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The ordered semilattice equivalence relations on ordered semihypergroups

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Abstract. The semilattice equivalence relations play an important role in investigating the structural properties of ordered semihypergroups. Such relations can be expressed in terms of hyperfilters. There are two concepts of (ordered) hyperfilters of (ordered) semihypergroups which were introduced by Tang et al. [16] and Kehayopulu[9]. In this paper, we prove that those two concepts coincide and characterize the least semilattice equivalence relations on ordered semihypergroups. Furthermore, we investigate the relationship between the semilattice equivalence relations and the strongly ordered regular equivalence relations on ordered semihypergroups. Finally, we introduce the concept of ρ-classes-chain on ordered semihypergroups and give the characterization of the strongly ordered regular equivalence relations via such concept.

1. Introduction

The investigation of ordered semihypergroups, which are a generalization of ordered semigroups, was initiated by Davvaz and Heidari [8] in 2011. As we known, the semilattice congruences on ordered semigroups play a significant role in studying the structural properties of ordered semigroups, see [12, 10, 13]. In case of ordered semihypergroups, the analogous role is played by the concept of semilattice equivalence relations which was defined through the strongly regular equivalence relations on ordered semihypergroups. Such semilattice equivalence relations can be expressed by means of hyperfilters and completely prime hyperideals. The concept of (ordered) hyperideals of ordered semihypergroups was studied by Changphas and Davvaz [1]. In 2015, Tang et al. [16] introduced the concept of (ordered) hyperfilters and completely prime (ordered) hyperideals on ordered semihypergroups. They gave the characterization of the (ordered) hyperfilters in terms of completely prime (ordered) hyperideals. Omidi and Davvaz [14] generalized some remarkable results concerning the semilattice congruences and the relation $\mathcal{N}$ on ordered semigroups to ordered semihypergroups, where $\mathcal{N}$ is generated by the same principal filters (ordered hyperfilters) of ordered semigroups (ordered semihypergroups, respectively). They showed that $\mathcal{N}$ is the semilattice equivalence relation

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on ordered semihypergroups. Also, they discussed the relationship between the Green's relation $\mathcal{J}$ and the relation $\mathcal{N}$ on ordered semihypergroups. Gu and Tang [7, 6] introduced the concept of ordered regular (strongly ordered regular) equivalence relations on ordered semihypergroups and discussed their related properties. Moreover, they introduced the notion of ordered semilattice equivalence relations on ordered semihypergroups and proved that $\mathcal{N}$ is the least ordered semilattice equivalence relation. In the meantime, they illustrated by counterexample that $\mathcal{N}$ is not the least semilattice equivalence relation on ordered semihypergroups in general. Recently, Kehayopulu[11] introduced a new concept of hyperfilters (in such paper, it is called filter) on ordered semihypergroups and used such hyperfilters to define the relation $\mathcal{N}^*$ (in such paper, the author use the notation $\mathcal{N}$). The author proved that $\mathcal{N}^*$ is the semilattice equivalence relation on ordered semihypergroups. Furthermore, the author introduced the concept of complete semilattice congruences on ordered semihypergroups and showed that $\mathcal{N}^*$ is the least complete semilattice congruence on ordered semihypergroups. From those works, the following question is natural: What is the smallest semilattice equivalence relation on ordered semihypergroups? In this paper, we attempt to solve this problem.

The present paper is organized as follows. In Section 2, we recall some basic notions and elementary results of ordered semihypergroups. We show that the Tang's hyperfilter and the Kehayopulu's hyperfilter coincide. This implies that $\mathcal{N} = \mathcal{N}^*$. In Section 3, we give the characterization of (complete) semilattice equivalence relations in terms of the (ordered) hyperfilters generated by their corresponding equivalence classes. In Section 4, we answer the previous question. Some our results are an extension and a generalization of the results on ordered semigroups given in [5]. The last section, we establish the connection between the complete semilattice equivalence relations and the strongly ordered regular equivalence relations on ordered semihypergroups. Furthermore, we introduce the concept of ρ-classes-chain on ordered semihypergroups and characterize the strongly ordered regular equivalence relation by means of such chain.

2. Preliminaries

In this section, we recall some basic results of ordered semihypergroups, see [4, 17, 18, 3].

Let S be a nonempty set and let $\mathcal{P}^*(S)$ be denoted as the set of all nonempty subsets of S. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$ is called a *hyperoperation*. A couple $(S, \circ)$ is called a *hypergroupoid*. For any $A, B \in \mathcal{P}^*(S)$ and $x \in S$, we write

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hyperoperation $\circ$ is called *associative* if $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in S$. In this case, the hypergroupoid $(S, \circ)$ is called a *semihypergroup*.

An *ordered semihypergroup* $(S, \circ, \leqslant)$ is a semihypergroup $(S, \circ)$ with a partial order $\leqslant$ that is *compatible* together with the hyperoperation $\circ$, i.e., for $x, y, z \in S$,

$$x \leqslant y \text{ implies } x \circ z \leqslant y \circ z \text{ and } z \circ x \leqslant z \circ y.$$

Note that, for any $A, B \in \mathcal{P}^*(S)$, $A \leqslant B$ means for any $a \in A$ there exists $b \in B$ such that $a \leqslant b$.

Throughout this paper, we denote S as an ordered semihypergroup $(S, \circ, \leqslant)$. For $A \in \mathcal{P}^*(S)$, A is said to be a *subsemihypergroup* of S if $A \circ A \subseteq A$. A is called a *right(left) hyperideal* of S if $A \circ S \subseteq A(S \circ A \subseteq A)$. If A is both a right and a left hyperideal of S, then A is a *hyperideal* of S. A right hyperideal (left hyperideal, hyperideal) of S is called an *ordered right hyperideal (ordered left hyperideal, ordered hyperideal)* of S if for any $y \in S$ and $x \in A$, $y \leqslant x$ implies $y \in A$.

A hyperideal A of S is said to be *prime* if, for any $x, y \in A$, $x \circ y \subseteq A$ implies $x \in A$ or $y \in A$. A hyperideal A of S is called *completely prime* if, for any $x, y \in A$, $x \circ y \cap A \neq \emptyset$ implies $x \in A$ or $y \in A$. Clearly, every completely prime hyperideal is always a prime hyperideal but the converse does not hold in general. Let $\mathcal{CP}(\mathcal{S})$ and $\mathcal{CPO}(\mathcal{S})$ denote the set of all completely prime hyperideals of S and the set of all completely prime ordered hyperideals of S, respectively. Evidently, $\emptyset \neq \mathcal{CPO}(\mathcal{S}) \subseteq \mathcal{CP}(\mathcal{S})$.

There are two concepts of (ordered) hyperfilters of ordered semihypergroups. In 2015, Tang et. al[16] introduced the following notion.

Definition 2.1 (cf. [16])**.** Let S be an ordered semihypergroup. A subsemihypergroup F of S is called an *ordered hyperfilter* of S if the following two conditions hold.

(T1) F is a hyperfilter of S, i.e., if $(x \circ y) \cap F \neq \emptyset$, for all $x, y \in S$, then $x, y \in F$.

(T2) If for any $x \in F$ and $y \in S$ such that $x \leqslant y$, then $y \in F$.

Also, they proved the following result.

Lemma 2.2. (cf. [16]) *Let S be an ordered semihypergroup and $A \in \mathcal{P}^*(S)$. Then, the following statements are equivalent.*

(i) *A is a (ordered) hyperfilter of S*

(ii) *$S \setminus A = \emptyset$ or $S \setminus A$ is a completely prime (ordered) hyperideal of S.*

In 2017, Kehayopulu[9] introduced the notion of hyperfilter (in such paper, it is called filter) on hypergroupoids and applied it to ordered hypergroupoids which presented in [11] as follows.

Definition 2.3 (cf. [11])**.** Let S be an ordered semihypergroup. A subsemihypergroup F of S is called a *hyperfilter* of S if it satisfies the conditions (K1) and (K2). Furthermore, F is called an *ordered hyperfilter* of S if it satisfies the conditions (K1), (K2) and (K3) as follows.

(K1) If $x \circ y \subseteq F$, for all $x, y \in S$, then $x, y \in F$.

(K2) For any $x, y \in S$, we have $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.

(K3) If for any $x \in F$ and $y \in S$ such that $x \leqslant y$, then $y \in F$.

To prevent the confusing, we call the Tang's (ordered) hyperfilter that *T-(ordered) hyperfilter* and call the Kehayopulu's (ordered) hyperfilter that *K-(ordered) hyperfilter*. Next, we show that the T-hyperfilter and K-hyperfilter coincide on ordered semihypergroups.

Theorem 2.4. *Let S be an ordered semihypergroup and $F \in \mathcal{P}^*(S)$. Then, F is the K-hyperfilter of S if and only if F is the T-hyperfilter of S.*

Proof. ($\Rightarrow$) Let F be a K-hyperfilter of S. Let $x, y \in S$ with $(x \circ y) \cap F \neq \emptyset$. By (K2), we have $x \circ y \subseteq F$. By (K1), we get $x, y \in F$. Thus F is a T-hyperfilter of S. ($\Rightarrow$) Suppose that F is a T-hyperfilter of S. Let $x, y \in S$ and $x \circ y \subseteq F$. It follows that $x \circ y \cap F \neq \emptyset$. Since F is a T-hyperfilter of S, by (T1), we have $x, y \in F$. Next, we show that F satisfies (K2). Let $x_1, x_2 \in S$. Since F is a T-hyperfilter of S, by Lemma 2.2, we obtain that $S \setminus F = \emptyset$ or $S \setminus F$ is a completely prime hyperideal of S. If $S \setminus F = \emptyset$, then $F = S$. So F satisfies (K2). If $S \setminus F$ is a completely prime hyperideal of S, then we consider the following two cases.
Case 1: $x_i \in S \setminus F$ for some $i \in \{1, 2\}$. Since $S \setminus F$ is a hyperideal of S, we get $x_1 \circ x_2 \subseteq S \setminus F$. So $(x_1 \circ x_2) \cap F = \emptyset$.
Case 2: $x_1, x_2 \notin S \setminus F$. Then $x_1, x_2 \in F$. Since F is a subsemihypergroup of S, we have $x_1 \circ x_2 \subseteq F$. From Case 1 and 2, we conclude that F satisfies (K2). Therefore F is a K-hyperfilter of S. □

From the previous theorem, we also conclude that the T-ordered hyperfilter and the K-ordered hyperfilter of S coincide. The present paper is based on the notion of hyperfilters of S which was defined by Tang et al. As we know, the intersection of all hyperfilters (ordered hyperfilters) of S is always a hyperfilter (an ordered hyperfilter, respectively), provided it is nonempty. The intersection of all hyperfilters (ordered hyperfilters) of S containing $A (A \in \mathcal{P}^*(S))$ is called a *hyperfilter* (an ordered hyperfilter) of S generated by A. For case $A = \{x\}$, let $n(x)$ denote the hyperfilter of S generated by x, $N(x)$ denote the ordered hyperfilter of S generated by x.

