the transitive closure $\stackrel{l}{\longrightarrow}^{\infty}$ of the relation $\stackrel{l}{\longrightarrow}$ on a semiring S, given by for $a, b \in S, a \stackrel{l}{\longrightarrow} b \Leftrightarrow b^n \in \overline{Sa}$ for some $n \in \mathbb{N}$.

1. Introduction

The notion of principal left k-radical plays a vital role in providing the decomposition of semirings with semilattice additive reduct. The distributive lattice decomposition of semirings is one of the elegant techniques for giving the structure of such semirings, and has been given in [1, 6, 7]. In [7], while giving the decompositions of semirings, the simpler components are found to be left k-Archimedean subsemirings, and that too, via k-radicals of left k-ideals. The notion of principal left k-radicals was introduced in [6] following the ideas of Ćirić and Bogdanović [2], and studied its important characteristics. Also, the very notion of principal left k-radicals induces an equivalence relation λ which was found to be the least distributive lattice congruence on a semiring through which those semirings were characterized which are distributive lattices of λ-simple subsemirings. In terms of principal left k-radical, the semirings in which the principal left k-radicals are the least completely semiprime k-ideals, have been decomposed into λ-simple subsemirings. In this paper, we continue to study the class of semirings with semilattice additive reduct, pick up the notion of principal left k-radicals, and show that a semiring S which is a distributive lattices of left k-Archimedean semirings, can be expressed as a union of principal k-radicals if and only if the relation $\stackrel{l}{\longrightarrow}^{\infty}$ is symmetric on S. During this decomposition we find that the principal left k-radicals becomes the least completely semiprime left k-ideals of S. The preliminaries and prerequisites for this article have been discussed in section 2. In section 3, we study principal left k-radicals $\Lambda(a)$, and show that they are the least completely semiprime left k-ideals of a semiring S containing a, where S is a distributive lat-

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tice of left k-Archimedean semirings. Finally, we characterize the semirings which are the disjoint union of principal left k-radicals via the transitive relation $\overset{l}{\longrightarrow}^{\infty}$.

2. Preliminaries and prerequisites

A *semiring* $(S,+,\cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S,+)$ and the *multiplicative reduct* $(S,\cdot)$ are semigroups and such that the following distributive laws hold: for $x,y\in S$,

$$x(y+z)=xy+xz \text{ and } (x+y)z=xz+yz.$$

Thus the semirings can be viewed as a common generalization of both rings and distributive lattices. Throughout this paper, unless otherwise stated, the author studies the semirings $(S,+,\cdot)$ such that $(S,+)$ is a semilattice. If A is a nonempty subset of a semiring S, then A is called *completely semiprime* if for $x\in S, x^2\in A$ implies $x\in A$. Let S be a semiring and $\phi\neq A\subseteq S$. Then the k-closure of A, denoted by $\overline{A}$, and is defined by $\overline{A}=\{x\in S\mid x+a_1=a_2 \text{ for some } a_i\in A\}$, and the k-radical of A by $\sqrt{A}=\{x\in S\mid (\exists\ n\in\mathbb{N})\ x^n\in\overline{A}\}$. Then by definition, one has $\overline{A}\subseteq\sqrt{A}$, and also $A\subseteq\overline{A}$ since the additive reduct $(S,+)$ is a semilattice. Moreover, if $(A,+)$ is a subsemigroup of $(S,+)$ then $\overline{A}=\{x\in S\mid x+a=a$ for some $a\in A\}$. An ideal A of S is said to be a k-ideal of S if and only if $\overline{A}=A$. An equivalence relation ρ on a semiring is said to be a congruence on S if ρ is compatible with both the operations, that is, for $a,b,c\in S, a\rho b$ implies $(a+c)\rho(b+c), ac\rho bc$, and $ca\rho cb$, or equivalently, for $a,b,c,d\in S, a\rho b$ and $c\rho d$ imply $(a+c)\rho(b+d), ac\rho bd$. A congruence ρ on a semiring S is said to be rectangular idempotent congruence if for all $a,b\in S, a^2\rho a$ and $aba\rho a$, and right zero idempotent congruence if for all $a,b\in S, ab\rho b$. A semiring S is said to be a right zero idempotent semiring if for all $a,b\in S$ one has $ab=b$. In particular, $a^2=a$ for every $a\in S$. Let $\mathcal{C}$ be a class of semirings, and we refer to semirings in $\mathcal{C}$ as $\mathcal{C}$-semirings. A semiring S is called a right zero idempotent of $\mathcal{C}$-semirings if there exists a congruence ρ on S such that the quotient semiring S/ρ is right zero idempotent, and each ρ-class is a semiring in $\mathcal{C}$.

