Quasigroups and Related Systems **30** (2022), 123 − 132
https://doi.org/10.56415/qrs.v30.10

Weak multiplication semimodule

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Abstract. The objective of this article is to introduce the concept of weak mutiplication semimodule and study several properties which are generalization of corresponding results for multiplication modules. We characterize full prime subsemimodules and full maximal subsemimodules and finally it is shown that in a finitely generated faithful weak multiplication semimodule, every proper full subsemimodule is contained in a maximal full subsemimodule.

1. Introduction

In 1988, EI-Bast [1] investigated some useful properties of multiplication module. The study of the multiplication semimodule has been carried out by many authors. Semimodules over semirings appear in many areas of mathematics and it has many applications in the area of computer science as well as in cryptography (see [5]). In this paper, we consider inverse semimodule over a semiring R such that R is a distributive lattice of rings. Let M be an R-semimodule such that $(Rm : M) \neq \emptyset$ for all $m \in M$. Then M is called a weak multiplication semimodule if for each full subsemimodule N of M there exists a full ideal I of R such that $N = IM$. Some basic definitions and preliminaries are discussed in Section 2 and finally in Section 3, we study some properties of finitely generated faithful weak multiplication R-semimodules.

2. Definitions and preliminaries

A *semiring* $(R, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(R, +)$ and $(R, \cdot)$ are connected by distributivity, i.e., $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in R$. We call a semiring $(R, +, \cdot)$ *additive regular* if for every element $a \in R$ there exists an element $x \in R$ such that $a + x + a = a$. Additive regular semirings were first studied by J. Zeleznekow [12] in 1981. We call a semiring $(R, +, \cdot)$ an *additive inverse semiring* if $(R, +)$ is an inverse semigroup, i.e., for each $a \in R$ there exists a unique element $a' \in R$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Additive inverse semirings were first studied by Karvellas [3] in 1974. Throughout the paper, $E^+(R)$ will always denote the set of all additive

2020 Mathematics Subject Classification: 15A03, 16Y60.
Keywords: semimodule, inverse semimodule, multiplication semimodule, weak multiplication semimodule.

idempotents of the semiring R. A subsemiring I of a semiring $(R,+,\cdot)$ is called an *ideal* of R if $RI, IR \subseteq I$. An ideal I of a semiring R is called a full ideal if $E^+(R) \subseteq I$. For each ideal I of a semiring R, the k-closure $\overline{I}$ of I is defined by $\overline{I} = \{a \in R : a + a_1 = a_2 \text{ for some } a_1,\, a_2 \in I\}$ and is an ideal of R satisfying $I \subseteq \overline{I}$ and $\overline{\overline{I}} = \overline{I}$. An ideal I of a semiring R is called a k-ideal of R if and only if $I = \overline{I}$ holds.

