Quasigroups and Related Systems **30** (2022), 1 − 10
https://doi.org/10.56415/qrs.v30.01

The Cayley sum graph of ideals of a semigroup

Mojgan Afkhami, Mehdi Hassankhani and Kazem Khashyarmanesh

Abstract. Let S be a regular semigroup, $\Im(S)$ be the set of ideals of S and M be a subset of $\Im(S)$. In this paper, we introduce an undirected Cayley graph of S, denoted by $\Gamma_{S,M}$, with elements of $\Im(S)$ as the vertex set, and, for two distinct vertices I and J, I is adjacent to J if and only if there is an element K of M such that $IK = J$ or $JK = I$. We study some basic properties of the graph $\Gamma_{S,M}$ such as connectivity, girth and clique number. Moreover, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{S,M}$.

1. Introduction

The Cayley sum graphs of ideals of a commutative ring was introduced by Afkhami et al. in [3]. Among all types of graphs related to various algebraic structures, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [2], [6], [12], [13], [14], [17] for examples of recent results and further references. Let us refer the readers to the survey article [15] for extensive bibliography devoted to various applications of Cayley graphs. A semigroup is an algebraic structure consisting of a set together with an associative binary operation. The Cayley graphs of semigroups are related to automata theory, as explained in [11] and the monograph [12]. For a semigroup S and a subset H of S, the Cayley graph $Cay(S, H)$ of S relative to H is defined as the digraph with vertex set S and edge set $E(S, H)$ consisting of those ordered pairs (x, y) such that $y = sx$, for some $s \in H$ (cf. [13]).

Let S be a regular semigroup, $\Im(S)$ be the set of ideals of S and M be a subset of $\Im(S)$. In this paper, we introduce an undirected Cayley graph associated to S, which is denoted by $\Gamma_{S,M}$. The elements of $\Im(S)$ are its vertices and two distinct vertices I and J are adjacent if and only if there is an element K of M such that $IK = J$ or $JK = I$. In Section 2, we recall some definitions and notations about semigroups. In Section 3, we study some basic properties of the graph $\Gamma_{S,M}$ such as connectivity, girth and clique number. For example we show that if $M = \{I, J\}$, where I and J are not minimal ideals and the graph $\Gamma_{S,M}$ is connected, then $\mathrm{diam}(\Gamma_{S,M}) \leqslant 4$ and $\mathrm{girth}(\Gamma_{S,M}) \leqslant 4$. Also, we prove that if $M = \{I_1, I_2, \ldots, I_n\}$, where non of the I_i's are minimal, then the graph $\Gamma_{S,M}$ is connected if and only

2010 Mathematics Subject Classification: 05C10, 13A99
Keywords: Semigroup, Cayley graph, ring graph, planar graph.

if $I_1I_2 \dots I_n = \mathfrak{J}$. Finally, in Section 4, we determine M for which $\Gamma_{S,M}$ is planar, outerplanar and a ring graph.