For information regarding undefined concepts in semigroup theory one can approach [4], and [3] for undefined notions in semiring theory. Here we state a lemma which is going to be used frequently throughout the paper.

Lemma 2.1. [7] *Let S be a semiring.*
(a) *For $a,b\in S$ the following statements are equivalent:*
  (i) there are $s_i\in S$ such that $b+s_1a=s_2a$.
  (ii) there are $s\in S$ such that $b+sa=sa$.
(b) *If $a,b,c\in S$ such that $c+xa=xa$ and $d+yb=yb$ for some $x,y\in S$, then there is some $z\in S$ such that $c+za=za$ and $d+zb=zb$.*

3. Union of principal left k-radicals

In this section, we study the principal left k-radicals of a semiring emerging from left k-radicals. We find the semirings in which the principal left k-radicals are the least completely semiprime k-ideals, and these are the very semirings which are distributive lattices of left k-Archimedean subsemirings[7]. We also study the semirings which are union of principal left k-radicals.

Let S be a semiring. As in [5], the author defined the division relation $|_l$ on S by: $a|_l b \Leftrightarrow b \in \overline{Sa}$ and the relation $\overset{l}{\longrightarrow}$ on S is defined as: $a \overset{l}{\longrightarrow} b \Leftrightarrow b^n \in \overline{Sa}$ for some $n \in \mathbb{N}$. For $n \in \mathbb{N}$, $\overset{l}{\longrightarrow}^n$ denotes the n^{th} power of $\overset{l}{\longrightarrow}$, and $a \overset{l}{\longrightarrow}^\infty b \Leftrightarrow a \overset{l}{\longrightarrow}^n b$ for some $n \in \mathbb{N}$. In general, $\overset{l}{\longrightarrow}^\infty$ is not symmetric relation on S. Let S be a semiring, $a \in S$ and $n \in \mathbb{N}$. In [6], the following sets have been introduced:

$$\Lambda(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^\infty x\},\ \Lambda_n(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^n x\}.$$

For every $a \in S$ the set $\Lambda(a)$ is called a principal left k-radical in S containing a. Here we state some basic characteristics of the above mentioned sets:

Lemma 3.1. *[6] Let S be a semiring and $a, b, c \in S$. Then*

1. $\Lambda_1(a) = \sqrt{Sa}$.
2. $\Lambda_n(a) \subseteq \sqrt{S\Lambda_n(a)} = \Lambda_{n+1}(a), n \in \mathbb{N}$.
3. $\Lambda(a) = \bigcup_{n\in\mathbb{N}} \Lambda_n(a)$.

