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Normal subgyrogroups of certain gyrogroups

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Abstract. Suppose that $(T, \star)$ is a groupoid with a left identity such that each element $a \in T$ has a left inverse. Then T is called a *gyrogroup* if and only if (i) there exists a function $gyr : T \times T \longrightarrow Aut(T)$ such that for all $a, b, c \in T$, $a \star (b \star c) = (a \star b) \star gyr[a, b]c$, where $gyr[a, b]c = gyr(a, b)(c)$; and (ii) for all $a, b \in T$, $gyr[a, b] = gyr[a \star b, b]$. In this paper, the structure of normal subgyrogroups of certain gyrogroups are investigated.

1. Introduction

Gyrogroup theory started in 1988 by Ungar [5] in which he proved that the set of all 3-dimensional relativistically admissible velocities possesses a group-like structure in which the group-like operation is given by the standard relativistic velocity composition law. In another paper [6], he has shown that the Thomas rotation, in turn, gives rise to a non-associative group-like structure for the set of relativistically admissible velocities. Nowadays this non-associative group-like structure is known as a gyrogroup. In an algebraic language, if $(T, \star)$ is a groupoid with a left identity such that each element $a \in T$ has a left inverse. Then T is called a *gyrogroup* if and only if (i) there exists a function $gyr : T \times T \longrightarrow Aut(T)$ such that for all $a, b, c \in T$, $a \star (b \star c) = (a \star b) \star gyr[a, b]c$, where $gyr[a, b]c = gyr(a, b)(c)$; and (ii) for all $a, b \in T$, $gyr[a, b] = gyr[a \star b, b]$. It is easy to see that it is a generalization of a group by defining the gyroautomorphisms to be the identity automorphism.

Let T be a gyrogroup and let H be a non-empty subset of T. If H is a group under the induced operation of T, then H is called a *subgroup* of T, and if H is a gyrogroup under the induced operation of T, then we use the term *subgyrogroup* for H. If H is the kernel of a homomorphism from T to another gyrogroup, then H is called a *normal subgyrogroup* of T; see [2] for more details.

Suppose that $(H^+, \oplus)$ is a gyrogroup, H^- is a non-empty set disjoint from H^+ such that $|H^+| = |H^-|$ and $\varphi : H^+ \longrightarrow H^-$ is bijective. Set $G = H^+ \cup H^-$ and $a^- = \varphi(a^+)$. For arbitrary elements $a^\varepsilon, b^\delta \in G$, we define:

$$a^\epsilon \otimes b^\delta = \begin{cases} a^+ \oplus b^+ & \epsilon = \delta = + \ or \ \epsilon = \delta = - \\ (a^+ \oplus b^+)^- & \epsilon = +, \ \delta = - \ or \ \epsilon = -, \ \delta = + \end{cases}.$$

The function $gyr_G : G \times G \longrightarrow Aut(G)$ is given by

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$$gyr_G[a^\epsilon, b^\delta](t^\gamma) = \begin{cases} gyr_{H^+}[a^+, b^+](t^+) & \gamma = + \\ (gyr_{H^+}[a^+, b^+](t^+))^- & \gamma = - \end{cases},$$

where a^ϵ, b^δ and t^γ are arbitrary elements of G.

The aim of this paper is to prove the following theorem:

Theorem 1.1. *A non-empty subset M of G is a normal subgyrogroup if and only if one of the following conditions are satisfied:*

1. $M \trianglelefteq H^+$;
2. *there exists* $N^+ \trianglelefteq H^+$ *and* $L^- \subseteq H^-$ *such that* $M = N^+ \cup L^-$ *and for each* $x, y \in L^-$, $x \otimes y \in N^+$. *Also,* $N^+ \cap L^+ = \emptyset$ *and* $N^+ \cup L^+ \trianglelefteq H^+$, *where* $L^+ = \varphi^{-1}(L^-)$;
3. $M = N^+ \cup N^-$ *such that* $N^+ \trianglelefteq H^+$ *and* $N^- = \varphi(N^+)$.

Throughout this paper, our notations are standard and can be taken mainly from the books [7, 9]. We refer the readers to consult the survey article [8] for a complete history of gyrogroups. Our calculations are checked by the aid of GAP [10].

2. Preliminary results

The following result of Suksumran [2, Theorem 32] is crucial throughout this paper:

Theorem 2.1. *Let $(T, \star)$ be a gyrogroup containing a subgyrogroup H. Then H is normal in T if and only if for all $a, b \in T$, $a \star (H \star b) = (a \star b) \star H = (a \star H) \star b$.*

The present authors [1] obtained the structure of the subgyrogroups of G which is important in finding its normal subgyrogroups.