An equivalence relation ρ on S is a *semilattice equivalence relation* [14] if ρ satisfies the following conditions.

(1) ρ is a strongly regular relation[4] on S, i.e.,

$$(x, y) \in \rho \text{ implies } z \circ x \ \bar{\bar{\rho}} \ z \circ y \text{ and } x \circ z \ \bar{\bar{\rho}} \ y \circ z \quad \text{for all } x, y, z \in S$$

where, for any $A, B \in \mathcal{P}^*(S)$, $A \bar{\bar{\rho}} B$ means $(a, b) \in \rho$ for all $a \in A$ and $b \in B$.

(2) $x \ \bar{\bar{\rho}} \ x \circ x$ and $x \circ y \ \bar{\bar{\rho}} \ y \circ x$ for all $x, y \in S$.

Let $\mathcal{SR}(\mathcal{S})$ be the set of all semilattice equivalence relations on S. In addition, any $\rho \in \mathcal{SR}(\mathcal{S})$ is called a *complete semilattice equivalence relation* [11] on S if

(3) for any $x, y \in S$, $x \leqslant y$ implies $x \ \bar{\bar{\rho}}\ x \circ y$.

Let $\mathcal{CSR}(\mathcal{S})$ be the set of all complete semilattice equivalence relations on S. Clearly, $\emptyset \neq \mathcal{CSR}(\mathcal{S}) \subseteq \mathcal{SR}(\mathcal{S})$. For any $A \in \mathcal{P}^*(S)$, we define the relations on S as follows:

$$\delta_A := \{(x,y) \in S \times S : x, y \in A \text{ or } x, y \notin A\},$$
$$\eta := \{(x,y) \in S \times S : n(x) = n(y)\},$$
$$\mathcal{N} := \{(x,y) \in S \times S : N(x) = N(y)\}.$$

Clearly, the relations δ_A, η and $\mathcal{N}$ are equivalence relations on S. Moreover, we have $\delta_A = \delta_{S \setminus A}$ for all $A \in \mathcal{P}^*(S)$. Omidi and Davvaz [14] established the remarkable properties concerning the relation $\mathcal{N}$ on ordered semihypergroups as follows.

Lemma 2.5 (cf. [14])**.** *If A is a completely prime (ordered) hyperideal of S, then $\delta_A \in \mathcal{SR}(\mathcal{S})$.*

Lemma 2.6 (cf. [14])**.** *Let S be an ordered semihypergroup. Then $\mathcal{N} \in \mathcal{SR}(\mathcal{S})$ and $\mathcal{N} = \bigcap \{\delta_A : A \in \mathcal{CPO}(\mathcal{S})\}$.*

Lemma 2.7. (cf.[4]) *Let $(S, \circ)$ be a semihypergroup and ρ be a strongly regular equivalence relation on S. Then, $(S/\rho, *_\rho)$ is a semigroup with respect to the following operation: $\rho(x) *_\rho \rho(y) = \rho(z)$ for all $z \in x \circ y$ where $S/\rho := \{\rho(x) : x \in S\}$ and $\rho(x)$ denotes the equivalence class of x ($x \in S$).*

3. Complete semilattice equivalence relations

In this section, we give some properties of (complete) semilattice equivalence relations on ordered semihypergroups in terms of the (ordered) hyperfilters generated by their corresponding equivalence classes. Firstly, we give the following results which are easily to prove by applying Lemma 2.7.

Lemma 3.1. *Let $\rho \in \mathcal{SR}(\mathcal{S})$. Then the following statements hold.*

(i) *For any $x \in S$, the ρ-class $\rho(x)$ is a subsemihypergroup of S.*

(ii) *The quotient set $S/\rho := \{\rho(x) : x \in S\}$ is a commutative semigroup under the multiplication $*_\rho$ defined by $\rho(x) *_\rho \rho(y) = \rho(z)$ for all $z \in x \circ y$.*

Note that, for any $x, y, z \in S$ and $\rho \in \mathcal{SR}(\mathcal{S})$, $(x \circ y) \cap \rho(z) \neq \emptyset$ if and only if $\rho(z) = \rho(x) *_\rho \rho(y)$. In fact, let $a \in (x \circ y) \cap \rho(z)$. Then $a \in x \circ y$ and $\rho(a) = \rho(z)$. Since $\rho \in \mathcal{SR}(\mathcal{S})$, we have $x \circ y \ \bar{\bar{\rho}}\ x \circ y$ and it implies that $(a, b) \in \rho$ for all $b \in x \circ y$. By Lemma 3.1(ii), we get $\rho(b) = \rho(x) *_\rho \rho(y)$ for all $b \in x \circ y$. Thus $\rho(z) = \rho(a) = \rho(b) = \rho(x) *_\rho \rho(y)$. Conversely, if $\rho(z) = \rho(x) *_\rho \rho(y)$ then by Lemma 3.1(ii), we have $\rho(x) *_\rho \rho(y) = \rho(c)$ for all $c \in x \circ y$. Consequently, $c \in \rho(c) = \rho(z)$ and then $c \in (x \circ y) \cap \rho(z) \neq \emptyset$.

Lemma 3.2. *Let S be an ordered semihypergroup and $A \in \mathcal{P}^*(S)$. Then the following statements are equivalent.*

(i) $\delta_A \in \mathcal{CSR}(\mathcal{S})$.

(ii) *One of A or $S \setminus A$ is a completely prime ordered hyperideal of S.*

Proof. $(i) \Rightarrow (ii)$ Let $\delta_A \in \mathcal{CSR}(\mathcal{S})$ and $A \in \mathcal{P}^*(S)$. If $A = S$, then we are done. Suppose that $A \neq S$. Firstly, we show that A and $S \setminus A$ are subsemihypergroups of S. Let $x, y \in A$. Then $(x, y) \in \delta_A$. Since $\delta_A \in \mathcal{CSR}(\mathcal{S})$, we have $x \circ y \ \overline{\overline{\delta_A}} \ y \circ y$ and $y \circ y \ \overline{\overline{\delta_A}} \ y$. It follows that $x \circ y \ \overline{\overline{\delta_A}} \ y$. Since $y \in A$, we get $x \circ y \subseteq A$. So A is a subsemihypergroup of S. Using the same process, we can show that $S \setminus A$ is a subsemihypergroup of S. Next, we show that A is an ordered hyperideal of S. Let $x \in A$ and $y \in S$. We consider two cases as follows.

<u>Case 1:</u> $x \circ y \subseteq A$. Since $\delta_A \in \mathcal{CSR}(\mathcal{S})$, we get $x \circ y \ \overline{\overline{\delta_A}} \ y \circ x$. Consequently, $y \circ x \subseteq A$ and so A is a hyperideal of S. Next, let $u \in S$ and $v \in A$ with $u \leqslant v$. Since $\delta_A \in \mathcal{CSR}(\mathcal{S})$, we have $u \ \overline{\overline{\delta_A}} \ u \circ v$. Since A is a hyperideal of S and $v \in A$, we get $u \circ v \subseteq A$. By the definition of δ_A, it follows that $u \in A$. So A is an ordered hyperideal of S.

<u>Case 2:</u> $x \circ y \nsubseteq A$. Then $y \notin A$. Indeed, if $y \in A$, then, since A is a subsemihypergroup of S, we have $x \circ y \subseteq A$. It is impossible. Hence $y \notin A$. We have $(y, z) \in \delta_A$ for all $z \in x \circ y \setminus A$. By Lemma 3.1(ii), we obtain that $\delta_A(y) = \delta_A(z) = \delta_A(x) *_{\delta_A} \delta_A(y)$. Since $x \in A$ and $y \notin A$, we get $\delta_A(x) = A$ and $\delta_A(y) = S \setminus A$. It follows that $S \setminus A = A \circ (S \setminus A)$. Furthermore, we have

$$\begin{aligned} S \circ (S \setminus A) &= (A \cup S \setminus A) \circ (S \setminus A) \\ &\subseteq (A \circ (S \setminus A)) \cup ((S \setminus A) \circ (S \setminus A)) \subseteq (S \setminus A) \cup (S \setminus A) = S \setminus A, \end{aligned}$$

since $S \setminus A$ is a subsemihypergroup of S. Consequently, $S \setminus A$ is a left hyperideal of S. As we known that, for any $x, y \in S$, if $y \circ x \subseteq A$, then, since $y \circ x \ \overline{\overline{\delta_A}} \ x \circ y$, we have $x \circ y \subseteq A$. Since $x \circ y \nsubseteq A$, it follows that $y \circ x \nsubseteq A$. Using the same process, we can show that $S \setminus A$ is a right hyperideal of S and hence $S \setminus A$ is a hyperideal of S. Next, let $u \in S$ and $v \in S \setminus A$. If $u \leqslant v$, then, since $\delta_{S \setminus A} = \delta_A \in \mathcal{CSR}(\mathcal{S})$, we have $u \ \overline{\overline{\delta_{S \setminus A}}} \ u \circ v$. Since $S \setminus A$ is a hyperideal of S and $v \in S \setminus A$, we get $u \circ v \subseteq S \setminus A$ and hence $u \in S \setminus A$. Consequently, $S \setminus A$ is an ordered hyperideal of S. Finally, we show that A is completely prime. Let $x, y \notin A$. Then $(x, y) \in \delta_A$. Since δ_A is a strongly regular, we obtain that $x \circ x \ \overline{\overline{\delta_A}} \ x \circ y$. Since $\delta_A \in \mathcal{CSR}(\mathcal{S})$, we have $x \ \overline{\overline{\delta_A}} \ x \circ x$. So $x \ \overline{\overline{\delta_A}} \ x \circ y$. Since $x \notin A$, by the definition of δ_A, we get $(x \circ y) \cap A = \emptyset$. Consequently, A is a completely prime ordered hyperideal of S. Similarly, we can show that $S \setminus A$ is also a completely prime ordered hyperideal.