Let $(M,+)$ be a commutative semigroup and $(R,+,\cdot)$ be a semiring with identity. Then M is called a *left R-semimodule* or simply an *R-semimodule* if there exists a mapping $R \times M \to M$, written as $(r,a) \mapsto ra$, for all $r \in R$ and for all $a \in M$, satisfying (i) $r(m+n) = rm + rn$, (ii) $(r+s)m = rm + sm$, (iii) $r(sm) = (rs)m$ and (iv) $1m = m$ for all $r, s \in R$ and $m, n \in M$. An R-semimodule M is said to be an *inverse semimodule* [11], if M is an inverse semigroup. A subset S of an R-semimodule M is said to be a *k-set* if $a, a+b \in S$ implies that $b \in S$. A subsemimodule N of an R-semimodule M is said to be full if $E(M) \subseteq N$, where $E(M)$ is the set of all idempotents of the semigroup M. A subsemimodule N of an R-semimodule M is said to be a k-subsemimodule of M if for $a, a+b \in N$ for some $b \in M$ imply that $b \in N$. For any subsemimodule N of an R-semimodule M, the k-closure of N, denoted by $\overline{N}$, is defined by $\overline{N} = \{m \in M : m + n_1 = n_2 \text{ for some } n_1, n_2 \in N\}$. One can easily prove that a subsemimodule N of an R-semimodule M is a k-subsemimodule if and only if $\overline{N} = N$. Let N and K be two subsemimodules of M. Then the set $\{a \in R : aK \subseteq N\}$ is denoted by $(N : K)$. It is easy to verify that $(N : K)$ is an ideal of R. Thus $(E(M) : M)$ is a full ideal of R. An R-semimodule M is said to be faithful if $(E(M) : M) = E^+(R)$. A proper subsemimodule N of M is said to be a prime subsemimodule if for any $r \in R$, $m \in M$, $rm \in N$ implies either $m \in N$ or $r \in (N : M)$. A proper subsemimodule N of M is said to be a maximal subsemimodule if it is not properly contained in any other proper subsemimodule of M. A semimodule M is said to be a multiplication semimodule if every subsemimodule of M is of the form IM, for some ideal I of R. Let M an R-semimodule such that $(Rm : M) \neq \emptyset$ for all $m \in M$. Then M is called a weak multiplication semimodule if for each full subsemimodule N of M there exists a full ideal I of R such that $N = IM$.

We need the following results.

Theorem 2.1. (cf. [4]) *A semiring R is a distributive lattice of skew-rings if and only if R is an additive inverse semiring satisfying the following conditions:*

(*i*) $a + a' = a' + a$,

(*ii*) $a(a + a') = a + a'$,

(*iii*) $a(b + b') = (b + b')a$,

(*iv*) $a + a(b + b') = a$, *for all* $a, b \in R$.

Theorem 2.2. (cf. [6]) *Let M be an inverse R-semimodule. Then*

(*i*) $(ra)' = ra'$ *for all* $r \in R$ *and for all* $a \in M$.

(*ii*) $ea \in E(M)$ *for all* $e \in E^+(R)$ *and for all* $a \in M$.

(*iii*) $ra \in E(M)$ *implies that* $ra = ru$ *for some* $u \in E(M)$.
If moreover, R is such that $(R, +)$ *is an inverse semigroup, then*
(*iv*) $(ra)' = r'a$ *and* $ra = r'a'$ *for all* $r \in R$ *and for all* $a \in M$.
(*v*) $ra \in E(M)$ *implies that* $ra = eu$ *for some* $e \in E^+(R)$, $u \in E(M)$.

Furthermore, if M is an inverse R-semimodule, then

$$E(M) = \{m + m' : m \in M\}$$

and for all $m, n \in M$, $m = (m')'$, $(m + n)' = n' + m'$.

Example 2.3. Let S be a ring and D be a distributive lattice. With respect to component wise addition and multiplication $R = S \times D$ is an additive commutative additive inverse semiring satisfying conditions (*ii*), (*iii*) and (*iv*) of Theorem 2.1 and $(R, +)$ is an inverse R-semimodule.

Throughout this paper, all semirings are assumed to be additive as well as multiplicative commutative which are distributive lattices of rings. This means R denotes an additive commutative and multiplicative commutative additive inverse semiring satisfying the conditions (*ii*) and (*iv*) of Theorem 2.1. Also, assume that R contains an identity element 1 such that $1 \notin E^+(R)$ and all semimodules are inverse semimodules with $M \neq E(M)$.

3. Weak-multiplication semimodule

An R-module M is said to be a multiplication module [1] if every submodule N of M is of the form $N = IM$ for some ideal I of R. Multiplication modules play an important role in the study of modules theory. Similar to module theory, in [10], the authors defined multiplication semimodule and studied some of its properties.

First we recall the definition of multiplication semimodule from [10].

Definition 3.1. An R-semimodule M is said to be a *multiplication semimodule* if for each subsemimodule N of M there exists an ideal I of R such that $N = IM$.