Lemma 3.2. *Let S be a semiring.*

1. *For $a, b \in S$, one has $(a + b) \overset{l}{\longrightarrow} a$.*
2. *If $x, y, s_1, s_2, t_1, t_2, u, v$ and $m, n \in \mathbb{N}$ such that $x^m + s_1u = t_1u$ and $y^n + s_2v = t_2v$, then there exist $k \in \mathbb{N}$ and $s \in S$ such that $x^k + su = su$ and $y^k + sv = sv$.*
3. *If $x, y \in \Lambda(a)$, then there exists $n \in \mathbb{N}$ such that $a \overset{l}{\longrightarrow}^n x$ and $a \overset{l}{\longrightarrow}^n y$.*

Proof. (1): Since $(S, +, \cdot)$ is a semilattice, $a^2 + a^2 = a^2$. Adding ab on both sides we get $a^2 + a(a + b) = a(a + b) \in \overline{S(a + b)}$, whence $(a + b) \overset{l}{\longrightarrow} a$.
(2): If $m = n$, then the result follows from Lemma 2.1. Otherwise, suppose $m < n$. Then from $x^m + s_1u = t_1u$, multiplying both sides on the left by x^{n-m} we get $x^n + x^{n-m}s_1u = x^{n-m}t_1u$. Now by Lemma 2.1, there exists $s \in S$ such that $x^n + su = su$ and $y^n + sv = sv$, the latter arises from the relation $y^n + s_2v = t_2v$.
(3): Let $x, y \in \Lambda(a)$. Then there are $m, n \in \mathbb{N}$ such that $a \overset{l}{\longrightarrow}^m x$ and $a \overset{l}{\longrightarrow}^n y$. If $m = n$, then we are done. If not, then suppose that $m < n$. Since the relation $\overset{l}{\longrightarrow}$ is reflexive, we have $a \overset{l}{\longrightarrow}^m x \underbrace{\overset{l}{\longrightarrow} x \overset{l}{\longrightarrow} \ldots. \overset{l}{\longrightarrow}}_{(n-m)times} x$. This implies $a \overset{l}{\longrightarrow}^n x$. $\square$

In the following we give two useful properties of a completely semiprime left k-ideal of a semiring S.

Lemma 3.3. *Let S be a semiring, L a completely semiprime left k-ideal of S and $n \in \mathbb{N}$. Then*

1. *$a^n \in L$ implies that $a \in L$.*
2. *$\sqrt{L} = L$.*

Proof. (1): The statement is true for $n = 1$. Also, if $a^2 \in L$, then $a \in L$, since L is completely semiprime. So the statement is true for $n = 2$. Suppose it is true for some $k(\geq 2) \in \mathbb{N}$, i.e. $a^k \in L$ implies that $a \in L$. Now let $a^{k+1} \in L$. Since L is a left k-ideal, $a^{k-1}a^{k+1} \in L$, that is, $(a^k)^2 = a^{2k} \in L$ so that $a^k \in L$, whence $a \in L$. Thus by induction, we have (1).
(2): For a completely semiprime left k-ideal L of a semiring S, and $a \in \sqrt{L}$ one gets $a^n + l = l$ for some $n \in \mathbb{N}$ and $l \in L$ so that $a^n \in \overline{L} = L$, and so $a \in L$, applying (1). Thus $\sqrt{L} \subseteq L$. The opposite inclusion is always true. Consequently, $\sqrt{L} = L$. □

In a semiring S, for $a, b, c \in S$, $a + b = c$ implies $a^n + b^n = c^n$ for all $n \in \mathbb{N}$. This observation was frequently used in [1]. Here too, we take this opportunity to make the tasks easy in the following two presentations. In the following lemma we study all those semirings which are distributive lattices of left k-Archimedean semirings, and in these semirings, one has $ab \in \sqrt{Sa}$ for any $a, b \in S$[7]. This equivalent condition we use here in the statement.