(ii) $\Rightarrow$ (i) Since $\delta_A = \delta_{S \setminus A}$ for all $A \in \mathcal{P}^*(S)$, by Lemma 2.5, we get $\delta_A \in \mathcal{SR}(\mathcal{S})$. Next, we show that $\delta_A \in \mathcal{CSR}(\mathcal{S})$. Firstly, suppose that A is a completely prime ordered hyperideal of S. Let $x, y \in S$ with $x \leqslant y$. If $y \in A$, then, since A is an ordered hyperideal of S, we get $x \in A$ and so $x \circ y \subseteq A$. It follows that $x \ \overline{\overline{\delta_A}} \ x \circ y$.

Hence $\delta_A \in \mathcal{CSR}(\mathcal{S})$. If $y \notin A$, then we consider the following two cases.
Case 1: $(x \circ y) \cap A \neq \emptyset$. Since A is completely prime and $y \notin A$, we have $x \in A$. Since A is an ordered hyperideal of S, we get $x \circ y \subseteq A$ and so $x \overline{\overline{\delta_A}} x \circ y$.
Case 2: $(x \circ y) \cap A = \emptyset$. Then $x \circ y \subseteq S \setminus A$. If $x \in A$, then, since A is an ordered hyperideal of S, we get $x \circ y \subseteq A$, which is a contradiction. So $x \notin A$. Hence $x \overline{\overline{\delta_A}} x \circ y$. From Case 1 and 2, we conclude that $\delta_A \in \mathcal{CSR}(\mathcal{S})$. Similarly, for case $S \setminus A$ is a completely prime ordered hyperideal of S, we can also show that $\delta_A = \delta_{S \setminus A} \in \mathcal{CSR}(\mathcal{S})$ and the proof is completed. $\square$

Corollary 3.3. *Let S be an ordered semihypergroup and $A \in \mathcal{P}^*(S)$. Then, the following statements are equivalent.*

(i) $\delta_A \in \mathcal{SR}(\mathcal{S})$.

(ii) *One of A or $S \setminus A$ is a completely prime hyperideal of S.*

Let ρ be an equivalence relation on S. For any $x \in S$, let $\langle \rho(x) \rangle_f$ denote the hyperfilter of S generated by ρ-class $\rho(x)$; t denote the hyperfilter of S generated by $\bigcup\limits_{y \in \rho(x)} n(y)$; $\langle \rho(x) \rangle_F$ denote the ordered hyperfilter of S generated by ρ-class $\rho(x)$; T denote the ordered hyperfilter of S generated by $\bigcup\limits_{y \in \rho(x)} N(y)$.

The following lemmas extend and generalize the results of filters and ordered filters on ordered semigroups, see Lemma 2.4 and Lemma 2.6 in [5].

Lemma 3.4. *Let S be an ordered semihypergroup, $\rho \in \mathcal{SR}(\mathcal{S})$ and $x \in S$. Then the following statements hold.*

(i) $\langle \rho(x) \rangle_f = \Big\{ a \in S : a \in \rho(x) \text{ or } u \circ a \cap \rho(x) \neq \emptyset \text{ for some } u \in \langle \rho(x) \rangle_f \Big\}$.

(ii) $\langle \rho(x) \rangle_f = t$.

(iii) *If $y \in \langle \rho(x) \rangle_f$, then $\langle \rho(y) \rangle_f \subseteq \langle \rho(x) \rangle_f$.*

(iv) $\rho = \Big\{ (x, y) \in S \times S : \langle \rho(x) \rangle_f = \langle \rho(y) \rangle_f \Big\}$.

Proof. (i) Let $N = \Big\{ a \in S : a \in \rho(x) \text{ or } u \circ a \cap \rho(x) \neq \emptyset \text{ for some } u \in \langle \rho(x) \rangle_f \Big\}$. Clearly, $\rho(x) \subseteq N \subseteq \langle \rho(x) \rangle_f$. Indeed, if $a \in N$, then $a \in \rho(x)$ or $u \circ a \cap \rho(x) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_f$. Since $\emptyset \neq u \circ a \cap \rho(x) \subseteq u \circ a \cap \langle \rho(x) \rangle_f$ and $\langle \rho(x) \rangle_f$ is a hyperfilter of S, we have $a \in \langle \rho(x) \rangle_f$. So $N \subseteq \langle \rho(x) \rangle_f$. The following assertions hold.

(1) N is a subsemihypergroup of S. Indeed, let $a, b \in N$. There exist 4 cases to be considered as follows.
Case 1.1 $a, b \in \rho(x)$. By Lemma 3.1(i), we have $a \circ b \subseteq \rho(x) \subseteq N$.
Case 1.2 $a \in \rho(x)$ and $u \circ b \cap \rho(x) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_f$. We have $\rho(x) = \rho(u) *_\rho \rho(b)$. Since $\rho \in \mathcal{SR}(\mathcal{S})$, we have $x \,\overline{\overline{\rho}}\, x \circ x$ and then $\rho(x) = \rho(x) *_\rho \rho(x)$. By Lemma

3.1(ii) we have $\rho(x) = \rho(x) *_\rho \rho(x) = (\rho(u) *_\rho \rho(b)) *_\rho \rho(a) = \rho(u) *_\rho (\rho(a) *_\rho \rho(b))$. Let $z \in a \circ b$. By Lemma 3.1(ii), we have $\rho(x) = \rho(u) *_\rho \rho(z)$ and it follows that $u \circ z \cap \rho(x) \neq \emptyset$. Hence $z \in N$ and it implies that $a \circ b \subseteq N$.
<u>Case 1.3</u> $b \in \rho(x)$ and $u \circ a \cap \rho(x) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_f$. The proof is similar to Case 1.2 and we get $a \circ b \subseteq N$.
<u>Case 1.4</u> $u_1 \circ a \cap \rho(x) \neq \emptyset$ and $u_2 \circ b \cap \rho(x) \neq \emptyset$ for some $u_1, u_2 \in \langle \rho(x) \rangle_f$. By Lemma 3.1(ii), we have $\rho(u_1) *_\rho \rho(a) = \rho(x) = \rho(u_2) *_\rho \rho(b)$. Since $\rho \in \mathcal{SR}(\mathcal{S})$, we have $\rho(x) = \rho(x) *_\rho \rho(x)$. This implies that $\rho(x) = (\rho(u_1) *_\rho \rho(a)) *_\rho (\rho(u_2) *_\rho \rho(b)) = (\rho(u_1) *_\rho \rho(u_2)) *_\rho (\rho(a) *_\rho \rho(b))$. Let $z \in a \circ b$. By Lemma 3.1(ii), we have $\rho(x) = (\rho(u_1) *_\rho \rho(u_2)) *_\rho \rho(z)$. It follows that $\rho(x) \cap ((u_1 \circ u_2) \circ z) \neq \emptyset$. Since $\langle \rho(x) \rangle_f$ is a subsemihypergroup of S, we have $u_1 \circ u_2 \subseteq \langle \rho(x) \rangle_f$. Consequently, $z \in N$ and so $a \circ b \subseteq N$. Thus N is a subsemihypergroup of S.

(2) We show that N is a hyperfilter of S. Before showing that, we first prove that, for any $a, b \in S$, if $a \circ b \cap N \neq \emptyset$ then $b \circ a \cap N \neq \emptyset$. Let $a, b \in S$ with $a \circ b \cap N \neq \emptyset$. Then there exists $c \in a \circ b \cap N$. Since $c \in N$, we obtain that $c \in \rho(x)$ or $v \circ c \cap \rho(x) \neq \emptyset$ for some $v \in \langle \rho(x) \rangle_f$. We consider the following two cases.
<u>Case 2.1:</u> $c \in \rho(x)$. By Lemma 3.1(ii), we have $\rho(x) = \rho(c) = \rho(a) *_\rho \rho(b) = \rho(b) *_\rho \rho(a)$. It follows that $\emptyset \neq b \circ a \cap \rho(x) \subseteq b \circ a \cap N$.
<u>Case 2.2:</u> $v \circ c \cap \rho(x) \neq \emptyset$ for some $v \in \langle \rho(x) \rangle_f$. Then $v \circ (a \circ b) \cap \rho(x) \neq \emptyset$. By Lemma 3.1(ii), we have $\rho(x) = \rho(v) *_\rho (\rho(a) *_\rho \rho(b)) = \rho(v) *_\rho (\rho(b) *_\rho \rho(a))$. It follows that $v \circ (b \circ a) \cap \rho(x) \neq \emptyset$. Consequently, $b \circ a \subseteq N$ and so $b \circ a \cap N \neq \emptyset$. From the previous two cases, we conclude that, for any $a, b \in S$, if $a \circ b \cap N \neq \emptyset$, then $b \circ a \cap N \neq \emptyset$.

Next, we show that N is a hyperfilter of S. Let $a, b \in S$ with $a \circ b \cap N \neq \emptyset$. Then there exists $d \in a \circ b \cap N$. Since $d \in N$, we obtain that $d \in \rho(x)$ or $v \circ d \cap \rho(x) \neq \emptyset$ for some $v \in \langle \rho(x) \rangle_f$. It follows that $a \circ b \cap \rho(x) \neq \emptyset$ or $((v \circ a) \circ b) \cap \rho(x) = (v \circ (a \circ b)) \cap \rho(x) \neq \emptyset$. Since $\emptyset \neq a \circ b \cap N \subseteq a \circ b \cap \langle \rho(x) \rangle_f$ and $\langle \rho(x) \rangle_f$ is a hyperfilter of S, we have $a, b \in \langle \rho(x) \rangle_f$. Since $\langle \rho(x) \rangle_f$ is a subsemihypergroup of S, we have $v \circ a \subseteq \langle \rho(x) \rangle_f$. Consequently, $b \in N$. As we know that $a \circ b \cap N \neq \emptyset$ implies $b \circ a \cap N \neq \emptyset$. Using the same process, we also get $a \in N$. Hence N is a hyperfilter of S containing $\rho(x)$ and so $\langle \rho(x) \rangle_f \subseteq N$. Therefore $N = \langle \rho(x) \rangle_f$.

(ii) Since $\rho(x) \subseteq \bigcup \{n(y) : y \in \rho(x)\}$, we obtain that t is a hyperfilter of S containing $\rho(x)$. Thus $\langle \rho(x) \rangle_f \subseteq t$. Conversely, for any $y \in \rho(x)$, since $n(y)$ is the hyperfilter of S generated by y and $y \in \langle \rho(x) \rangle_f$, we have $n(y) \subseteq \langle \rho(x) \rangle_f$. So $\bigcup \{n(y) : y \in \rho(x)\} \subseteq \langle \rho(x) \rangle_f$. This follows that $t \subseteq \langle \rho(x) \rangle_f$ and hence $t = \langle \rho(x) \rangle_f$.