Theorem 3.2. (cf. [10]) *An R-semimodule M is a multiplication semimodule if and only if for each* $m \in M$ *there exists an ideal I of R such that* $Rm = IM$.

Recall that a subsemimodule N of an R-semimodule M is said to be *full* if $E(M) \subseteq N$. For an R-semimodule M, let $\mathscr{L}(M)$ denote the set of all full subsemimodules of M. In [8], we proved that $\mathscr{L}(M)$ forms a modular lattice. Therefore, to study the structure of an R-semimodule, full subsemimodules play a crucial role.

Definition 3.3. Let M be an R-semimodule satisfying the property $(Rm : M) \neq \emptyset$ for all $m \in M$. Then M is said to be a *weak multiplication semimodule* if for each full subsemimodule N of M, there exists a full ideal I of R such that $N = IM$.

Remark 3.4. Let M be a multiplication R-semimodule. Then for every $m \in M$, we have Rm is a subsemimodule of M and hence there is an ideal I of R such that $Rm = IM$. This implies $I \subseteq (IM : M) = (Rm : M)$ and thus $(Rm : M) \neq \emptyset$ for all $m \in M$. Moreover, for any full subsemimodule N of M, there exists an ideal J of R such that $N = JM$ and thus $J + E^+(R)$ is a full ideal of R such that $N = (J + E^+(R))M$. Therefore, it follows that every multiplication semimodule is a weak multiplication semimodule. But the converse is not true in general. This follows from the following example.

Example 3.5. We consider the set $R = \{0, 1, a, b\}$. Then $(R, +, \cdot)$ is a semiring, where addition and multiplication are defined by the following Cayley tables :

$+$	0	1	a	b
0	0	1	a	b
1	1	b	1	1
a	a	1	0	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	0	b

Now, it is easy to verify that $M = \{0, a, b\}$ is a subsemimodule of the R-semimodule R and thus M is an R-semimodule with $E(M) = \{0, b\}$. Also, it is easy to check that $(Rm : M) \neq \emptyset$ for all $m \in M$. Clearly, $N = \{0, a\}$ is a subsemimodule of M. One can easily verify that there is no ideal I of R such that $N = IM$. Hence M is not a multiplication semimodule. Now, $E(M)$ and M itself are only two full subsemimodules of M. Moreover, $E(M) = \{0, b\}M$ and $M = RM$. Therefore, M is a weak multiplication R semimodule but not a multiplication semimodule.

Theorem 3.6. *Let M be an R-semimodule such that $(Rm : M) \neq \emptyset$ for all $m \in M$. Then M is a weak multiplication semimodule if and only if for each $m \in M$ there exists a full ideal I of R such that $Rm + E(M) = IM + E(M)$.*

Proof. First suppose that an R-semimodule M is a weak multiplication semimodule. Now, for each $m \in M$, we have $Rm + E(M)$ is a subsemimodule of M. We first prove that $Rm + E(M)$ is full. For this, let $x \in E(M)$. Since $(Rm : M) \neq \emptyset$, let $r \in (Rm : M)$. Then $x = 1 \cdot x = (1 + r + r')x = x + (r + r')x$. Also $r \in (Rm : M)$ implies $rx = sm$ for some $s \in R$ and so $(rx)' = (sm)'$. Then by Theorem 2.2, we have $r'x = s'm$. Therefore, $x = x + (s + s')m = (s + s')m + x \in Rm + E(M)$. Hence $E(M) \subseteq Rm + E(M)$ and thus $Rm + E(M)$ is a full subsemimodule of M. Consequently, there exists a full ideal I of R such that $Rm + E(M) = IM$. Now $Rm + E(M) = IM \subseteq IM + E(M) \subseteq Rm + E(M) + E(M) = Rm + E(M)$ implies $Rm + E(M) = IM + E(M)$.

Conversely, suppose that the given condition holds. To show M is a weak multiplication semimodule, let N be a full subsemimodule of M. Then for each $m \in N$, there exists a full ideal I_m of R such that $Rm + E(M) = I_m M + E(M)$.