Theorem 2.2. *With above notations, $(G, \otimes)$ is a gyrogroup. A non-empty subset B of G is a subgyrogroup of $(G, \otimes)$ if and only if one of the following three conditions hold:*

(a) $B \leq H^+$;

(b) *there exists* $A^+ \leq H^+$ *and* $L^- \subseteq H^-$ *such that* $B = A^+ \cup L^-$ *and for each* $x, y \in L^-$, $x \otimes y \in A^+$. *Moreover,* $A^+ \cap L^+ = \emptyset$ *and* $A^+ \cup L^+ \leq H^+$, *where* $L^+ = \varphi^{-1}(L^-)$;

(c) $B = A^+ \cup A^-$ *such that* $A^+ \leq H^+$ *and* $A^- = \varphi(A^+)$.

Corollary 2.3. *The gyrogroup $(G, \otimes)$ satisfies the following conditions:*

1. $H^+ \trianglelefteq G$.
2. *If* $N^+ \trianglelefteq H^+$, *then* $N^+ \trianglelefteq G$

Proof. The case (1) is [1, Theorem 2.10]. To prove (2), we assume that a^ϵ, b^δ are arbitrary elements of G. We consider two cases as follows:

(i) $\epsilon = \delta = +$ *or* $\epsilon = \delta = -$. By definition,

$$\begin{aligned}
(a^+ \otimes b^+) \otimes N^+ &= (a^+ \oplus b^+) \otimes N^+ = (a^+ \oplus b^+) \oplus N^+ \\
(a^+ \otimes N^+) \otimes b^+ &= (a^+ \oplus N^+) \otimes b^+ = (a^+ \oplus N^+) \oplus b^+ \\
a^+ \otimes (N^+ \otimes b^+) &= a^+ \otimes (N^+ \oplus b^+) = a^+ \oplus (N^+ \oplus b^+) \\
(a^- \otimes b^-) \otimes N^+ &= (a^+ \oplus b^+) \otimes N^+ = (a^+ \oplus b^+) \oplus N^+ \\
(a^- \otimes N^+) \otimes b^- &= (a^+ \oplus N^+)^- \otimes b^- = (a^+ \oplus N^+) \oplus b^+ \\
a^- \otimes (N^+ \otimes b^-) &= a^- \otimes (N^+ \oplus b^+)^- = a^+ \oplus (N^+ \oplus b^+)
\end{aligned}$$

Note that by our assumption, $N^+ \trianglelefteq H^+$ and so for all $a^+, b^+ \in H^+$,

$$(a^+ \oplus b^+) \oplus N^+ = (a^+ \oplus N^+) \oplus b^+ = a^+ \oplus (N^+ \oplus b^+).$$

This shows that

$$(a^\epsilon \otimes b^\delta) \otimes N^+ = (a^\epsilon \otimes N^+) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta).$$

(ii) $(\epsilon, \delta) = (+, -)$ *or* $(-, +)$. By definition,

$$\begin{aligned}
(a^+ \otimes b^-) \otimes N^+ &= (a^+ \oplus b^+)^- \otimes N^+ = ((a^+ \oplus b^+) \oplus N^+)^- \\
(a^+ \otimes N^+) \otimes b^- &= (a^+ \oplus N^+) \otimes b^- = ((a^+ \oplus N^+) \oplus b^+)^- \\
a^+ \otimes (N^+ \otimes b^-) &= a^+ \otimes (N^+ \oplus b^+)^- = (a^+ \oplus (N^+ \oplus b^+))^- \\
(a^- \otimes b^+) \otimes N^+ &= (a^+ \oplus b^+)^- \otimes N^+ = ((a^+ \oplus b^+) \oplus N^+)^- \\
(a^- \otimes N^+) \otimes b^+ &= (a^+ \oplus N^+)^- \otimes b^+ = ((a^+ \oplus N^+) \oplus b^+)^- \\
a^- \otimes (N^+ \otimes b^+) &= a^- \otimes (N^+ \oplus b^+) = (a^+ \oplus (N^+ \oplus b^+))^-
\end{aligned}$$

By our assumption, $N^+ \trianglelefteq H^+$ and so for all $a^+, b^+ \in H^+$,

$$(a^+ \oplus b^+) \oplus N^+ = (a^+ \oplus N^+) \oplus b^+ = a^+ \oplus (N^+ \oplus b^+).$$