(iii) Let $y \in \langle \rho(x) \rangle_f$. Then $y \in \rho(x)$ or $u \circ y \cap \rho(x) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_f$. We consider two cases as follows.
<u>Case 3.1</u> $y \in \rho(x)$. We have $\rho(y) = \rho(x) \subseteq \langle \rho(x) \rangle_f$. Thus $\langle \rho(y) \rangle_f \subseteq \langle \rho(x) \rangle_f$.
<u>Case 3.2</u> $u \circ y \cap \rho(x) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_f$. We have $\rho(x) = \rho(u) *_\rho \rho(y)$. Let $z \in \rho(y)$. Then $\rho(x) = \rho(u) *_\rho \rho(z)$. It follows that $\emptyset \neq u \circ z \cap \rho(x) \subseteq u \circ z \cap \langle \rho(x) \rangle_f$. Since $\langle \rho(x) \rangle_f$ is a hyperfilter, we have $z \in \langle \rho(x) \rangle_f$. Consequently, $\rho(y) \subseteq \langle \rho(x) \rangle_f$ and hence $\langle \rho(y) \rangle_f \subseteq \langle \rho(x) \rangle_f$.

(iv) Let $\sigma = \Big\{(x, y) \in S \times S : \langle\rho(x)\rangle_f = \langle\rho(y)\rangle_f\Big\}$. Obviously, $\rho \subseteq \sigma$. Conversely, let $(x, y) \in \sigma$. Then $\langle\rho(x)\rangle_f = \langle\rho(y)\rangle_f$. Hence $x \in \langle\rho(y)\rangle_f$ and $y \in \langle\rho(x)\rangle_f$. By (i), if $x \in \rho(y)$ or $y \in \rho(x)$, then $\rho(x) = \rho(y)$ and so $\sigma \subseteq \rho$. Next, assume that $x \notin \rho(y)$ and $y \notin \rho(x)$. By (i), there exist $u, v \in \langle\rho(x)\rangle_f = \langle\rho(y)\rangle_f$ such that $u \circ x \cap \rho(y) \neq \emptyset$ and $v \circ y \cap \rho(x) \neq \emptyset$. By Lemma 3.1(ii), we have $\rho(y) = \rho(u) *_\rho \rho(x)$ and $\rho(x) = \rho(v) *_\rho \rho(y)$. Since $\rho \in \mathcal{SR}(\mathcal{S})$, we have $x \ \bar{\bar{\rho}}\ x \circ x$ and $y \ \bar{\bar{\rho}}\ y \circ y$. By Lemma 3.1(ii), we obtain that $\rho(x) = \rho(x) *_\rho \rho(x)$ and $\rho(y) = \rho(y) *_\rho \rho(y)$. We have $\rho(x) = \rho(v) *_\rho \rho(y) = \rho(v) *_\rho (\rho(y) *_\rho \rho(y)) = (\rho(v) *_\rho \rho(y)) *_\rho \rho(y) = \rho(x) *_\rho \rho(y) = \rho(x) *_\rho (\rho(u) *_\rho \rho(x)) = \rho(u) *_\rho (\rho(x) *_\rho \rho(x)) = \rho(u) *_\rho \rho(x) = \rho(y)$. Consequently, $(x, y) \in \rho$. Therefore $\sigma \subseteq \rho$ and the proof is completed. □

Corollary 3.5. *Let S be an ordered semihypergroup and $x \in S$. Then the following assertions hold.*

(i) $\eta \in \mathcal{SR}(\mathcal{S})$ *and* $\langle\eta(x)\rangle_f = n(x)$.

(ii) $n(x) = \{a \in S : a \in \eta(x)$ *or* $u \circ a \cap \eta(x) \neq \emptyset$ *for some* $u \in n(x)\}$.

Proof. (i) As in the proof of Theorem 5.5 in [14], it is not difficult to verify that $\eta \in \mathcal{SR}(\mathcal{S})$. By Lemma 3.4(ii), we have $\langle\eta(x)\rangle_f = t$. Since $n(y) = n(x)$ for all $y \in \eta(x)$. So $\bigcup_{y\in\eta(x)} n(y) = n(x)$. Since t is a hyperfilter of S generated by $\bigcup_{y\in\eta(x)} n(y) = n(x)$, we have $n(x) \subseteq t \subseteq n(x)$. Consequently, $\langle\eta(x)\rangle_f = t = n(x)$.

(ii) By Lemma 3.4 (i), we have
$n(x) = \langle\eta(x)\rangle_f = \{a \in S : a \in \eta(x)$ or $u \circ a \cap \eta(x) \neq \emptyset$ for some $u \in n(x)\}$. □

For any $A \in \mathcal{P}^*(S)$, denote by $[A) = \{x \in S : a \leqslant x$ for some $a \in A\}$.

Lemma 3.6. *Let S be an ordered semihypergroup, $\rho \in \mathcal{CSR}(\mathcal{S})$ and $x \in S$. Then*

(i) $\langle\rho(x)\rangle_F = \{a \in S : a \in [\rho(x))$ *or* $u \circ a \cap [\rho(x)) \neq \emptyset$ *for some* $u \in \langle\rho(x)\rangle_F\}$.

(ii) $\langle\rho(x)\rangle_F = T$.

(iii) *If* $y \in \langle\rho(x)\rangle_F$, *then* $\langle\rho(y)\rangle_F \subseteq \langle\rho(x)\rangle_F$.

(iv) $\rho = \{(x, y) \in S \times S : \langle\rho(x)\rangle_F = \langle\rho(y)\rangle_F\}$.

Proof. (i) Let $N = \{a \in S : a \in [\rho(x))$ or $u \circ a \cap [\rho(x)) \neq \emptyset$ for some $u \in \langle\rho(x)\rangle_F\}$. Clearly, $\rho(x) \subseteq N \subseteq \langle\rho(x)\rangle_F$. Firstly, we show that N is an ordered hyperfilter

(1) N is a subsemihypergroup of S. Let $a_1, a_2 \in N$. We consider the following two cases.

Case 1.1 $a_1, a_2 \notin [\rho(x))$. Then, there exist $u_1, u_2 \in \langle\rho(x)\rangle_F$ such that $u_i \circ a_i \cap [\rho(x)) \neq \emptyset$ for all $i = 1, 2$. For any $i = 1, 2$, we put $b_i \in u_i \circ a_i \cap [\rho(x)) \neq \emptyset$. Then, there exists $v_i \in \rho(x)$ such that $v_i \leqslant b_i$. Since $\leqslant$ is compatible, we get $v_1 \circ v_2 \leqslant b_1 \circ b_2$ and $v_1 \circ v_2 \subseteq \rho(x)$ by Lemma 3.1(i). Let $c \in v_1 \circ v_2$. Then, there exists $d \in b_1 \circ b_2$ such that $c \leqslant d$. Since $\rho \in \mathcal{CSR}(\mathcal{S})$, we have $c \ \bar{\bar{\rho}}\ c \circ d$ and then

$\rho(c) = \rho(c) *_\rho \rho(d)$. Since $c \in v_1 \circ v_2 \subseteq \rho(x)$, we have $\rho(c) = \rho(x)$. By Lemma 3.1(ii), we have

$$\begin{aligned}\rho(x) = \rho(c) &= \rho(c) *_\rho \rho(d) = \rho(x) *_\rho \rho(d), && \text{since } d \in b_1 \circ b_2,\\ &= \rho(x) *_\rho (\rho(b_1) *_\rho \rho(b_2)), && \text{since } b_i \in u_i \circ a_i \text{ for all } i = 1, 2,\\ &= \rho(x) *_\rho ((\rho(u_1) *_\rho \rho(a_1)) *_\rho (\rho(u_2) *_\rho \rho(a_2))), && \text{since } *_\rho \text{ is commutat.},\\ &= (\rho(x) *_\rho (\rho(u_1) *_\rho \rho(u_2))) *_\rho (\rho(a_1) *_\rho \rho(a_2)).\end{aligned}$$

Let $y \in a_1 \circ a_2$. By Lemma 3.1(ii), we have $\rho(x) = (\rho(x) *_\rho (\rho(u_1) *_\rho \rho(u_2))) *_\rho \rho(y)$. Consequently, $\emptyset \neq ((x \circ (u_1 \circ u_2)) \circ y) \cap \rho(x) \subseteq ((x \circ (u_1 \circ u_2)) \circ y) \cap [\rho(x))$. Since $x \circ (u_1 \circ u_2) \subseteq \langle \rho(x) \rangle_F$, we have $y \in N$ and so $a_1 \circ a_2 \subseteq N$.
Case 1.2 $a_i \in [\rho(x))$ for some $i \in \{1, 2\}$. There exists $v_i \in \rho(x)$ such that $v_i \leqslant a_i$. Since $\leqslant$ is compatible, we have $v_i \circ v_i \leqslant v_i \circ a_i$, i.e., for any $b \in v_i \circ v_i$ there exists $c \in v_i \circ a_i$ such that $b \leqslant c$. It follows that $c \in [\rho(x))$ and so $v_i \circ a_i \cap [\rho(x)) \neq \emptyset$. As we known that $v_i \in \rho(x) \subseteq \langle \rho(x) \rangle_F$, using the similar process as in Case 1.1, we also conclude that $a_1 \circ a_2 \subseteq N$. Thus N is a subsemihypergroup of S.

(2) Using the similar process as in the proof of Lemma 3.4(ii), we obtain that N is a hyperfilter of S. Next, let $a \in N$ and $b \in S$ with $a \leqslant b$. Then $a \in [\rho(x))$ or $u \circ a \cap [\rho(x)) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_F$. We consider the following two cases.
Case 2.1: $a \in [\rho(x))$. Since $a \leqslant b$, we get $b \in [\rho(x))$. So $b \in N$.
Case 2.2: $u \circ a \cap [\rho(x)) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_F$. Since $\leqslant$ is compatible, we get $u \circ a \leqslant u \circ b$, i.e., for any $c \in u \circ a$, there exists $d \in u \circ b$ such that $c \leqslant d$. Put $c' \in u \circ a \cap [\rho(x))$, there exists $d' \in u \circ b$ such that $c' \leqslant d'$. Since $c' \in [\rho(x))$, we have $d' \in [\rho(x))$. It implies that $u \circ b \cap [\rho(x)) \neq \emptyset$. Consequently, $b \in N$.
From Case 2.1 and 2.2, we conclude that N is an ordered hyperfilter of S containing $\rho(x)$. Hence $\langle \rho(x) \rangle_F \subseteq N$ and so $\langle \rho(x) \rangle_F = N$.