Let $I = \sum_{m \in N} I_m$. Then I is a full ideal of R such that $N = IM$. Consequently, M is a weak multiplication semimodule. □

Lemma 3.7. *Let M be a weak multiplication R-semimodule. If N is a full subsemimodule of M, then $N = (N : M)M$.*

Proof. Let N be a full subsemimodule of M. Then $N = IM$, for some full ideal I of R. Let $r \in I$. Then for any $m \in M$, we have $rm \in IM = N$ and so $r \in (N : M)$. This implies $I \subseteq (N : M)$. Therefore $N = IM \subseteq (N : M)M \subseteq N$. Hence $N = (N : M)M$. □

Lemma 3.8. *Homomorphic image of a weak multiplication semimodule is again a weak multiplication semimodule.*

Proof. Let M and M' be R-semimodules and $f : M \to M'$ be an R-epimorphism. Also, let M be a weak multiplication R-semimodule. To show M' is a weak multiplication R-semimodule, we prove that M' satisfies the property $(Rz : M') \neq \emptyset$ for all $z \in M'$. If $z \in M'$, we have $z = f(x)$ for some $x \in M$. Since M is a weak multiplication semimodule, by definition we have $(Rx : M) \neq \emptyset$. Let $r \in (Rx : M)$ and so $rM \subseteq Rx$. Then $f(rM) \subseteq f(Rx)$ implies $rf(M) \subseteq Rf(x)$. Thus $rM' = rf(M) \subseteq Rf(x) = Rz$. Hence $r \in (Rz : M')$. Consequently, $(Rz : M') \neq \emptyset$ for all $z \in M'$. The remaining part of this proof is trivially holds. □

Definition 3.9. Let M be an R-semimodule and I be an ideal of R. We define

$$T_I(M) = \{m \in M\colon \text{there exists } r \in I \text{ such that } (1 + r')m \in E(M)\}.$$

Remark 3.10. One can easily prove that $T_I(M)$ is a subsemimodule of M.

Lemma 3.11. *Let M be a weak multiplication R-semimodule. Then for every maximal ideal P of R either $M = T_P(M)$ or there exist $q \in P$ and $m \in M$ such that $(1 + q')M \subseteq Rm + E(M)$.*

Proof. Clearly PM is a subsemimodule of M. First let $M = PM$. In this case, we show that $M = T_P(M)$. For this, let $m \in M$. Now, from the proof of Theorem 3.6, it follows that $Rm + E(M)$ is a full subsemimodule of M. Since M is a weak multiplication semimodule, so there exists a full ideal A of R such that $Rm + E(M) = AM$. Then $Rm + E(M) = AM = APM = PAM = P(Rm + E(M)) \subseteq Pm + E(M)$ and thus $m = pm + m_1$ for some $p \in P$ and $m_1 \in E(M)$. This implies $(1 + p')m = (p + p')m + m_1 \in E(M)$ and thus $m \in T_P(M)$. Hence $M = T_P(M)$. On the other hand, if $M \neq PM$, then there exists an element $x \in M$ such that $x \notin PM$. Now, for the full subsemimodule $Rx + E(M)$, from the definition of weak multiplication semimodule, there is a full ideal B of R such that $Rx + E(M) = BM$. Then $B \nsubseteq P$, otherwise $BM \subseteq PM$ and so $x \in Rx + E(M) = BM \subseteq PM$ which is not possible. Now $B + P$ is an ideal of R such that for all $p_1 \in P$, we have

$p_1 = (p_1 + p_1') + p_1 \in B + P$, since B is a full ideal of R. Hence $P \subseteq B + P$. We claim that $P \neq B + P$. On the contrary if $P = B + P$, then for all $b \in B$, we have $b = b + b(p_2 + p_2') \in B + P = P$ for any $p_2 \in P$ and thus $B \subseteq P$ which is a contradiction. Since P is maximal, we must have $B + P = R$ and hence $1 = b + q$ for some $b \in B$ and $q \in P$. This implies $1 + q' = b + q + q' \in B + E^+(R)$ and thus $(1 + q')M \subseteq BM + E^+(R)M = Rx + E(M)$. □

Definition 3.12. An R-semimodule M is said to be *faithful* if $(E(M) : M) = E^+(R)$.