Since ϕ is bijective,

$$((a^+ \oplus b^+) \oplus N^+)^- = ((a^+ \oplus N^+) \oplus b^+)^- = (a^+ \oplus (N^+ \oplus b^+))^-.$$

This shows that

$$(a^\epsilon \otimes b^\delta) \otimes N^+ = (a^\epsilon \otimes N^+) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta).$$

This proves that $N^+ \trianglelefteq G$, as desired. ∎

3. Proof of the main result

The aim of this section is to prove the main result of this paper. To do this, we assume that M is a normal subgyrogroup of the gyrogroup G introduced in Section 1. By Theorem 2.1, for each $a^\epsilon, b^\delta \in G = H^+ \cup H^-$,

$$(a^\epsilon \otimes b^\delta) \otimes M = (a^\epsilon \otimes M) \otimes b^\delta = a^\epsilon \otimes (M \otimes b^\delta). \tag{1}$$

By Theorem 2.2, one of the following conditions hold:

(a) $M \leq H^+$;

(b) there exists $N^+ \leq H^+$ and $L^- \subseteq H^-$ such that $M = N^+ \cup L^-$ and for all $x, y \in L^-$, $x \otimes y \in N^+$. Also, $N^+ \cap L^+ = \emptyset$ and $N^+ \cup L^+ \leq H^+$, where $L^+ = \varphi^{-1}(L^-)$;

(c) $M = N^+ \cup N^-$ such that $N^- = \varphi(N^+)$.

Suppose that the condition (a) is satisfied. Then by considering $\delta = \varepsilon = +$ in Equation (1), $M \trianglelefteq H^+$. If condition (b) is satisfied, then $H^+ \cap M = (H^+ \cap N^+) \cup (H^+ \cap L^-) = H^+ \cap N^+ = N^+$. Note that by our assumption, M is normal in G and by Corollary 2.3(1), $H^+ \trianglelefteq G$. Hence, by [3, Theorem 2.2], $N^+ \trianglelefteq G$, which implies that $N^+ \trianglelefteq H^+$. To complete this part, it is enough to prove that $N^+ \cup L^+ \trianglelefteq H^+$, where $L^+ = \varphi^{-1}(L^-)$. By our assumption, $M = N^+ \cup L^- \trianglelefteq G$ and by Equation (1), $(a^\epsilon \otimes b^\delta) \otimes (N^+ \cup L^-) = (a^\epsilon \otimes (N^+ \cup L^-)) \otimes b^\delta = a^\epsilon \otimes ((N^+ \cup L^-) \otimes b^\delta)$, where a^ϵ, b^δ are arbitrary elements of G. Therefore,

$$\begin{aligned}((a^\epsilon \otimes b^\delta) \otimes N^+) \cup ((a^\epsilon \otimes b^\delta) \otimes L^-) &= ((a^\epsilon \otimes N^+) \otimes b^\delta) \cup ((a^\epsilon \otimes L^-) \otimes b^\delta) \\ &= (a^\epsilon \otimes (N^+ \otimes b^\delta)) \cup (a^\epsilon \otimes (L^- \otimes b^\delta)). \quad (2)\end{aligned}$$

We know that $N^+ \trianglelefteq H^+$, and by Corollary 2.3(2), $N^+ \trianglelefteq G$. So for all $a^\epsilon, b^\delta \in G$,

$$(a^\epsilon \otimes b^\delta) \otimes N^+ = (a^\epsilon \otimes N^+) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta) \tag{3}$$

Since $N^+ \cap L^- = \emptyset$, by Equations (2) and (3),

$$(a^\epsilon \otimes b^\delta) \otimes L^- = (a^\epsilon \otimes L^-) \otimes b^\delta = a^\epsilon \otimes (L^- \otimes b^\delta).$$

In the previous equation, we set $\epsilon = \delta = +$, then

$$(a^+ \otimes b^+) \otimes L^- = (a^+ \otimes L^-) \otimes b^+ = a^+ \otimes (L^- \otimes b^+).$$