Lemma 3.4. *Let S be a semiring and $ab \in \sqrt{Sa}$ for every pair $a, b \in S$. Then $\Lambda(a)$ is the least completely semiprime left k-ideal of S containing a.*

Proof. Let $x, y \in \Lambda(a)$ and $s \in S$ such that $s + x = y$. Then $a \overset{l}{\longrightarrow}^{\infty} y = (s + x) \overset{l}{\longrightarrow} s$, by (1) of Lemma 3.2 yielding $a \overset{l}{\longrightarrow}^{\infty} s$. Consequently $s \in \Lambda(a)$. Thus $\Lambda(a)$ is a k-set. Again, by (3) of Lemma 3.2, there exists $n \in \mathbb{N}$ such that $a \overset{l}{\longrightarrow}^{n} x$ and $a \overset{l}{\longrightarrow}^{n} y$. Then there are $x_i, y_i (i = 1, 2, ..., n-1)$ in S such that $a \overset{l}{\longrightarrow} x_1 \overset{l}{\longrightarrow} x_2 \overset{l}{\longrightarrow} ... \overset{l}{\longrightarrow} x_{n-2} \overset{l}{\longrightarrow} x_{n-1} \overset{l}{\longrightarrow} x$ and $a \overset{l}{\longrightarrow} y_1 \overset{l}{\longrightarrow} y_2 \overset{l}{\longrightarrow} ... \overset{l}{\longrightarrow} y_{n-2} \overset{l}{\longrightarrow} y_{n-1} \overset{l}{\longrightarrow} y$. Now by (2) of Lemma 3.2, their exist $m \in \mathbb{N}$ and $s \in S$ such that $x_1^m + sa = sa, x_{i+1}^m + sx_i = sx_i (i = 1, 2, ..., n-2), x^m + sx_{n-1} = sx_{n-1}$ and $y_1^m + sa = sa, y_{i+1}^m + sy_i = sy_i (i = 1, 2, ..., n-2), y^m + sy_{n-1} = sy_{n-1}$. Now $(x + y)^m = y^m + ux + xv + \sum_{i=1}^{k} u_i x v_i$ for some $u, v, u_i, v_i \in S$. Adding sy_{n-1} on both sides we get $(x+y)^m + sy_{n-1} = sy_{n-1} + ux + xv + \sum_{i=1}^{k} u_i x v_i$. From this we write $(x+y)^{m+2} + (x+y)s(x+y_{n-1})(x+y) = (x+y)s(x+y_{n-1})(x+y) + (x+y)u(x+y_{n-1})(x+y) + (x+y)(x+y_{n-1})v(x+y) + \sum_{i=1}^{k}(x+y)u_i(x+y_{n-1})v_i(x+y)$. Now for $w = (x+y)s + (x+y) + (x+y)u + v(x+y) + \sum_{i=1}^{k}(x+y)u_i + \sum_{i=1}^{k} v_i(x+y) + x + y$ we obtain $(x+y)^{m+2} + w(x+y_{n-1})w = w(x+y_{n-1})w$ since $(S, +)$ is a semilattice.