Now we recall some definitions and notations about undirected graphs. We use the standard terminology of graphs following [4]. In a graph G, the distance between two distinct vertices a and b, denoted by $d(a,b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise, we set $d(a,b) := \infty$. The diameter of a graph G is $\mathrm{diam}(G) = \sup\{d(a,b) : a$ and b are distinct vertices of $G\}$. The girth of G, denoted by girth(G), is the length of a shortest cycle in G, if G contains a cycle; otherwise, we set $\mathrm{girth}(G) := \infty$. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A vertex a in a graph G is said to be a pendant vertex if $\deg(a) = 1$, where $\deg(a)$ denotes the number of vertices which are adjacent to a. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, the complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. We say that G is totally disconnected if no two vertices of G are adjacent. Also, G is called an empty graph if its vertex set is empty. A clique of a graph is a complete subgraph of it and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the clique number of G. A subset X of the vertices of G is called an independent set if the induced subgraph on X has no edges. A vertex a of G is called a cutvertex if the number of connected components of $G \setminus \{a\}$ is larger than that of G. A graph G is 2-connected if $|V(G)| > 2$ and G has no cutvertices. A graph is said to be planar if it can be drawn in the plane, so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Let G be a graph with n vertices and q edges. We denote the vertex set and edge set of G by $V(G) = \{x_1, \cdots, x_n\}$ and $E(G) = \{t_1, \cdots, t_q\}$ respectively. Recall that a 0-chain (resp. 1-chain) of G is a formal linear combination $\sum a_i x_i$(resp. $\sum b_i t_i$) of vertices (resp. edges), where $a_i \in \mathbb{Z}_2$ (resp. $b_i \in \mathbb{Z}_2$). The boundary operator is the linear map $\partial : C_1 \to C_0$ defined by $\partial(\{x,y\}) = x + y$, where C_i is the $\mathbb{Z}_2$-vector space of i-chains. A cycle vector is a 1-chain of the form $t_1 + \cdots + t_r$ where $t_1, \cdots, t_r$ are the edges of a cycle of G. The cycle space $\mathfrak{Z}(G)$ of G over $\mathbb{Z}_2$ is equal to $\ker(\partial)$.

Let C be a cycle of G. A chord in G is any edge joining two nonadjacent vertices in C. A primitive cycle is a cycle without chords. Moreover, we say that a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The free rank of G, denoted by frank(G), is the number of primitive cycles of G. Also, the number $\mathrm{rank}(G) := q - n + r$, where r is the number of connected components of G, is called the cycle rank of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. These two numbers satisfy the inequality $\text{rank}(G) \leqslant \text{frank}(G)$, as is seen in [7, Proposition 2.2]. In the second section of [7], the authors provided a characterization of graphs such that the equality occurs. The precise definition of a ring graph can be found in Section 2 of [7]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. In [7], it is showed that, for the graph G, the following conditions are equivalent:

(i) G is a ring graph,

(ii) $\text{rank}(G) = \text{frank}(G)$,

(iii) G satisfies PCP and G does not contain a subdivision of K_4 as a subgraph.

The following lemma is useful.

Lemma 1.1. [1, Lemma 7.78] *Let G be a graph with vertex set V. If G is 2-connected and $\deg(v) \geqslant 3$ for all $v \in V$, then G contains a subdivision of K_4 as a subgraph.*

2. Preliminaries

In this section we recall some basic definitions and notations on a semigroup S. For more details on semigroups see [5], [8], [9] and [16].

Let A be a nonempty subset of a semigroup S. We say that A is a subsemigroup of S, denoted by $A \leqslant S$, if A is closed under the product of S, that is, $A \leqslant S \Leftrightarrow A^2 \subseteq A$. Also, a nonempty subset I of S is a left ideal, if $SI \subseteq I$, and it is a right ideal, if $IS \subseteq I$. Moreover I is called an ideal, if it is both a left and a right ideal.

An ideal I of S is said to be minimal, if for any ideal J of S, $J \subseteq I$ implies that $J = I$.

Theorem 2.1. [9, Theorem 2.5] *If a semigroup S has a minimal ideal, then it is unique.*

Lemma 2.2. [9, Lemma 2.11] *If I is a minimal ideal, and J is any ideal of S, then $I \subseteq J$.*

Every finite semigroup S has a minimal ideal. Indeed, consider an ideal I, which has the least number of elements. Such an ideal exists because S is finite and S is its own ideal. An element $a \in S$ is regular if $a = axa$, for some $x \in S$. S is regular if every $a \in S$ is regular. Also $b \in S$ is an inverse of a if $a = aba$ and $b = bab$. We denote $V(a)$ to be the set of inverses of a.

The following two theorems, provide a condition under which a semigroup S is regular.