(ii) The proof is similar to Lemma 3.4(ii).

(iii) Let $y \in \langle \rho(x) \rangle_F$. By (i), we obtain that $y \in [\rho(x))$ or $u \circ y \cap [\rho(x)) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_F$. We analyze two cases as follows.
Case 3.1: $y \in [\rho(x))$. Then, there exists $a \in \rho(x)$ such that $a \leqslant y$. Since $\rho \in \mathcal{CSR}(\mathcal{S})$, we have $a \,\overline{\overline{\rho}}\, a \circ y$. It follows that $\rho(x) = \rho(a) = \rho(a) *_\rho \rho(y)$. Let $y' \in \rho(y)$. We have $\rho(x) = \rho(a) *_\rho \rho(y')$ and it follows that $\emptyset \neq (a \circ y') \cap \rho(x) \subseteq (a \circ y') \cap [\rho(x))$. Since $a \in \rho(x) \subseteq \langle \rho(x) \rangle_F$ and by (i), we have $y' \in \langle \rho(x) \rangle_F$. It follows that $\rho(y) \subseteq \langle \rho(x) \rangle_F$ and so $\langle \rho(y) \rangle_F \subseteq \langle \rho(x) \rangle_F$.
Case 3.2: $u \circ y \cap [\rho(x)) \neq \emptyset$ for some $u \in \langle \rho(x) \rangle_F$. Then, there exists $b \in u \circ y \cap [\rho(x))$. There is $a \in \rho(x)$ such that $a \leqslant b$. Since $\rho \in \mathcal{CSR}(\mathcal{S})$, we get $a \,\overline{\overline{\rho}}\, a \circ b$. By Lemma 3.1(ii), we have $\rho(x) = \rho(a) = \rho(a) *_\rho \rho(b) = \rho(a) *_\rho (\rho(u) *_\rho \rho(y)) = (\rho(a) *_\rho \rho(u)) *_\rho \rho(y)$. Let $y' \in \rho(y)$. It follows that $\rho(x) = (\rho(a) *_\rho \rho(u)) *_\rho \rho(y')$. We have $\emptyset \neq ((a \circ u) \circ y') \cap \rho(x) \subseteq ((a \circ u) \circ y') \cap [\rho(x))$. Since $a \in \rho(x) \subseteq \langle \rho(x) \rangle_F$ and $\langle \rho(x) \rangle_F$ is a subsemihypergroup of S, we have $a \circ u \subseteq \langle \rho(x) \rangle_F$. By (i), we have $y' \in \langle \rho(x) \rangle_F$ and hence $\rho(y) \subseteq \langle \rho(x) \rangle_F$. Thus $\langle \rho(y) \rangle_F \subseteq \langle \rho(x) \rangle_F$.

(iv) Let $\sigma = \{(x, y) \in S \times S : \langle \rho(x) \rangle_F = \langle \rho(y) \rangle_F\}$. Evidently, $\rho \subseteq \sigma$. Conversely, let $(x, y) \in \rho$. Then $\langle \rho(x) \rangle_F = \langle \rho(y) \rangle_F$. It follows that $x \in \langle \rho(y) \rangle_F$ and $y \in \langle \rho(x) \rangle_F$. We consider the following two cases.

Case 4.1: $x \notin [\rho(y))$ and $y \notin [\rho(x))$. Then there exist $u_1, u_2 \in \langle \rho(x) \rangle_F = \langle \rho(y) \rangle_F$ such that $u_1 \circ x \cap [\rho(y)) \neq \emptyset$ and $u_2 \circ y \cap [\rho(x)) \neq \emptyset$. Since $u_1 \circ x \cap [\rho(y)) \neq \emptyset$, there exists $a \in u_1 \circ x \cap [\rho(y))$ and $a' \in \rho(y)$ such that $a' \leqslant a$. Since $\rho \in \mathcal{CSR}(\mathcal{S})$, we have $a' \ \bar{\bar{\rho}}\ a' \circ a$. Since $a \in u_1 \circ x$ and $a' \in \rho(y)$, by Lemma 3.1(ii), we have $\rho(y) = \rho(a') = \rho(a') *_\rho \rho(a) = \rho(y) *_\rho \rho(a) = \rho(y) *_\rho (\rho(u_1) *_\rho \rho(x))$. Similarly, since $u_2 \circ y \cap [\rho(x)) \neq \emptyset$, we have $\rho(x) = \rho(x) *_\rho (\rho(u_2) *_\rho \rho(y))$. It follows that

$$\begin{aligned}\rho(u_2) *_\rho \rho(y) &= \rho(u_2) *_\rho (\rho(y) *_\rho (\rho(u_1) *_\rho \rho(x))), \quad \text{ since } *_\rho \text{ is commutative,}\\ &= \rho(u_1) *_\rho (\rho(x) *_\rho (\rho(u_2) *_\rho \rho(y))) = \rho(u_1) *_\rho \rho(x).\end{aligned}$$

Since $\rho \in \mathcal{CSR}(\mathcal{S})$, we have $\rho(x) = \rho(x) *_\rho \rho(x)$ and $\rho(y) = \rho(y) *_\rho \rho(y)$. Then $\rho(x) = \rho(x) *_\rho (\rho(u_2) *_\rho \rho(y)) = \rho(x) *_\rho (\rho(u_1) *_\rho \rho(x)) = \rho(u_1) *_\rho (\rho(x) *_\rho \rho(x)) = \rho(u_1) *_\rho \rho(x) = \rho(u_2) *_\rho \rho(y) = \rho(u_2) *_\rho (\rho(y) *_\rho \rho(y)) = (\rho(u_2) *_\rho \rho(y)) *_\rho \rho(y) = (\rho(u_1) *_\rho \rho(x)) *_\rho \rho(y) = \rho(y)$. Consequently, $(x, y) \in \rho$.
Case 4.2: $x \in [\rho(y))$ or $y \in [\rho(x))$. Without loss of generality, assume that $x \in [\rho(y))$. Using the similar poof as in Case 1.2, there exists $v \in \langle \rho(x) \rangle_F = \langle \rho(y) \rangle_F$ such that $v \circ x \cap [\rho(y)) \neq \emptyset$. Using the analogous processes as in Case 4.1, we also obtain that $(x, y) \in \rho$. From Case 4.1 and 4.2, we conclude that $\sigma \subseteq \rho$ and the proof is completed. □

Applying Lemma 3.6 and Corollary 3.5, we obtain the following result.

Corollary 3.7. *Let S be an ordered semihypergroup and $x \in S$. Then*

(i) $\langle \mathcal{N}(x) \rangle_F = N(x)$.

(ii) $N(x) = \{a \in S : a \in [\mathcal{N}(x))$ *or* $u \circ a \cap [\mathcal{N}(x)) \neq \emptyset$ *for some* $u \in N(x)\}$.

4. The least semilattice equivalence relation

We will show that the relation η is the least semilattice equivalence relation on ordered semihypergroups which is used to answer the question in Section 1.

Theorem 4.1. *Let S be an ordered semihypergroup. Then*

(i) $\eta = \bigcap \{\delta_A : A \in \mathcal{CP}(\mathcal{S})\}$.

(ii) $\eta \subseteq \mathcal{N}$.

Proof. (i) Let $\tau = \bigcap \{\delta_A : A \in \mathcal{CP}(\mathcal{S})\}$. By Corollary 3.3, we have $\delta_A \in \mathcal{SR}(\mathcal{S})$ for all $A \in \mathcal{CP}(\mathcal{S})$. So $\tau \in \mathcal{SR}(\mathcal{S})$. By using the similar proof as in Theorem 2.8 in [14], we can show that $\eta = \tau$.

(ii) Since $\mathcal{CPO}(\mathcal{S}) \subseteq \mathcal{CP}(\mathcal{S})$, by (i) and Lemma 2.6 (ii), we obtain

$$\eta = \bigcap \{\delta_A : A \in \mathcal{CP}(\mathcal{S})\} \subseteq \bigcap \{\delta_A : A \in \mathcal{CPO}(\mathcal{S})\} = \mathcal{N}.$$ □

Theorem 4.2. *Let S be an ordered semihypergroup and $\rho \in \mathcal{SR}(\mathcal{S})$. Then*

(i) $\rho = \bigcap\limits_{x \in S} \delta_{\langle \rho(x) \rangle_f}$.

(ii) *η is the least semilattice equivalence relation on S.*

Proof. (i) Let $\tau = \bigcap \left\{ \delta_{\langle \rho(x) \rangle_f} : x \in S \right\}$. Since $\langle \rho(x) \rangle_f$ is a hyperfilter of S, by Lemma 2.2, we have $S \backslash \langle \rho(x) \rangle_f$ is a completely prime hyperideal of S. By Corollary 3.3, we have $\delta_{\langle \rho(x) \rangle_f} = \delta_{S \backslash \langle \rho(x) \rangle_f} \in \mathcal{SR}(\mathcal{S})$. It follows that $\tau \in \mathcal{SR}(\mathcal{S})$. Let $(x, y) \in \tau$. Then $(x, y) \in \delta_{\langle \rho(a) \rangle_f}$ for all $a \in S$. To show that $(x, y) \in \rho$, assume that $(x, y) \notin \rho$. By Lemma 3.4(iv), we have $\langle \rho(x) \rangle_f \neq \langle \rho(y) \rangle_f$. It follows that $\langle \rho(x) \rangle_f \nsubseteq \langle \rho(y) \rangle_f$ or $\langle \rho(y) \rangle_f \nsubseteq \langle \rho(x) \rangle_f$.
<u>Case 1.1:</u> $\langle \rho(x) \rangle_f \nsubseteq \langle \rho(y) \rangle_f$. By Lemma 3.4(iii), we have $x \notin \langle \rho(y) \rangle_f$. Since $y \in \langle \rho(y) \rangle_f$, it follows that $(x, y) \notin \delta_{\langle \rho(y) \rangle_f}$. This is a contradiction.
<u>Case 1.2:</u> $\langle \rho(y) \rangle_f \nsubseteq \langle \rho(x) \rangle_f$. Using the similar proof as in Case 1.1, we obtain a contradiction. Hence $(x, y) \in \rho$ and so $\tau \subseteq \rho$. Conversely, let $(x, y) \in \rho$. By Lemma 3.4(iv), we have $\langle \rho(x) \rangle_f = \langle \rho(y) \rangle_f$. To show that $(x, y) \in \tau$, assume that $(x, y) \notin \tau$. Then, there exists $a \in S$ such that $(x, y) \notin \delta_{S \backslash \langle \rho(a) \rangle_f}$. We obtain the following two cases.
<u>Case 2.1:</u> $x \notin \langle \rho(a) \rangle_f$ and $y \in \langle \rho(a) \rangle_f$. By Lemma 3.4(iii), we have $\langle \rho(y) \rangle_f \subseteq \langle \rho(a) \rangle_f$. and so $x \in \langle \rho(x) \rangle_f = \langle \rho(y) \rangle_f \subseteq \langle \rho(a) \rangle_f$. It is impossible.
<u>Case 2.2:</u> $x \in \langle \rho(a) \rangle_f$ and $y \notin \langle \rho(a) \rangle_f$. The proof is similar to Case 2.1, we also get a contradiction. Consequently, $(x, y) \in \tau$. Therefore $\rho = \tau$.