By definition, $((a^+ \oplus b^+) \oplus L^+)^- = ((a^+ \oplus L^+) \oplus b^+)^- = (a^+ \oplus (L^+ \oplus b^+))^-$. Since φ is bijective, $(a^+ \oplus b^+) \oplus L^+ = (a^+ \oplus L^+) \oplus b^+ = a^+ \oplus (L^+ \oplus b^+)$. By the last equality and Equation (3), $((a^+ \oplus b^+) \oplus N) \cup ((a^+ \oplus b^+) \oplus L^+) = ((a^+ \oplus N) \oplus b^+) \cup ((a^+ \oplus L^+) \oplus b^+) = (a^+ \oplus (N \oplus b^+)) \cup (a^+ \oplus (L^+ \oplus b^+))$ and hence $(a^+ \oplus b^+) \oplus (N^+ \cup L^+) = (a^+ \oplus (N^+ \cup L^+)) \oplus b^+ = a^+ \oplus ((N^+ \cup L^+) \oplus b^+)$. Then $N^+ \cup L^+ \trianglelefteq H^+$. This completes the proof of (2). To prove (3), we have to show that $N^+ \trianglelefteq H^+$. By condition (c), $H^+ \cap M = (H^+ \cap N^+) \cup (H^+ \cap N^-) = H^+ \cap N^+ = N^+$. By our assumption, $M \trianglelefteq G$ and by Corollary 2.3, $H^+ \trianglelefteq G$. We now apply [3, Theorem 2.2] to deduce that $N^+ \trianglelefteq G$. Therefore, $N^+ \trianglelefteq H^+$, as desired.

Conversely, we assume that M satisfies one of the conditions (1), (2) or (3) in Theorem 1.1. It will be shown that $M \trianglelefteq G$. To do this, the following three cases will be considered:

(A) $M \trianglelefteq H^+$. By Corollary 2.3(2), $M \trianglelefteq G$.

(B) *There exists* $N^+ \trianglelefteq H^+$ *and* $L^- \subseteq H^-$ *such that* $M = N^+ \cup L^-$ *and for all* $x, y \in L^-$, $x \otimes y \in N^+$. *Also,* $N^+ \cap L^+ = \emptyset$ *and* $N^+ \cup L^+ \trianglelefteq H^+$, *where* $L^+ = \varphi^{-1}(L^-)$. Since $N^+, N^+ \cup L^+ \trianglelefteq H^+$, by Corollary 2.3(2), $N^+, N^+ \cup L^+ \trianglelefteq G$. Then by Theorem 2.1, for all $a^\epsilon, b^\delta \in G$,

$$(a^\epsilon \otimes b^\delta) \otimes N^+ = (a^\epsilon \otimes N^+) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta) \tag{4}$$

and $(a^\epsilon \otimes b^\delta) \otimes (N^+ \cup L^+) = (a^\epsilon \otimes (N^+ \cup L^+)) \otimes b^\delta = a^\epsilon \otimes ((N^+ \cup L^+) \otimes b^\delta)$. Hence, $((a^\epsilon \otimes b^\delta) \otimes N^+) \cup ((a^\epsilon \otimes b^\delta) \otimes L^+) = ((a^\epsilon \otimes N^+) \otimes b^\delta) \cup ((a^\epsilon \otimes L^+) \otimes b^\delta)$ $= (a^\epsilon \otimes (N^+ \otimes b^\delta)) \cup (a^\epsilon \otimes (L^+ \otimes b^\delta))$. Since $N^+ \cap L^+ = \emptyset$, by Equation (4) and the last equality,

$$(a^\epsilon \otimes b^\delta) \otimes L^+ = (a^\epsilon \otimes L^+) \otimes b^\delta = a^\epsilon \otimes (L^+ \otimes b^\delta). \tag{5}$$

Since φ is bijective,

$$((a^\epsilon \otimes b^\delta) \otimes L^+)^- = ((a^\epsilon \otimes L^+) \otimes b^\delta)^- = (a^\epsilon \otimes (L^+ \otimes b^\delta))^-. \tag{6}$$

Now, we claim that for all $a^\epsilon, b^\delta \in G$, the following equality holds:

$$(a^\epsilon \otimes b^\delta) \otimes L^- = (a^\epsilon \otimes L^-) \otimes b^\delta = a^\epsilon \otimes (L^- \otimes b^\delta). \tag{7}$$

We consider two cases as follows:

(B1) $(\epsilon, \delta) = (+,+)$ *or* $(-,-)$. By definition,

$$\begin{aligned}
(a^+ \otimes b^+) \otimes L^- &= (a^+ \oplus b^+) \otimes L^- = ((a^+ \oplus b^+) \oplus L^+)^- \\
(a^+ \otimes L^-) \otimes b^+ &= (a^+ \oplus L^+)^- \otimes b^+ = ((a^+ \oplus L^+) \oplus b^+)^- \\
a^+ \otimes (L^- \otimes b^+) &= a^+ \otimes (L^+ \oplus b^+)^- = (a^+ \oplus (L^+ \oplus b^+))^- \\
(a^- \otimes b^-) \otimes L^- &= (a^+ \oplus b^+) \otimes L^- = ((a^+ \oplus b^+) \oplus L^+)^- \\
(a^- \otimes L^-) \otimes b^- &= (a^+ \oplus L^+)^- \otimes b^- = ((a^+ \oplus L^+) \oplus b^+)^- \\
a^- \otimes (L^- \otimes b^-) &= a^- \otimes (L^+ \oplus b^+) = (a^+ \oplus (L^+ \oplus b^+))^-.
\end{aligned}$$

In this case, by Equation (6), our claim is true.

(B2) $(\epsilon, \delta) = (+, -)$ *or* $(-, +)$. By definition,

$$\begin{aligned}
(a^+ \otimes b^-) \otimes L^- &= (a^+ \oplus b^+)^- \otimes L^- = (a^+ \oplus b^+) \oplus L^+ \\
(a^+ \otimes L^-) \otimes b^- &= (a^+ \oplus L^+)^- \otimes b^- = (a^+ \oplus L^+) \oplus b^+ \\
a^+ \otimes (L^- \otimes b^-) &= a^+ \otimes (L^+ \oplus b^+) = a^+ \oplus (L^+ \oplus b^+) \\
(a^- \otimes b^+) \otimes L^- &= (a^+ \oplus b^+)^- \otimes L^- = (a^+ \oplus b^+) \oplus L^+ \\
(a^- \otimes L^-) \otimes b^+ &= (a^+ \oplus L^+) \otimes b^+ = (a^+ \oplus L^+) \oplus b^+ \\
a^- \otimes (L^- \otimes b^+) &= a^- \otimes (L^+ \oplus b^+)^- = a^+ \oplus (L^+ \oplus b^+)
\end{aligned}$$

Also, in this case, by Equation (5), our claim is true.

By Equations (4) and (7), we can see that $((a^\epsilon \otimes b^\delta) \otimes N^+) \cup ((a^\epsilon \otimes b^\delta) \otimes L^-) = ((a^\epsilon \otimes N^+) \otimes b^\delta) \cup ((a^\epsilon \otimes L^-) \otimes b^\delta) = (a^\epsilon \otimes (N^+ \otimes b^\delta)) \cup (a^\epsilon \otimes (L^- \otimes b^\delta))$, which is equivalent to $((a^\epsilon \otimes b^\delta) \otimes (N^+ \cup L^-)) = ((a^\epsilon \otimes (N^+ \cup L^-)) \otimes b^\delta) = (a^\epsilon \otimes ((N^+ \cup L^-) \otimes b^\delta))$ or $((a^\epsilon \otimes b^\delta) \otimes M) = ((a^\epsilon \otimes M) \otimes b^\delta) = (a^\epsilon \otimes (M \otimes b^\delta))$. This proves that $M \trianglelefteq G$.

(C) $M = N^+ \cup N^-$ *such that* $N^+ \trianglelefteq H^+$ *and* $N^- = \varphi(N^+)$. By Theorem 2.2, $M \leq G$. Since $N^+ \trianglelefteq H^+$, by Corollary 2.3, $N^+ \trianglelefteq G$. By Theorem 1.1, for all $a^\epsilon, b^\delta \in G$,

$$(a^\epsilon \otimes b^\delta) \otimes N^+ = (a^\epsilon \otimes N^+) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta). \tag{8}$$

Since φ is bijective, $((a^\epsilon \otimes b^\delta) \otimes N^+)^- = ((a^\epsilon \otimes N^+) \otimes b^\delta)^- = (a^\epsilon \otimes (N^+ \otimes b^\delta))^-$. Similar to the part B, we can show that $(a^\epsilon \otimes b^\delta) \otimes N^- = (a^\epsilon \otimes N^-) \otimes b^\delta = a^\epsilon \otimes (N^- \otimes b^\delta)$. By the last equality and Equation (8), $(a^\epsilon \otimes b^\delta) \otimes N^+ \cup (a^\epsilon \otimes b^\delta) \otimes N^- = (a^\epsilon \otimes N^+) \otimes b^\delta \cup (a^\epsilon \otimes N^-) \otimes b^\delta = a^\epsilon \otimes (N^+ \otimes b^\delta) \cup a^\epsilon \otimes (N^- \otimes b^\delta)$ and so $(a^\epsilon \otimes b^\delta) \otimes (N^+ \cup N^-) = (a^\epsilon \otimes (N^+ \cup N^-)) \otimes b^\delta = a^\epsilon \otimes ((N^+ \cup N^-) \otimes b^\delta)$. We now apply our assumption to deduce that $(a^\epsilon \otimes b^\delta) \otimes M = (a^\epsilon \otimes M) \otimes b^\delta = a^\epsilon \otimes (M \otimes b^\delta)$, which means that $M \trianglelefteq G$.