Lemma 3.13. *Let M be a faithful weak multiplication R-semimodule such that $E(M)$ is a k-set. Then $IM \cap JM = (I \cap J)M$ for any two full k-ideals I, J of R.*

Proof. Clearly, $(I \cap J)M \subseteq IM \cap JM$. For the reverse inclusion, let $x \in IM \cap JM$ and $x \notin (I \cap J)M$. We consider $K = \{r \in R : rx \in (I \cap J)M\}$. Then $1 \notin K$ and thus K is a proper ideal of R. Then there exists a maximal ideal P of R such that $K \subseteq P$. We claim that $x \notin T_P(M)$. Otherwise there exists $p \in P$ such that $(1+p')x \in E(M) \subseteq (I \cap J)M$ and so $1+p' \in K \subseteq P$ implies $1 = 1+p+p' \in P$ and thus $P = R$, which is a contradiction. Then by Lemma 3.11, there exist elements $m \in M$ and $p_1 \in P$ such that $(1 + p_1')M \subseteq Rm + E(M)$. Now $x \in IM$ implies $x = r_1 m_1 + r_2 m_2 + \cdots + r_k m_k$, where $r_i \in I$ and $m_i \in M$. Then $(1 + p_1')x = (1 + p_1')(r_1 m_1 + r_2 m_2 + \cdots + r_k m_k) = r_1(1+p_1')m_1 + r_2(1+p_1')m_2 + \ldots + r_k(1+p_1')m_k = r_1(s_1 m + t_1) + r_2(s_2 m + t_2) + \ldots + r_k(s_k m + t_k) = r_1 s_1 m + r_2 s_2 m + \ldots + r_k s_k m + n \in Im + E(M)$, where $s_i \in R$ and $n = r_1 t_1 + r_2 t_2 + \ldots + r_k t_k \in E(M)$. Similarly, $(1+p_1')x \in Jm + E(M)$. So $(1+p_1')x = a_1 m + n_1 = a_2 m + n_2$, where $a_1 \in I, a_2 \in J$ and $n_1, n_2 \in E(M)$. Then $(a_1 + a_2')m + n_1 = (a_2 + a_2')m + n_2 \in E(M)$. Since $E(M)$ is a k-set, it follows that $(a_1 + a_2')m \in E(M)$. Now $(1 + p_1')(a_1 + a_2')M = (a_1 + a_2')(1 + p_1')M \subseteq (a_1 + a_2')(Rm + E(M)) \subseteq E(M)$. So $(1 + p_1')(a_1 + a_2') \in (E(M) : M) = E^+(R) \subseteq J$. Therefore $(1 + p_1')a_1 + (1 + p_1')(a_2 + a_2') = (1 + p_1')(a_1 + a_2') + (1 + p_1')a_2 \in J$. This implies $(1 + p_1')a_1 \in J$ as J is a k-ideal. So $(1 + p_1')^2 x = (1 + p_1')(a_1 m + n_1) \in (I \cap J)M + E(M) \subseteq (I \cap J)M$ and therefore, $(1 + p_1')^2 \in K \subseteq P$ and hence $1 \in P$, which is a contradiction. Therefore, $IM \cap JM \subseteq (I \cap J)M$ and thus $IM \cap JM = (I \cap J)M$. □

Theorem 3.14. *Let M be a finitely generated weak multiplication R-semimodule. Then for every maximal ideal P of R, there exist $m \in M$, $q \in P + E^+(R)$ such that $(1 + q')M \subseteq Rm + E(M)$.*