By hypothesis, $w(x+y_{n-1})w \in \sqrt{Sw(x+y_{n-1})}$ so that there exists $k \in \mathbb{N}$ such that $[w(x+y_{n-1})w]^k \in \overline{Sw(x+y_{n-1})}$, and thus $(x+y)^{k(m+2)} \in \overline{Sw(x+y_{n-1})} \subseteq \overline{S(x+y_{n-1})}$. This shows that $(x+y_{n-1}) \overset{l}{\longrightarrow} (x+y)$. Applying the same process one gets $(x+a) \overset{l}{\longrightarrow} (x+y_1) \overset{l}{\longrightarrow} (x+y_2) \overset{l}{\longrightarrow} \dots \overset{l}{\longrightarrow} (x+y_{n-2}) \overset{l}{\longrightarrow} (x+y_{n-1})$, and so $(x+a) \overset{l}{\longrightarrow}^{\infty} (x+y)$. Similarly $a \overset{l}{\longrightarrow}^{\infty} a$ and $a \overset{l}{\longrightarrow}^{\infty} x$ give $(a+a) \overset{l}{\longrightarrow}^{\infty} (a+x)$, i.e. $a \overset{l}{\longrightarrow}^{\infty} (a+x)$. Then by transitivity of $\overset{l}{\longrightarrow}^{\infty}$ we get $a \overset{l}{\longrightarrow}^{\infty} (x+y)$, i.e. $x+y \in \Lambda(a)$. Let $c \in S$. Then $a \overset{l}{\longrightarrow}^{\infty} x$ and since $x \overset{l}{\longrightarrow} cx$, one has $cx \in \Lambda(a)$. Let $x \in S$ such that $x^2 \in \Lambda(a)$. Then $a \overset{l}{\longrightarrow}^{\infty} x^2 \overset{l}{\longrightarrow} x$ implies $x \in \Lambda(a)$. Thus $\Lambda(a)$ is a completely semiprime left k-ideal of S containing a, since $\overset{l}{\longrightarrow}$ is reflexive. Let L be a completely semiprime left k-ideal of S containing a. Then $Sa \subseteq SL \subseteq L$, so $\Lambda_1(a) = \sqrt{Sa} \subseteq \sqrt{L} = L$, by Lemma 3.3. As an induction hypothesis, we assume that $\Lambda_n(a) \subseteq L$. Then $S\Lambda_n(a) \subseteq SL \subseteq L$, so $\Lambda_{n+1}(a) = \sqrt{S\Lambda_n(a)} \subseteq \sqrt{L} = L$, by Lemmas 3.1 and 3.3. Hence by induction $\Lambda(a) = \bigcup_{n\in\mathbb{N}} \Lambda_n(a) \subseteq L$. Consequently, $\Lambda(a)$ is the least completely semiprime left k-ideal of S. □

The author in [6] introduced the following equivalence relation λ on a semiring S, which is induced from the principal left k-radicals:

$$a\lambda b \Leftrightarrow \Lambda(a) = \Lambda(b).$$

Finally, we are in a position to characterize the semirings which are union of principal left k-radicals via the relation $\overset{l}{\longrightarrow}^{\infty}$, where we see that the decomposition occurs if and only if $\overset{l}{\longrightarrow}^{\infty}$ is symmetric.

Theorem 3.5. *Let S be a semiring S such that $ab \in \sqrt{Sa}$ for all $a, b \in S$ holds. Then the following conditions are equivalent on S:*

1. *λ is a rectangular idempotent congruence on S;*
2. *λ is a right zero idempotent congruence on S;*
3. *$abc \overset{l}{\longrightarrow}^{\infty} ac$;*
4. *$aba \overset{l}{\longrightarrow}^{\infty} a$;*
5. *$ab \overset{l}{\longrightarrow}^{\infty} b$ for all $a, b \in S$;*
6. *S is a disjoint union of its principal left k-radicals;*
7. *The relation $\overset{l}{\longrightarrow}^{\infty}$ is symmetric on S.*

Proof. (1) $\Rightarrow$ (3): For $a, b, c \in S$, by hypothesis, one has $c\lambda cac$. Since λ is left compatible with multiplication, one gets $(ab)c\lambda(ab)cac$. This can be written as $abc\lambda(abca)c\lambda ac$, as again by hypothesis, $a(bc)a\lambda a$. Since λ is transitive, one has $abc\lambda ac$. Then by definition of λ, we have $\Lambda(abc) = \Lambda(ac)$. Consequently, $abc \overset{l}{\longrightarrow}^{\infty} ac$ by the definition of principal left k-radicals.

(3) $\Rightarrow$ (4): For $a, b \in S$ one has $aba \overset{l}{\longrightarrow}^{\infty} a^2 \overset{l}{\longrightarrow} a$, that is, $aba \overset{l}{\longrightarrow}^{\infty} a$, since $\overset{l}{\longrightarrow}^{\infty}$ is transitive on S.

(4) $\Rightarrow$ (5): For $a, b \in S, bab \in \overline{Sab}$ implies that $ab \overset{l}{\longrightarrow} bab$. Also $bab \overset{l}{\longrightarrow}^{\infty} b$. Then by transitivity of $\overset{l}{\longrightarrow}^{\infty}$, one gets $ab \overset{l}{\longrightarrow}^{\infty} b$.