Theorem 2.3. [10] *A semigroup S is regular if and only if $IJ = I \cap J$, for every right ideal I and every left ideal J of S.*

3. Basic properties of the Cayley graph $\Gamma_{S,M}$

Let S be a finite regular semigroup, $\mathfrak{I}(S)$ be the set of all ideals of S, and $\mathfrak{J}$ be the minimal ideal of S. Let M be a nonempty subset of $\mathfrak{I}(S)$. We define the graph $\Gamma_{S,M}$, as an undirected graph with $\mathfrak{I}(S)$ as the vertex set, and two distinct vertices I and J are adjacent if and only if there is a vertex K in M such that $IK = J$ or $JK = I$. Hence if I is adjacent to J, then, for some vertex K in M, either $I \subseteq J \cap K$ or $J \subseteq I \cap K$. Thus, the set of maximal ideals is an independent set. Also, for each vertex I, $\mathfrak{J}I = \mathfrak{J}$, that is, if $\mathfrak{J} \in M$, then $\mathfrak{J}$ is adjacent to all vertices of $\Gamma_{S,M}$ and $\Gamma_{S,M}$ is a refinement of a star graph. Thus, in the rest of the paper, we assume that $\mathfrak{J} \notin M$ and we put $\mathfrak{I}^*(S) = \mathfrak{I}(S) \setminus \{\mathfrak{J}\}$.

Lemma 3.1. *Let $M = \{I\} \subseteq \mathfrak{I}(S)$. Then there is no path of length greater than 2 in $\Gamma_{S,M}$.*

Proof. First we claim that if there is a path $K_1 - K_2 - K_3$ of length 2 in $\Gamma_{S,M}$, then $K_2 \subseteq K_1, K_3$. Since K_1 is adjacent to K_2, we have $K_1 I = K_2$ or $K_2 I = K_1$. Also K_3 is adjacent to K_2. So $K_3 I = K_2$ or $K_2 I = K_3$. Assume that $K_2 I = K_1$. Thus we have $K_3 I = K_2$ which is impossible. Hence $K_1 I = K_2$ and $K_3 I = K_2$. Therefore $K_2 \subseteq K_1, K_3$. Now suppose that there is a path $K_1 - K_2 - K_3 - K_4$ of length three in $\Gamma_{S,M}$. By the above discussion, we have $K_2 \subseteq K_1, K_3$ and $K_3 \subseteq K_2, K_4$ which is again impossible. □

Proposition 3.2. *Let $M \subseteq \mathfrak{I}(S)$. Then $\Gamma_{S,M}$ has no cycle if and only if $M = \{I\}$, for some $I \in \mathfrak{I}(S)$.*

Proof. Assume that $|M| \geqslant 2$ and $I, J \in M$. Put $F = IJ$ and $G = I \cup J$. Then it is clear that $F - I - G - J - F$ is a cycle in $\Gamma_{S,M}$. Now let $M = \{I\}$. Then, by Lemma 3.1, there is no cycle in $\Gamma_{S,M}$. □

Proposition 3.3. *Let M be a singleton subset of $\mathfrak{I}(S)$. Then $\Gamma_{S,M}$ is disconnected.*

Proof. Suppose that $M = \{I\}$, and let J be any vertex distinct from I. If $I \subseteq J$, then I is adjacent to J and J is not adjacent to any vertex of $\Gamma_{S,M}$, and if $J \subseteq I$, then I is not adjacent to J. Now suppose that I and J are not comparable. Then clearly I is not adjacent to J. Therefore the set $A = \{J : I \subseteq J\}$ forms a component of $\Gamma_{S,M}$ and hence the graph $\Gamma_{S,M}$ is not connected. □

Lemma 3.4. *Let $M = \{I, J\} \subseteq \mathfrak{I}^*(S)$. Then the graph $\Gamma_{S,M}$ is connected if and only if $IJ = \mathfrak{J}$.*