Proof. Since M is finitely generated, there exists a positive integer n and elements $m_i \in M$ such that $M = Rm_1 + Rm_2 + \cdots + Rm_n$. Let P be a maximal ideal of R. Then by Lemma 3.11, we have either $M = T_P(M)$ or there exist $q \in P$, $m \in M$ such that $(1 + q')M \subseteq Rm + E(M)$. Suppose $M = T_P(M)$. Then for each $i = 1, 2, \ldots, n$, there exists $p_i \in P$ such that $(1 + p_i')m_i \in E(M)$. Let $q = 1 + [(1 + p_1') \cdots (1 + p_n')]' = 1 + 1' + p_0$ for some $p_0 \in P$. Then $q \in P + E^+(R)$. Now $(1 + q')m_i = [1 + 1' + (1 + p_1') \cdots (1 + p_n')]m_i$ implies

$$(1 + q')M = (1 + q')(Rm_1 + Rm_2 + \cdots + Rm_n) \subseteq E(M) \subseteq Rm + E(M),$$

for any m. Thus, there exist $m \in M$, $q \in P + E^+(R)$ such that $(1 + q')M \subseteq Rm + E(M)$. $\square$

Lemma 3.15. *Let M be a faithful weak multiplication R-semimodule such that $E(M)$ is a k-set. Then PM is a full prime subsemimodule of M for every full prime k-ideal P of R such that $PM \neq M$.*

Proof. Clearly, PM is a full subsemimodule of M. To show PM is prime, let $ax \in PM$ for some $a \in R$ and $x \in M$. Let $a \notin P$. We consider the ideal $K = \{r \in R : rx \in PM\}$. If $K \neq R$, then there exist a maximal ideal Q of R such that $K \subseteq Q$. Note that $x \notin T_Q(M)$, otherwise, there exist $q \in Q$ such that $(1 + q')x \in E(M) \subseteq PM$ and so $1 + q' \in K \subseteq Q$ which is not possible. Then by Lemma 3.11, there exist $q \in Q$ and $m \in M$ such that $(1 + q')M \subseteq Rm + E(M)$. Now $(1 + q')x \in (1 + q')M \subseteq Rm + E(M)$. So $(1 + q')x = sm + m_1$, for some $s \in R$ and $m_1 \in E(M)$. Also $(1 + q')ax \in (1 + q')PM \subseteq Pm + E(M)$ and so $(1 + q')ax = pm + m_2$, for some $p \in P$ and $m_2 \in E(M)$. Therefore, $asm + am_1 = pm + m_2$. Thus $(as + p')m + am_1 \in E(M)$. Since $E(M)$ is k-set, we must have $(as + p')m \in E(M)$. Now we claim that $(1 + q')(E(M) : m) \subseteq (E(M) : M)$. To show this, let $(1 + q')r \in (1 + q')(E(M) : m)$ for some $r \in (E(M) : m)$. Then for any $u \in M$, $(1 + q')u \in (1 + q')M \subseteq Rm + E(M)$. So we have $(1 + q')u = tm + w$ for some $t \in R$ and $w \in E(M)$. Then $(1 + q')ru = r(tm + w) \in E(M)$. Therefore, $(1 + q')(E(M) : m) \subseteq (E(M) : M)$ and hence $(1 + q')(as + p') \in (E(M) : M) = E^+(R) \subseteq P$. Thus $(1 + q')as + (1 + q')(p + p') \in P$ and so $(1 + q')as \in P$, since P is a k-set. Hence $s \in P$ and thus $(1 + q')x \in PM$ which gives $(1 + q') \in K$ which is a contradiction. Thus $K = R$ and so $x \in PM$. Therefore, $ax \in PM$ implies either $a \in P$ or $x \in PM$. Consequently, PM is a prime subsemimodule of M. $\square$

Following Lemma 3.7 and Lemma 3.15, we have the following two results.