(ii) Let $\rho \in \mathcal{SR}(\mathcal{S})$. Since $\left\{ S \setminus \langle \rho(x) \rangle_f : x \in S \right\} \subseteq \mathcal{CP}(\mathcal{S})$, we obtain that

$$\eta = \bigcap \{\delta_A : A \in \mathcal{CP}(\mathcal{S})\} \subseteq \bigcap \left\{ \delta_{S \backslash \langle \rho(x) \rangle_f} : x \in S \right\} = \bigcap \left\{ \delta_{\langle \rho(x) \rangle_f} : x \in S \right\} = \rho.$$

Therefore η is the least semilattice equivalence relation on S. □

Applying Lemma 3.2, 3.6, and Corollary 3.7, by using the similar proof of Theorem 4.2, we obtain that $\mathcal{N}$ is the least complete semilattice equivalence relation on ordered semihypergroups which analogous to the Kehayopulu's results, see Corollary 4.11 in [11].

Theorem 4.3. *Let S be an ordered semihypergroup and $\rho \in \mathcal{CSR}(\mathcal{S})$. Then*

(i) $\rho = \bigcap\limits_{x \in S} \delta_{\langle \rho(x) \rangle_F}$.

(ii) *$\mathcal{N}$ is the least complete semilattice equivalence relation on S.*

Example 4.4. Let $S = \{a, b, c, d\}$. Define $\circ : S \times S \to \mathcal{P}^*(S)$ as in the table. Then $(S, \circ)$ is a semihypergroup(see Example 1 [15]). Define a partial order $\leqslant$ by

$$\leqslant := \{(a, a), (b, b), (c, a), (c, c), (d, a), (d, d)\}.$$

Then $(S, \circ, \leqslant)$ is an ordered semihypergroup. Clearly, $H_1 = \{a, b, d\}$, $H_2 = \{a, c, d\}$, $H_3 = \{a, d\}$ and S are all hyperideals of S. On the other hand, H_2

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

and S are all ordered hyperideals of S. Furthermore, H_1, H_2, S are completely prime and H_3 is not completely prime because $(b \circ c) \cap H_3 = \{a,d\} \cap \{a,d\} \neq \emptyset$ but $b, c \notin H_3$. By Lemma 2.6 and Theorem 4.1, we have

$$\eta = \delta_{H_1} \cap \delta_{H_2} \cap \delta_S = \{(a,a),(a,d),(b,b),(c,c),(d,a),(d,d)\},$$
$$\mathcal{N} = \delta_{H_2} \cap \delta_S = \{(a,a),(a,c),(a,d),(b,b),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}.$$

It is not difficult to show that all semilattice equivalence relations on S are defined as follows:

$$\rho_1 = \{(a,a),(a,d),(b,b),(c,c),(d,a),(d,d)\} = \eta,$$
$$\rho_2 = \{(a,a),(a,b),(a,d),(b,a),(b,b),(b,d),(c,c),(d,a),(d,b),(d,d)\},$$
$$\rho_3 = \{(a,a),(a,c),(a,d),(b,b),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\} = \mathcal{N}$$
$$\rho_4 = \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d)\}.$$

Moreover, all complete semilattice equivalence relations on S are ρ_3 and ρ_4. Consequently, $\eta \subseteq \mathcal{N}$. Therefore η is the least semilattice equivalence relation on S and $\mathcal{N}$ is the least complete semilattice equivalence relation on S.

5. Ordered regular equivalence relations

In this section, we describe the relationship between complete semilattice equivalence relations and strongly ordered regular equivalence relations on an ordered semihypergroup S via the (ordered) hyperfilter of S generated by its equivalence classes. Firstly, we recall the notion of strongly ordered regular equivalence relations on ordered semihypergroups which was introduced by Gu and Tang in [7].

Definition 5.1. Let $(S_1, \circ_1, \leqslant_1)$ and $(S_2, \circ_2, \leqslant_2)$ be two ordered semihypergroups. A mapping $\varphi : S_1 \to S_2$ is called a *normal homomorphism* if the following two conditions hold.

(i) $\varphi(x \circ_1 y) = \varphi(x) \circ_2 \varphi(y)$ for all $x, y \in S$, where $\varphi(H) = \{\varphi(a) : a \in H\}$.

(ii) If $x \leqslant_1 y$, then $\varphi(x) \leqslant_2 \varphi(y)$.

Definition 5.2. Let $(S, \circ, \leqslant)$ be an ordered semihypergroup and ρ be a strongly regular equivalence relation on S. The relation ρ is called *strongly ordered regular* if there exists an *order relation* $\preccurlyeq$ on S/ρ which satisfies the following two conditions.

(i) $(S/\rho, *_\rho, \preccurlyeq)$ is an ordered semigroup.

(ii) the mapping $\varphi : S \to S/\rho$ by $x \mapsto \rho(x)$ is a normal homomorphism.

The following theorems extend and generalize Theorem 2.1 and 2.2 in [19].

Theorem 5.3. *Let $\rho \in \mathcal{SR}(\mathcal{S})$. Define an order relation $\preccurlyeq_f$ on S/ρ as follows*

$$\rho(x) \preccurlyeq_f \rho(y) \text{ if and only if } \langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f .$$

Then $(S/\rho, *_\rho, \preccurlyeq_f)$ is an ordered semigroup.

Proof. Let $\rho(x) = \rho(x')$ and $\rho(y) = \rho(y')$ with $\rho(x) \preccurlyeq_f \rho(y)$. Then $\langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f$. It follows that $\langle \rho(y')\rangle_f = \langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f = \langle \rho(x')\rangle_f$ and hence $\rho(x') \preccurlyeq_f \rho(y')$. So $\preccurlyeq_f$ is well-defined. Next, we show that $\preccurlyeq_f$ is a partial order on S/ρ. Obviously, $\rho(x) \preccurlyeq_f \rho(x)$ for all $x \in S$, so $\preccurlyeq_f$ is reflexive. Let $\rho(x) \preccurlyeq_f \rho(y)$ and $\rho(y) \preccurlyeq_f \rho(x)$ for all $x, y \in S$. Then $\langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f$ and $\langle \rho(x)\rangle_f \subseteq \langle \rho(y)\rangle_f$. Hence $\langle \rho(x)\rangle_f = \langle \rho(y)\rangle_f$. By Lemma 3.4(iv), we have $(x, y) \in \rho$. Consequently, $\rho(x) = \rho(y)$ and so $\preccurlyeq_f$ is anti-symmetric. Let $\rho(x) \preccurlyeq_f \rho(y)$ and $\rho(y) \preccurlyeq_f \rho(z)$ for all $x, y, z \in S$. Then $\langle \rho(z)\rangle_f \subseteq \langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f$. Hence $\rho(x) \preccurlyeq_f \rho(z)$ and $\preccurlyeq_f$ is transitive. Thus $\preccurlyeq_f$ is a partial order on S/ρ. Next, we show that $\preccurlyeq_f$ is compatible with the operation $*_\rho$ on S/ρ which was defined in Lemma 3.1(ii). Let $\rho(x) \preccurlyeq_f \rho(y)$ and $\rho(c) \in S/\rho$. Then $\langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f$. Firstly, we show that $\rho(x) *_\rho \rho(c) \preccurlyeq_f \rho(y) *_\rho \rho(c)$. Let $u \in x \circ c$. By Lemma 3.1(ii), we have $\rho(u) = \rho(x) *_\rho \rho(c)$. Since $\rho \in \mathcal{SR}(\mathcal{S})$, we have $\rho(x) = \rho(x) *_\rho \rho(x)$ and $\rho(c) = \rho(c) *_\rho \rho(c)$. Since $u \in \langle \rho(u)\rangle_f$, for any $a \in u \circ c$, we have $\rho(a) = \rho(u) *_\rho \rho(c) = (\rho(x) *_\rho \rho(c)) *_\rho \rho(c) = \rho(x) *_\rho (\rho(c) *_\rho \rho(c)) = \rho(x) *_\rho \rho(c) = \rho(u)$. It follows that $u \circ c \cap \rho(u) \neq \emptyset$. By Lemma 3.4(i), we get $c \in \langle \rho(u)\rangle_f$. Similarly, far any $b \in u \circ x$, we have $\rho(b) = \rho(u) *_\rho \rho(x) = (\rho(x) *_\rho \rho(c)) *_\rho \rho(x) = (\rho(x) *_\rho \rho(x)) *_\rho \rho(c) = \rho(x) *_\rho \rho(c) = \rho(u)$. It follows that $u \circ x \cap \rho(u) \neq \emptyset$. By Lemma 3.4(i), we get $x \in \langle \rho(u)\rangle_f$. By Lemma 3.4(iii), we have $y \in \langle \rho(y)\rangle_f \subseteq \langle \rho(x)\rangle_f \subseteq \langle \rho(u)\rangle_f$. Since $\langle \rho(u)\rangle_f$ is a subsemihypergroup of S, we have $y \circ c \subseteq \langle \rho(u)\rangle_f$. By Lemma 3.4(iii), we obtain that $\langle \rho(v)\rangle_f \subseteq \langle \rho(u)\rangle_f$ for all $v \in y \circ c$. It follows that $\rho(u) \preccurlyeq_f \rho(v)$ for all $u \in x \circ c$ and all $v \in y \circ c$. By Lemma 3.1(ii), we get $\rho(x) *_\rho \rho(c) \preccurlyeq_f \rho(y) *_\rho \rho(c)$. Similarly, we can show that $\rho(c) *_\rho \rho(x) \preccurlyeq_f \rho(c) *_\rho \rho(y)$. Therefore $(S/\rho, *_\rho, \preccurlyeq_f)$ is an ordered semigroup. □