Proof. Suppose that $IJ = \mathfrak{J}$. Clearly $\mathfrak{J}$ is adjacent to both vertices I and J. We claim that $\Gamma_{S,M}$ has no isolated vertex. Now, if $K \in \mathfrak{I}^*(S)$ and K is an isolated vertex, then $KI = K$ and $KJ = K$, and hence $K \subseteq I, J$. Therefore $K = \mathfrak{J}$, which is a contradiction. Thus it is enough to show that for any vertex K there is a path between K and $\mathfrak{J}$. As K is not an isolated vertex, there is a vertex K'

such that K is adjacent to K'. Hence $KI = K'$ or $K'I = K$, for some $I \in M$. If $KI = K'$, then $K' \subseteq I$ and so $K'J = \mathfrak{J}$, which means that K' is adjacent to $\mathfrak{J}$. If $K'I = K$, then $K \subseteq I$ and $KJ = \mathfrak{J}$, which implies that K is adjacent to $\mathfrak{J}$. A similar argument for $KJ = K'$ or $K'J = K$, shows that for any vertex K, there is a path between K and $\mathfrak{J}$.

Conversely assume that $\Gamma_{S,M}$ is connected. Suppose on the contrary that $IJ \neq \mathfrak{J}$. Let $K = IJ$ and $B = \{F : F \in \mathfrak{J}(S) \text{ and } K \subseteq F\}$. Suppose that $F \in B$ and $T \notin B$. It is clear that FI and FJ lies in B, and TI and TJ are not in B, and hence F is not adjacent to T. Therefore B forms a component of $\Gamma_{S,M}$ and hence the graph $\Gamma_{S,M}$ is not connected. □

Theorem 3.5. *Let $M = \{I, J\} \subseteq \mathfrak{J}^*(S)$ and the graph $\Gamma_{S,M}$ be connected. Then* $\operatorname{diam}(\Gamma_{S,M}) \leqslant 4$ *and* $\operatorname{girth}(\Gamma_{S,M}) \leqslant 4$.

Proof. By the proof of Lemma 3.4, for every vertex K that is not adjacent to $\mathfrak{J}$, there is a vertex K' such that K' is adjacent to K and $\mathfrak{J}$. Now let N and T be two distinct non adjacent vertices such that they are not adjacent to $\mathfrak{J}$. Then there are vertices N' and T' such that we have the path $N - N' - \mathfrak{J} - T' - T$, and hence its diameter is less than or equal to four.

Since we have the cycle, $I - \mathfrak{J} - J - (I \cup J) - I$ of length four, therefore the girth of $\Gamma_{S,M}$ is less than or equal to 4. □

Proposition 3.6. *Let $M = \{I_1, I_2, \ldots, I_n\} \subseteq \mathfrak{J}^*(S)$. Then the graph $\Gamma_{S,M}$ is connected if and only if $I_1 I_2 \ldots I_n = \mathfrak{J}$.*

Proof. First assume that $I_1 I_2 \ldots I_n = \mathfrak{J}$. Suppose that there are two ideals I_j and I_k in M such that $I_j I_k = \mathfrak{J}$, for some $1 \leqslant j \neq k \leqslant n$. Therefore, by Lemma 3.4, the result holds. So we assume that for each vertex I_j in M, $\prod(M \setminus \{I_j\}) \neq \mathfrak{J}$. Now let K be a vertex such that K is not adjacent to $\mathfrak{J}$. Hence $KI_j \neq \mathfrak{J}$, for $j = 1, 2, \ldots, n$. Put $K_j = (KI_1 \ldots I_{j-1})I_j$. Therefore there is a path of length at most n between K and $K_n = \mathfrak{J}$, and hence the graph is connected.