(5) $\Rightarrow$ (2): For $a \in S$, by hypothesis, one gets $a^2 \overset{l}{\longrightarrow}^{\infty} a$. Also $a \overset{l}{\longrightarrow}^{\infty} a^2$. These two relations imply $a \in \Lambda(a^2)$ and $a^2 \in \Lambda(a)$ so that $\Lambda(a) \subseteq \Lambda(a^2)$ and $\Lambda(a^2) \subseteq \Lambda(a)$ because of the least virtue of $\Lambda(a)$, for any $a \in S$. Thus one gets $\Lambda(a^2) = \Lambda(a)$, whence $a^2\lambda a$.

Now consider $a, b \in S$ such that $a\lambda b$ and $c \in S$. By hypothesis, $ac \overset{l}{\longrightarrow}^{\infty} c$ so that one has $c \in \Lambda(ac)$. By Lemma 3.4, $\Lambda(c)$ is the least completely semiprime left k-ideal of S, and so $c \in \Lambda(ac)$ implies $\Lambda(c) \subseteq \Lambda(ac)$. The opposite inclusion follows since $\Lambda(c)$ is a left k-ideal of S, and $ac \in \Lambda(c)$. Thus $\Lambda(ac) = \Lambda(c)$. Replacing a by b in $ac \overset{l}{\longrightarrow}^{\infty} c$, and proceeding as above one gets $\Lambda(bc) = \Lambda(c)$. Consequently, $\Lambda(ac) = \Lambda(bc)$, that is, $ac\lambda bc$. Also, $\Lambda(ca) = \Lambda(a) = \Lambda(b) = \Lambda(cb)$ so that $ca\lambda cb$.

Again, we have $\Lambda(a) = \Lambda(b)$ so that $a \in \Lambda(b)$ and $b \in \Lambda(a)$. By definition of the principal left k-radical, one has $b \overset{l}{\longrightarrow}^{\infty} a$ and $a \overset{l}{\longrightarrow}^{\infty} b$. Then there exist $m, n \in \mathbb{N}$ such that $b \overset{l}{\longrightarrow}^{m} a$ and $a \overset{l}{\longrightarrow}^{n} b$. For the first relation, there are $k \in \mathbb{N}$ and $s, a_i (i = 1, 2, ..., m-1) \in S$ such that $a_1^k + sb = sb, a_r^k + sa_{r-1} = sa_{r-1} (r = 2, 3, ..., m-1), a^k + sa_{m-1} = sa_{m-1}$. Also, $(a_1 + c)^k = a_1^k + uc + cv + \sum_{i=1}^{q} u_i c v_i$ for some $u, v, u_i, v_i \in S$. Then as in the proof of Lemma 3.4, one gets $(b + c) \overset{l}{\longrightarrow} (a_1 + c)$. Similarly, from the remaining equalities we can obtain $(a_1 + c) \overset{l}{\longrightarrow} (a_2 + c), (a_2 + c) \overset{l}{\longrightarrow} (a_3 + c)$,..., $(a_{m-1} + c) \overset{l}{\longrightarrow} (a + c)$. From these, one obtains $(b + c) \overset{l}{\longrightarrow}^{\infty} (a + c)$ so that $(a + c) \in \Lambda(b + c)$. Then one gets $\Lambda(a+c) \subseteq \Lambda(b+c)$. Similarly, starting with $a \overset{l}{\longrightarrow}^{n} b$ one can get $(b+c) \in \Lambda(a+c)$ so that $\Lambda(b+c) \subseteq \Lambda(a+c)$. Thus $\Lambda(a+c) = \Lambda(b+c)$. Consequently, $(a+c)\lambda(b+c)$.