Table 1: The Cayley Table of $K(1)$ such that $A = (4, 5)(6, 7)$.

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	gyr_K	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	1	2	3	4	5	6	7	**0**	I	I	I	I	I	I	I	I
1	1	0	3	2	5	4	7	6	**1**	I	I	I	I	I	I	I	I
2	2	3	0	1	6	7	4	5	**2**	I	I	I	I	A	A	A	A
3	3	2	1	0	7	6	5	4	**3**	I	I	I	I	A	A	A	A
4	4	5	6	7	0	1	2	3	**4**	I	I	A	A	I	I	A	A
5	5	4	7	6	1	0	3	2	**5**	I	I	A	A	I	I	A	A
6	6	7	4	5	3	2	1	0	**6**	I	I	A	A	A	A	I	I
7	7	6	5	4	2	3	0	1	**7**	I	I	A	A	A	A	I	I

4. Concluding remarks

In this paper, the normal subgyrogroups of a class of finite gyrogroups is characterized. In this section, we check our main result by a Gap code on three examples that is introduced in [1]. Our Gap code is accessible from the authors upon request. Suppose that $K(1)$ is the gyrogroup such that its Cayley table is given in Table 1. We apply our method to construct the gyrogroups $K(2)$ from $K(1)$ and hence $|K(1)| = 8$ and $|K(2)| = 16$. Our calculations show that all normal

Table 2: The addition Table of $K(2)$.

$\oplus$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	3	2	1	0	14	15	12	13	11	10	9	8
7	7	6	5	4	2	3	0	1	15	14	13	12	10	11	8	9
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	11	10	9	8	6	7	4	5	3	2	1	0
15	15	14	13	12	10	11	8	9	7	6	5	4	2	3	0	1

Table 3: The gyration table of $K(2)$ such that $A = (4,5)(6,7)(12,13)(14,15)$.

gyr	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
1	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
2	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
3	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
4	I	I	A	A	I	I	A	A	I	I	A	A	I	I	A	A
5	I	I	A	A	I	I	A	A	I	I	A	A	I	I	A	A
6	I	I	A	A	A	A	I	I	I	I	A	A	A	A	I	I
7	I	I	A	A	A	A	I	I	I	I	A	A	A	A	I	I
8	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
9	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
10	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
11	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
12	I	I	A	A	I	I	A	A	I	I	A	A	I	I	A	A
13	I	I	A	A	I	I	A	A	I	I	A	A	I	I	A	A
14	I	I	A	A	A	A	I	I	I	I	A	A	A	A	I	I
15	I	I	A	A	A	A	I	I	I	I	A	A	A	A	I	I

subgyrogroups of $K(1)$ and $K(2)$ are in Table 4, respectively. In both cases, these normal subgyrogroups can be obtained from our main result. A nondegenerate gyrogroup is a gyrogroup that is not a group. In Table 4, the nondegenerate normal subgyrogroups are bolded.

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Table 4: The normal subgyrogroups of $K(1)$ and $K(2)$.

Gyrogroup	The normal subgyrogroups
K(1)	$\{0\}$, $\{0,1\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$, $\{0,1,6,7\}$, **{ 0,1,2,3,4,5,6,7}**
K(2)	$\{0\}$, $\{0,1\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$, $\{0,1,6,7\}$, **{ 0,1,2,3,4,5,6,7}**, $\{0,8\}$, $\{0,9\}$, $\{0,1,8,9\}$, $\{0,1,10,11\}$, $\{0,1,12,13\}$, $\{0,1,14,15\}$, $\{0,1,2,3,8,9,10,11\}$,**{0,1,2,3,12,13,14,15}**, $\{0,1,4,5,8,9,12,13\}$, **{ 0,1,4,5,10,11,14,15}**, $\{0,1,6,7,8,9,14,15\}$, **{ 0,1,6,7,10,11,12,13}**, **{0,1,2,3, ..., 13,14,15}**

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