Theorem 5.4. *Every complete semilattice equivalence relation on S is a strongly ordered regular equivalence relation.*

Proof. Let $\rho \in \mathcal{CSR}(\mathcal{S})$. We define an order relation $\preccurlyeq_F$ on S/ρ by $\rho(x) \preccurlyeq_F \rho(y)$ if and only if $\langle \rho(y)\rangle_F \subseteq \langle \rho(x)\rangle_F$. By Lemma 3.6 and using the similar proof as in Theorem 5.3, we obtain that $(S/\rho, *_\rho, \preccurlyeq_F)$ is an ordered semigroup. Next, we consider the mapping $\varphi : S \to S/\rho$ which is defined by $\varphi(x) = \rho(x)$ for all $x \in S$. Clearly, φ is a normal homomorphism. Indeed, if $x \leqslant y$, then $y \in \langle \rho(x)\rangle_F$. By Lemma 3.6(iii), we have $\langle \rho(y)\rangle_F \subseteq \langle \rho(x)\rangle_F$. Consequently, $\rho(x) \preccurlyeq_F \rho(y)$ and then $\varphi(x) \preccurlyeq_F \varphi(y)$. Therefore ρ is a strongly ordered regular equivalence relation on S. □

From Theorem 5.4, we obtain the following result.

Corollary 5.5. *$\mathcal{N}$ is a strongly ordered regular equivalence relation.*

Definition 5.6. (cf. [7]) Let S be an ordered semihypergroup and ρ be an equivalence relation on S. Then, ρ is an *ordered semilattice equivalence relation* if

(i) S/ρ is a semilattice. Note that $(S/\rho, *_\rho, \lhd)$ is an ordered semigroup with the well-known partial order $\lhd$ on semilattice which is defined by

$$\rho(x) \lhd \rho(y) \text{ if and only if } \rho(x) = \rho(x) *_\rho \rho(y),$$

(ii) ρ satisfies the condition (ii) in Definition 5.2.

Clearly, $\mathcal{N}$ is an ordered semilattice equivalence relation on S.

On the other hand, a semilattice equivalence relation on an ordered semihypergroup S is called a *weak ordered semilattice equivalence relation* if it is a strongly ordered regular equivalence relation on S. Consequently, every complete semilattice equivalence relation and $\mathcal{N}$ are the weak ordered semilattice equivalence relations on S. In [7], Gu and Tang proved that $\mathcal{N}$ is the least ordered semilattice equivalence relation on S. The following example shows that the relation $\mathcal{N}$ does not need to be the least weak ordered semilattice equivalence relation in general.

Example 5.7. Let $S = \{a, b, c, d\}$. Define $\circ : S \times S \to \mathcal{P}^*(S)$ by the following table.

$\circ$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, c, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{c\}$	$\{d\}$

Then $(S, \circ)$ is a semihypergroup (see Example 3 [15]). Define a partial order $\leqslant$ by

$$\leqslant := \{(a, a), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Then $(S, \circ, \leqslant)$ is an ordered semihypergroup. Clearly, all completely prime hyperideals of S are $A = \{a, c, d\}$ and S. Furthermore, A is not an ordered hyperideal of S since $b \leqslant a \in A$ but $b \notin A$. Consequently, S is only a completely prime ordered hyperideal of S. By Lemma 2.6 and Corollary 3.3, we have

$$\begin{aligned}
\rho := \delta_A &= \{(a, a), (a, c), (a, d), (b, b), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d)\} \\
&\in \mathcal{SR}(\mathcal{S}), \\
\mathcal{N} = \delta_S &= \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), \\
&\quad (c, d), (d, a), (d, b), (d, c), (d, d)\} \in \mathcal{CSR}(\mathcal{S}).
\end{aligned}$$

$*_\rho$	$\rho(a)$	$\rho(b)$
$\rho(a)$	$\rho(a)$	$\rho(a)$
$\rho(b)$	$\rho(a)$	$\rho(b)$

Next, we show that ρ is a weak ordered semilattice equivalence relation on S. Clearly, $S/\rho = \{\rho(a), \rho(b)\}$ where $\rho(a) = \{a, c, d\} = \rho(c) = \rho(d)$, $\rho(b) = \{b\}$. By Lemma 3.1(ii), we have $(S/\rho, *_\rho)$ is a semigroup where the hyperoperation $*_\rho$ is defined by the following table.
Next, we define a partial order $\leqslant_\rho$ on S/ρ by

$$\leqslant_\rho := \{(\rho(a), \rho(a)), (\rho(b), \rho(a)), (\rho(b), \rho(b))\}.$$

Then, $(S/\rho, *_\rho, \leqslant_\rho)$ is an ordered semigroup. Furthermore, we have

$$\begin{aligned} b \leqslant a &\text{ implies } \rho(b) \leqslant_\rho \rho(a) \\ d \leqslant a &\text{ implies } \rho(d) = \rho(a) \leqslant_\rho \rho(a). \end{aligned}$$

Therefore ρ is a weak ordered semilattice equivalence relation on S and $\rho \subseteq \mathcal{N}$.

From the previous results, the following question is natural: Is every semilattice equivalence relation the weak ordered semilattice equivalence relation on ordered semihypergroups ?

In order to solve the problem, we first present Theorem 5.8 that gives the characterization of the smallest order relation with respect to the given strongly ordered regular equivalence relation.

Theorem 5.8. *Let ρ be a strongly regular equivalence relation on S. Define a relation $\preccurlyeq_\rho$ on S/ρ as follows:*

$$\alpha := \{(\rho(x), \rho(y)) \in S/\rho \times S/\rho : \text{ there exist } x' \in \rho(x) \text{ and } y' \in \rho(y) \text{ such that } x' \leqslant y'\}$$

$$\preccurlyeq_\rho = \{(\rho(x), \rho(y)) \in S/\rho \times S/\rho : (\rho(x), \rho(y)) \in \alpha^m \text{ for some } m \in \mathbb{N}\}.$$

If ρ is a strongly ordered regular equivalence relation on S, then $\preccurlyeq_\rho$ is the smallest order relation on S/ρ with respect to ρ.

Proof. Let ρ be a strongly ordered regular equivalence relation on S. First of all we show that $\preccurlyeq_\rho$ is a partial order on S/ρ.

(1) Reflexive. Since $x \leqslant x$ for all $x \in S$, we have $(\rho(x), \rho(x)) \in \alpha \subseteq \preccurlyeq_\rho$.

(2) Anti-symmetric. Let $(\rho(x), \rho(y)) \in \preccurlyeq_\rho$ and $(\rho(y), \rho(x)) \in \preccurlyeq_\rho$. Then, there exist $m, n \in \mathbb{N}$ such that $(\rho(x), \rho(y)) \in \alpha^m$ and $(\rho(y), \rho(x)) \in \alpha^n$. There are $\rho(a_1), ..., \rho(a_{m-1}), \rho(b_1), ..., \rho(b_{n-1}) \in S/\rho$ such that $\rho(x)\ \alpha\ \rho(a_1)\ \alpha\ ...\ \alpha\ \rho(a_{m-1})\ \alpha\ \rho(y)$ and $\rho(y)\ \alpha\ \rho(b_1)\ \alpha\ ...\ \alpha\ \rho(b_{n-1})\ \alpha\ \rho(x)$. By the definition of α, there exist $x', x'' \in \rho(x), y', y'' \in \rho(y), a'_i, a''_i \in \rho(a_i)$ and $b'_j, b''_j \in \rho(b_j)$, for all $i = 1, ..., m-1$ and $j = 1, ..., n-1$, such that $x' \leqslant a'_1, a''_1 \leqslant a'_2, ..., a''_{m-2} \leqslant a'_{m-1}, a''_{m-1} \leqslant y'$ and $y'' \leqslant b'_1, b''_1 \leqslant b'_2, ..., b''_{n-2} \leqslant b'_{n-1}, b''_{n-1} \leqslant x''$. Since ρ is a strongly ordered regular

equivalence relation, there exists an order regular $\preccurlyeq$ on S/ρ such that $(S/\rho, *_\rho, \preccurlyeq)$ is an ordered semigroup and the mapping $\varphi : S \to S/\rho$ which is defined by $x \mapsto \rho(x)$ is a normal homomorphism. It follows that $\rho(x) = \rho(x') \preccurlyeq \rho(a_1') = \rho(a_1) = \rho(a_1'') \preccurlyeq ... \preccurlyeq \rho(a'_{m-1}) = \rho(a_{m-1}) = \rho(a''_{m-1}) \preccurlyeq \rho(y') = \rho(y)$ and $\rho(y) = \rho(y'') \preccurlyeq \rho(b_1') = \rho(b_1) = \rho(b_1'') \preccurlyeq ... \preccurlyeq \rho(b'_{n-1}) = \rho(b_{n-1}) = \rho(b''_{n-1}) \preccurlyeq \rho(x'') = \rho(x)$. Since $\preccurlyeq$ is transitive, we have $\rho(x) \preccurlyeq \rho(y)$ and $\rho(y) \preccurlyeq \rho(x)$. Since $\preccurlyeq$ is anti-symmetric, we have $\rho(x) = \rho(y)$.

(3) Transitive. Let $(\rho(x), \rho(y)) \in \preccurlyeq_\rho$ and $(\rho(y), \rho(z)) \in \preccurlyeq_\rho$. Then, there exist $m, n \in \mathbb{N}$ and there are $\rho(u_1), ..., \rho(u_{m-1}), \rho(v_1), ..., \rho(v_{n-1}) \in S/\rho$ such that $\rho(x)\ \alpha\ \rho(u_1)\ \alpha ...\ \alpha\ \rho(u_{m-1})\ \alpha\ \rho(y)$ and $\rho(y)\ \alpha\ \rho(v_1)\ \alpha\ ...\ \alpha\ \rho(v_{n-1})\ \alpha\ \rho(z)$. Since $\rho(y)\ \alpha\ \rho(y)$, we have $(\rho(x), \rho(z)) \in \alpha^{m+n+1} \subseteq \preccurlyeq_\rho$.