Also, since $\Lambda(b)$ is the least completely semiprime left k-ideal, for $a \in S, ab\lambda b$, and so $\Lambda(ab) \subseteq \Lambda(b)$. Now, by hypothesis, one has $ab \overset{l}{\longrightarrow}^{\infty} b$, yielding $b \in \Lambda(ab)$. Then $\Lambda(b) \subseteq \Lambda(ab)$. Thus we have the equality $\Lambda(ab) = \Lambda(b)$ so that $ab\lambda b$. Thus λ is a right zero idempotent congruence on S.

(2) $\Rightarrow$ (6): Let S be a right zero idempotent I of semirings $S_i, i \in I$, which are λ-classes of S. Let $a \in S$, then $a \in S_i = \lambda_a$ for some $i \in I$. Let $x \in \lambda_a, c \in S$. Then $\Lambda(x) = \Lambda(a)$, and so $\Lambda(cx) = \Lambda(x) = \Lambda(a)$ so that $cx\lambda a$, whence $cx \in \lambda_a$.

Thus λ_a is a left ideal of S.

Also for $x \in S$ and $x^2 \in \lambda_a$, one has $\Lambda(x) = \Lambda(x^2) = \Lambda(a)$, since λ is idempotent. This yields $x\lambda a$ so that $x \in \lambda_a$. Thus λ_a is a completely semiprime left ideal of S containing a. Hence $\Lambda(a) \subseteq \lambda_a$. The opposite inclusion follows from the fact that whenever $b \in \lambda_a, a\lambda b$ implies $\Lambda(a) = \Lambda(b)$. Consequently, $S = \bigcup_{a\in S} \Lambda(a), \Lambda(a) = S_i$, and $S_i's$ are disjoint.

(6) $\Rightarrow$ (7): Let S be a disjoint union of its principal left k-radicals, that is, $S = \bigcup_{a\in S} \Lambda(a)$, and $\Lambda(a) \neq \Lambda(b)$ for $a \neq b$. Let $a, b \in S$ such that $a \overset{l}{\longrightarrow}^{\infty} b$. Then $b \in \Lambda(a)$, that yields $\Lambda(a) \cap \Lambda(b) \neq \phi$. This implies $\Lambda(a) = \Lambda(b)$ so that $a \in \Lambda(b)$. Then one gets $b \overset{l}{\longrightarrow}^{\infty} a$ by the definition of $\overset{l}{\longrightarrow}^{\infty}$. Consequently, the relation $\overset{l}{\longrightarrow}^{\infty}$ is symmetric on S.

(7) $\Rightarrow$ (5): Since $(S, +, \cdot)$ is a semilattice, for $a, b \in S$, one has $ab + ab = ab$ which implies $ab \in \overline{Sb}$ then we have $b \overset{l}{\longrightarrow} ab$. Since $\overset{l}{\longrightarrow} \subseteq \overset{l}{\longrightarrow}^{\infty}$, one gets $b \overset{l}{\longrightarrow}^{\infty} ab$ and since $\overset{l}{\longrightarrow}^{\infty}$ is symmetric, one gets $ab \overset{l}{\longrightarrow}^{\infty} b$.

(2) $\Rightarrow$ (1): Suppose that λ be a right zero idempotent congruence on S. Then, for $a, b \in S$ one has $a(ba)\lambda ba\lambda a$. Also, $a^2 = aa\lambda a$. Consequently, λ is a rectangular idempotent congruence on S. $\square$

Remark. We see that a semiring which is a distributive lattice of left k-Archimedean semirings, can also be treated as a semiring which is a right zero idempotent of principal left k-radicals that follows from the proof of (2) $\Rightarrow$ (6). Since the relation $\overset{l}{\longrightarrow}^{\infty}$ is reflexive and transitive on S by definition, the last theorem shows that the decomposition of semirings into principal left k-radicals makes the relation $\overset{l}{\longrightarrow}^{\infty}$ an equivalence relation on S. Moreover, the $\overset{l}{\longrightarrow}^{\infty}$-classes are exactly the λ-classes, and $\overset{l}{\longrightarrow}^{\infty} = \lambda$ on S.

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