For the converse statement, assume that $I_1 \ldots I_n \neq \mathfrak{J}$. Put $K = I_1 \ldots I_n$ and let $B = \{F : F \in \mathfrak{J}(S) \text{ and } K \subseteq F\}$. Now let $F \in B$ and $T \notin B$. It is clear that for $i = 1, \ldots, n$, FI_i lies in B, and TI_i are not in B, and hence F is not adjacent to T. Therefore the graph $\Gamma_{S,M}$ is not connected. □

Corollary 3.7. *Let $M = \{I_1, I_2, \ldots, I_n\} \subseteq \mathfrak{J}^*(S)$ and the graph $\Gamma_{S,M}$ be connected. Then* $\operatorname{diam}(\Gamma_{S,M}) \leqslant 2n$ *and also* $\operatorname{girth}(\Gamma_{S,M}) \leqslant 4$.

Proposition 3.8. *Let $\Gamma_{S,M}$ be connected and $K \in \mathfrak{J}^*(S)$ be a pendant vertex. Then K is adjacent to $\mathfrak{J}$.*

Proof. Suppose that for some I, J in M, $KI \neq KJ$. Then $\deg(K) \geqslant 2$, and hence for all I, J in S, $KI = KJ$. Put $F = KI$. So for all I in M, $F \subseteq I$, and hence $F = \mathfrak{J}$. □

Lemma 3.9. *If $K_1 - K_2 - K_3 - K_1$ is a cycle of length three in the graph $\Gamma_{S,M}$, then $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(S)$.*

Proof. If two vertices are adjacent in $\Gamma_{S,M}$, then one of them is a subset of another. Hence $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(S)$. □

Proposition 3.10. *Assume that M is a finite subset of $\mathfrak{I}(S)$ and that $\Gamma_{S,M}$ has a clique of size n. Then $|S| \geqslant n$.*

Proof. By the definition of adjacency of vertices in $\Gamma_{S,M}$, K_1 is adjacent to K_2 only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Thus if the graph $\Gamma_{S,M}$ has a clique with n vertices $K_1, K_2, \ldots, K_n$, then, by Lemma 3.9, the set $\{K_1, K_2, \ldots, K_n\}$ is a chain in $\mathfrak{I}(S)$. Without loss of generality, we may assume that $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n$. Hence if $|S| < n - 1$, then K_1 is not adjacent to at least one vertex K_i, for $i = 2, \ldots, n$, and hence $\{K_1, K_2, \ldots, K_n\}$ is not a clique, which is a contradiction. □

We say that a vertex I has the property $(*)$ if I is comparable with at least one of the elements in M or I is adjacent to $\mathfrak{I}$ in $\Gamma_{S,M}$.

Proposition 3.11. *Let $M = \{I, J\}$ and $\Gamma_{S,M}$ be connected. If all vertices of $\Gamma_{S,M}$ has the property $(*)$, then $M \cup \{\mathfrak{I}\}$ is a dominating set in $\Gamma_{S,M}$.*

Proof. Let F be an arbitrary vertex in $\Gamma_{S,M}$. Then we show that F is adjacent to $\mathfrak{I}$, I or J. Since F has the property $(*)$, there is a vertex in M, say I, such that $I \subseteq F$ or $F \subseteq I$. If $I \subseteq F$, then clearly F is adjacent to I. Also if $F \subseteq I$, then $FJ = \mathfrak{I}$, which means that F is adjacent to $\mathfrak{I}$. □

4. Planarity of $\Gamma_{S,M}$

Let M be a subset of $\mathfrak{I}^*(S)$. We say that M has a property $(*)$, if for all ideals M_i and M_j in M, $M_i \cap M_j = \mathfrak{I}$.

Example 4.1. Let S be the usual multiplicative semigroup $(\mathbb{Z}_6, \cdot)$ and let $M = \{2\mathbb{Z}_6, 3\mathbb{Z}_6\}$. Then $\mathfrak{I} = 0$ and $2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = 0$ and S has the property $(*)$.