Theorem 3.16. *Let M be a faithful weak multiplication R-semimodule such that $E(M)$ is a k-set. If P be a full prime k-ideal of R such that $PM \neq M$, then $(PM : M) = P$.*

Theorem 3.17. *Let N be a proper full k-subsemimodule of a faithful weak multiplication R-semimodule M such that $E(M)$ is a k-set. Then the following statements are equivalent :*

(i) *N is prime.*
(ii) *$(N : M)$ is full prime k-ideal of R.*
(iii) *$N = PM$ for some full prime k-ideal P of R with $(E(M) : M) \subseteq P$.*

Lemma 3.18. *If M is a finitely generated faithful weak multiplication R-semimodule and A, B are two full ideals of R such that $AM \subseteq BM$. Then $\overline{A} \subseteq \overline{B}$.*

Proof. To complete the proof, it is enough to prove that $A \subseteq \overline{B}$. Let $a \in A$ be an element such that $a \notin \overline{B}$. We consider $K = \{r \in R : ra \in \overline{B}\}$. Then K is a proper full ideal of R and hence there exists a maximal ideal P of R such that

$K \subseteq P$. Since M is finitely generated, so by Theorem 3.14, there exist $m \in M$ and $p \in P + E^+(R)$ such that $(1+p')M \subseteq Rm + E(M)$. Since K is full, it follows that $p \in P$. Now $(1+p')am \in (1+p')AM \subseteq (1+p')BM = B(1+p')M \subseteq B(Rm + E(M)) \subseteq Bm + E(M)$ implies $((1+p')a + b')m \in E(M)$ for some $b \in B$ and hence $(1+p')a + b' \in (E(M) : m)$. Now we show that $(1+p')(E(M) : m) \subseteq (E(M) : M)$. For this, let $(1+p')r \in (1+p')(E(M) : m)$ for some $r \in (E(M) : m)$. Now for any $x \in M$, $(1+p')x \in (1+p')M \subseteq Rm + E(M)$ implies $(1+p')x = tm + w$ for some $w \in E(M)$ and $t \in R$. Then $(1+p')rx = r(tm + w) \in E(M)$ and hence $(1+p')r \in (E(M) : M)$. Therefore, $(1+p')(E(M) : m) \subseteq (E(M) : M)$ and thus $(1+p')((1+p')a + b') \in (E(M) : M) = E^+(R) \subseteq B$. Hence $(1+p')^2 a + (1+p')(b+b') \in B$. This implies $(1+p')^2 a \in \overline{B}$, so $(1+p')^2 \in K \subseteq P$ and $1 + p' \in P$, a contradiction. This contradiction ensures that $a \in \overline{B}$ and hence $A \subseteq \overline{B}$. Consequently, $\overline{A} \subseteq \overline{B}$. □

Lemma 3.19. *Let M be a finitely generated faithful weak multiplication R-semimodule. If I is a full ideal of R, then $\overline{(IM : M)} = \overline{I}$.*

Proof. Clearly, $\overline{I} \subseteq \overline{(IM : M)}$. For the reverse inclusion, let $(IM : M) = Q$. Then $QM \subseteq IM$ and hence by Lemma 3.18, we have $\overline{(IM : M)} = \overline{Q} \subseteq \overline{I}$. Consequently, $\overline{(IM : M)} = \overline{I}$. □

Theorem 3.20. *Let M be a finitely generated faithful weak multiplication R-semimodule and K be a proper subset of M. Then K is a full maximal subsemimodule of M if and only if there exists a full maximal ideal P of R such that $K = PM$.*

Proof. First suppose that there exists a full maximal ideal P of R such that $K = PM$. Then clearly K is a full subsemimodule of M. To show K is maximal, let N be any other proper subsemimodule of M such that $K \subseteq N$. As M is a weak multiplication R-semimodule, we have $N = IM$, for some full ideal I of R. Since $N \neq M$, it follows that $I \neq R$. Now, $K \subseteq N$ implies $PM \subseteq IM$ and hence by Lemma 3.18, we have $P \subseteq \overline{P} \subseteq \overline{I}$. Now we prove that $\overline{I} \neq R$. Otherwise $1 \in \overline{I}$ and so $1 + a \in I$ for some $a \in I$. This gives $1 = 1 + a + a' \in I$ and thus $I = R$, a contradiction. Since P is a full maximal ideal of R, it follows that $P = \overline{I}$. Therefore $N = IM \subseteq \overline{I}M = PM = K$and hence $K = N$. Consequently, K is a full maximal subsemimodule of M.