(4) $\preccurlyeq_\rho$ is compatible with the operation $*_\rho$ of S/ρ. Before showing that, we first prove that the relation α is compatible with $*_\rho$. Let $(\rho(x), \rho(y)) \in \alpha$ and $\rho(z) \in S/\rho$. Then, there exist $x' \in \rho(x)$ and $y' \in \rho(y)$ such that $x' \leqslant y'$. Since $\leq$ is compatible with the hyperoperation $\circ$ of S, we have $x' \circ z \leqslant y' \circ z$, i.e., for any $a \in x' \circ z$ there exists $b \in y' \circ z$ such that $a \leqslant b$. So $(\rho(a), \rho(b)) \in \alpha$. By Lemma 3.1(ii), we have $\rho(a) = \rho(x') *_\rho \rho(z) = \rho(x) *_\rho \rho(z)$ and $\rho(b) = \rho(y') *_\rho \rho(z) = \rho(y) *_\rho \rho(z)$. Consequently, $(\rho(x) *_\rho \rho(z), \rho(y) *_\rho \rho(z)) \in \alpha$. Similarly, we can show that $(\rho(z) *_\rho \rho(x), \rho(z) *_\rho \rho(y)) \in \alpha$. Furthermore, it is not difficult to show that the relation α^m is compatible with $*_\rho$ for all $m \in \mathbb{N}$. Thus $\preccurlyeq_\rho$ is compatible. Consequently, $(S/\rho, *_\rho, \preccurlyeq_\rho)$ is an ordered semigroup. Clearly, the mapping $\varphi : S \to S/\rho$ by $x \mapsto \rho(x)$, for all $x \in S$, is a normal homomorphism. In fact, if $x \leqslant y$ for all $x, y \in S$, then $(\rho(x), \rho(y)) \in \alpha \subseteq \preccurlyeq_\rho$. Therefore $\preccurlyeq_\rho$ is an order relation on S/ρ.

Finally, we show that $\preccurlyeq_\rho$ is the smallest order relation on S/ρ with respect to ρ. Let $\preccurlyeq'$ be an order relation on S/ρ with respect to a strongly ordered regular equivalence relation ρ. Let $(\rho(x), \rho(y)) \in \preccurlyeq_\rho$. Then, there exists $m \in \mathbb{N}$ such that $(\rho(x), \rho(y)) \in \alpha^m$. There are $\rho(w_1), ..., \rho(w_{m-1}) \in S/\rho$ such that $\rho(x)\ \alpha\ \rho(w_1)\ \alpha\ ...\ \alpha\ \rho(w_{m-1})\ \alpha\ \rho(y)$.

By the definition of α, there exist $x' \in \rho(x), w_i', w_i'' \in \rho(w_i)$ and $y' \in \rho(y)$, for all $i = 1, ..., m-1$, such that $x' \leqslant w_1', w_1'' \leqslant w_2', ..., w''_{m-2} \leqslant w'_{m-1}$ and $w''_{m-1} \leqslant y'$. Using the similar proof of (2), we obtain that $\rho(x) \preccurlyeq' \rho(w_1) \preccurlyeq' ... \preccurlyeq' \rho(w_{m-1}) \preccurlyeq' \rho(y)$. Since $\preccurlyeq'$ is transitive, we have $(\rho(x), \rho(y)) \in \preccurlyeq'$ and the proof is completed. □

Example 5.9. Let $S = \{a, b, c, d, e, f\}$. Define $\circ : S \times S \to \mathcal{P}^*(S)$ by the following table.
Then $(S, \circ)$ is a semihypergroup(see page 63 in [4]) and a partial order $\leqslant$ is defined by

$\leqslant := \{(a, a), (a, b), (a, f), (a, e), (b, b), (b, f), (c, a), (c, b), (c, c), (c, f), (c, e),$
$(d, b), (d, d), (d, f), (e, e), (e, f), (f, f)\}$. hen $(S, \circ, \leqslant)$ is an ordered semihypergroup. We will show that the semilattice equivalence relation on S is not an ordered semilattice in general. Let $A = \{c, d, e, f\}$. Clearly, A is a completely prime hyperideal of S. By Corollary 3.3, we have

$\circ$	a	b	c	d	e	f
a	a	$\{a,b\}$	c	$\{c,d\}$	e	$\{e,f\}$
b	b	b	d	d	f	f
c	c	$\{c,d\}$	c	$\{c,d\}$	c	$\{c,d\}$
d	d	d	d	d	d	d
e	e	$\{e,f\}$	c	$\{c,d\}$	e	$\{e,f\}$
f	f	f	d	d	f	f

$\rho := \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(c,e),(c,f),(d,c),(d,d),(d,e),$ $(d,f),(e,c),(e,d),(e,e),(e,f),(f,c),(f,d),(f,e),(f,f)\} \in \mathcal{SR}(\mathcal{S})$.

By Lemma 3.1(ii), $(S/\rho, *_\rho)$ is a semigroup where $S/\rho := \{\rho(a), \rho(c)\}$, $\rho(a) = \{a,b\} = \rho(b)$ and $\rho(c) = \{c,d,e,f\} = \rho(d) = \rho(e) = \rho(f)$.

Assume that ρ is a weak ordered semilattice equivalence relation on S. By Theorem 5.8, we have the smallest order relation $\preccurlyeq_\rho$ on S/ρ with respect to ρ. Then

$$\alpha := \{(\rho(a),\rho(a)),(\rho(a),\rho(c)),(\rho(c),\rho(a)),(\rho(c),\rho(c))\} \subseteq \preccurlyeq_\rho .$$

Since $(\rho(a),\rho(c)),(\rho(c),\rho(a)) \in \preccurlyeq_\rho$ and $\preccurlyeq_\rho$ is anti-symmetric, we have $\rho(a) = \rho(c)$ which is impossible. Consequently, ρ is not a weak ordered semilattice. Therefore, any semilattice equivalence relation on ordered semihypergroups does not need to be weak ordered semilattice in general which leads to answer our problem.

Finally, we give a characterization of the strongly ordered regular and the weak ordered semilattice equivalence relation on ordered semihypergroups.

Definition 5.10. Let $(S, \circ, \leqslant)$ be an ordered semihypergroup and ρ be a (strongly) regular equivalence relation on S. A sequence $\rho(x), \rho(a_1), \rho(a_2), ..., \rho(a_n), \rho(y)$ of S/ρ is called a *ρ-classes-chain* if $\rho(x)\ \alpha\ \rho(a_1)\ \alpha\ \rho(a_2)\ \alpha\ ...\ \alpha\ \rho(a_n)\ \alpha\ \rho(y)$ where α is defined in Theorem 5.8. A ρ-classes-chain is called a *ρ-classes-cyclic* if and only if $(x,y) \in \rho$ and $a_1, a_2, ..., a_n \notin \rho(x)$.

Theorem 5.11. *Let S be an ordered semihypergroup and ρ be a strongly regular equivalence relation on S. Then, ρ is strongly ordered regular if and only if S/ρ does not contain the ρ-classes-cyclic.*

Proof. $(\Rightarrow)$ Let ρ be a strongly ordered regular equivalence relation on S. Then, there exists an order relation $\preccurlyeq$ on S/ρ such that $(S/\rho, *_\rho, \preccurlyeq)$ is an ordered semigroup and the mapping $\varphi : S \to S/\rho$ by $x \mapsto \rho(x)$ is a normal homomorphism. Assume that S/ρ has a ρ-classes-cyclic $\rho(a)\ \alpha\ \rho(b_1)\ \alpha\ \rho(b_2)\ \alpha\ ...\ \alpha\ \rho(b_n)\ \alpha\ \rho(a)$. Then $b_1, ..., b_n \notin \rho(a)$. By the definition of α, there exist $a', a'' \in \rho(a), b_i', b_i'' \in \rho(b_i)$, for all $i = 1, ..., n$, such that $a' \leqslant b_1'$, $b_i'' \leqslant b_{i+1}'$ and $b_n'' \leqslant a''$ for all $i = 1, ..., n-1$. It follows that $\varphi(a') \preccurlyeq \varphi(b_1')$, $\varphi(b_i'') \preccurlyeq \varphi(b_{i+1}')$ and $\varphi(b_n'') \preccurlyeq \varphi(a'')$ for all $i = 1, ..., n-1$. We have

$$\rho(a) = \rho(a') \preccurlyeq \rho(b_1') = \rho(b_1) = \rho(b_1'') \preccurlyeq ... \preccurlyeq \rho(b_n') = \rho(b_n) = \rho(b_n'') \preccurlyeq \rho(a'') = \rho(a).$$

Since $\preccurlyeq$ is anti-symmetric on S/ρ, we have $\rho(a) = \rho(b_1) = ... = \rho(b_n)$. This implies that $b_1, ..., b_n \in \rho(a)$ which is a contradiction. Thus S/ρ does not contain the ρ-classes-cyclic.

($\Leftarrow$) Suppose that S/ρ does not contain the ρ-classes-cyclic. Since ρ is a strongly regular equivalence relation on S, by Lemma 3.1(ii), we obtain that $(S/\rho, *_\rho)$ is a semigroup. Let $\preccurlyeq_\rho$ be the relation which was defined in Theorem 5.8. Consequently, the rest of the proof in this theorem that $\preccurlyeq_\rho$ is reflexive. Let $(\rho(x), \rho(y)) \in \preccurlyeq_\rho$ and $(\rho(y), \rho(x)) \in \preccurlyeq_\rho$. Then, there exist $m, n \in \mathbb{N}$ such that $(\rho(x), \rho(y)) \in \alpha^m$ and $(\rho(y), \rho(x)) \in \alpha^n$. There are $\rho(a_1), ..., \rho(a_{m-1}), \rho(b_1), ...,$ $\rho(b_{n-1}) \in S/\rho$ such that

$$\rho(x)\ \alpha\ \rho(a_1)\ \alpha\ ...\ \alpha\ \rho(a_{m-1})\ \alpha\ \rho(y)\ \alpha\ \rho(b_1)\ \alpha\ ...\ \alpha\ \rho(b_{n-1})\ \alpha\ \rho(x).$$

Since S/ρ does not contain the ρ-classes-cyclic, we get $a_1, ..., a_{m-1}, y, b_1, ..., b_{n-1} \in \rho(x)$ and then $\rho(x) = \rho(y)$. Using the similar proof of Theorem 5.8, we conclude that ρ is strongly ordered regular on S. □

Corollary 5.12. *Let $\rho \in \mathcal{SR}(\mathcal{S})$. Then, ρ is a weak ordered semilattice if and only if S/ρ does not contain the ρ-classes-cyclic.*

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