Conversely, let K be a full maximal subsemimodule of M. Then by Lemma 3.7, it follows that $K = (K : M)M$. Let $Q = (K : M)$. Then Q is a full ideal of R. To complete the proof, it remains to show that Q is a full maximal ideal of R. For this, let Q_1 be any other proper full ideal of R such that $Q \subseteq Q_1$. Then $(K : M) \subseteq Q_1$. This gives $K \subseteq Q_1 M$. If $Q_1 M = M$, then $(Q_1 M : M) = (M : M) = R$. This implies $R = (Q_1 M : M) \subseteq \overline{(Q_1 M : M)} \subseteq \overline{Q_1}$, by Lemma 3.19 and hence $\overline{Q_1} = R$. This implies $1 \in \overline{Q_1}$, i.e., $1 + q_1 \in Q_1$ for some $q_1 \in Q_1$. Therefore, $1 = 1 + q_1 + q_1' \in Q_1$ and thus $Q_1 = R$, a contradiction. This contradiction ensures that $Q_1 M$ is a proper full subsemimodule of M such that $K \subseteq Q_1 M$. Since K

is a full maximal subsemimodule of M, we must have $K = Q_1M$. From this it follows that $Q_1 \subseteq (Q_1M : M) = (K : M) = Q$ and thus $Q = Q_1$. Hence Q is a full maximal ideal of R. Consequently, $K = QM$, where Q is a full maximal ideal of R. □

Theorem 3.21. *Let M be a finitely generated faithful weak multiplication R-semimodule. Then every proper full subsemimodule of M is contained in a maximal full subsemimodule of M.*

Proof. Let N be a proper full subsemimodule of M. Then by Lemma 3.7, we have $N = (N : M)M$. Let $I = (N : M)$. Then I is a full ideal of R and hence contained in a maximal ideal P of R. By Theorem 3.20, it follows that PM is a full maximal subsemimodule of M and $N = IM \subseteq PM$. Hence every proper full subsemimodule of M is contained in a full maximal subsemimodule of M. □

Theorem 3.22. *Let M be an R-semimodule such that $(Rm : M) \neq \emptyset$ for all $m \in M$. Then every full maximal subsemimodule of M is a prime subsemimodule of M.*

Proof. Let N be a full maximal subsemimodule of M. To show N is prime, let $rx \in N$ for some $r \in R$, $x \in M$ such that $x \notin N$. Now we show that $N \subset N + \langle x \rangle$. For this, let $y \in N$. Since $(Rm : M) \neq \emptyset$ for all $m \in M$, let $s \in (Rx : M)$. Then $y = 1 \cdot y = (1 + s + s')y = y + (s + s')y$. Also, $s \in (Rx : M)$ implies $sy = tx$ for some $t \in R$ and so $s'y = t'x$. Then we have $y = y + (t + t')x \in N + \langle x \rangle$. Therefore, $N \subseteq N + \langle x \rangle$. Again, $x = x + x' + x \in N + \langle x \rangle$ and $x \notin N$. Consequently, N is properly contained in $N + \langle x \rangle$. Since N is a maximal subsemimodule of M, we have $N + \langle x \rangle = M$. Therefore, $rN + r\langle x \rangle = rM$. Since $rx \in N$, it follows that $rM \subseteq N$ and hence $r \in (N : M)$. Consequently, N is a prime subsemimodule of M. □

Acknowledgement. The authors are thankful to the Referee for the valuable suggestions which have definitely improved the presentation of the article.

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Received April 25, 2021

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