Notation 1. To simplify notations, let $M = \{M_1, M_2, \ldots, M_n\}$ has the property $(*)$. We set $S_i := \{F | F \supsetneqq M_i \text{ and } F \not\supseteq \bigcup_{j \neq i} M_j\}$ and

$$S_{ij} := \{F | F \supsetneqq M_i \cup M_j \text{ and } F \not\supseteq \bigcup_{k \neq i,j} M_k\}$$

and similarly $S_{12\ldots n} := \{F | F \supsetneqq M_1 \cup M_2 \cup \ldots \cup M_n\}$.

Note that if $S_{12\ldots n} = \varnothing$, then $M_1 \cup M_2 \ldots \cup M_n = S$.

Remark 1. Let $M = \{M_1, M_2, \ldots, M_n\}$ has the property $(*)$. Then for $n \geqslant 4$, the graph $\Gamma_{S,M}$ has a subdivision of $K_{3,3}$, and therefore it is not planar as it is shown in Figure 1. For $n = 2$, the graph is planar as it is shown in Figure 2, where F_i in S_{12}, G_{1j} in S_1 and G_{2k} in S_2. Note that if for some t, $G_{1t} \cap M_2 \neq \mathfrak{I}$ or $G_{2t} \cap M_1 \neq \mathfrak{I}$, then the graph $\Gamma_{S,M}$ is a subdivision of the graph in Figure 2 and clearly it is planar.

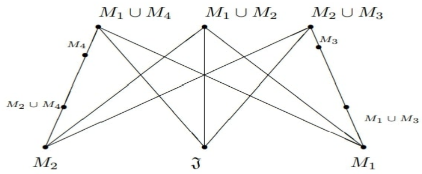

Figure 1.

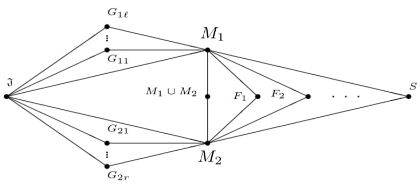

Figure 2.

Proposition 4.2. *Let $M = \{M_1, M_2, M_3\}$ be a subset of $\mathfrak{I}^*(S)$ which has the property $(*)$. Then $S_{123} \neq \varnothing$ implies that $\Gamma_{S,M}$ is not planar. Moreover, if $S_{123} = \varnothing$, then we have the following statements:*

1. *If for some i, j, $S_{ij} \neq \varnothing$, then $\Gamma_{S,M}$ is not planar.*

2. *If for all i, j, $S_{ij} = \varnothing$, then $\Gamma_{S,M}$ is a planar graph.*

Proof. If $S_{123} \neq \varnothing$, then $K_{3,3}$ is a subgraph of $\Gamma_{S,M}$ with two partitions $X = \{M_1, M_2, M_3\}$ and $Y = \{F, \mathfrak{I}, M_1 \cup M_2 \cup M_3\}$, where $F \in S_{123}$. Now let $S_{123} = \varnothing$, and for some i, j, $S_{ij} \neq \varnothing$. Without loss of generality, we may assume that $S_{12} \neq \varnothing$ and $F \in S_{12}$. Therefore $\Gamma_{S,M}$ has a subgraph isomorphic to $K_{3,3}$ as it is shown in Figure 3, and hence it is not planar.

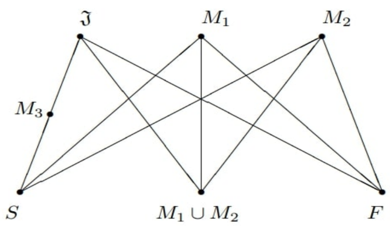

Figure 3.

For the second statement, let $S_{ij} = \varnothing$ for all i, j. Then $\Gamma_{S,M}$ is a planar graph, as it is shown in Figure 4, where $F_{ij} \in S_i$. Note that if, for some t, $F_{1t} \cap (M_2 \cup M_3) \neq \mathfrak{I}$, $F_{2t} \cap (M_1 \cup M_3) \neq \mathfrak{I}$ or $F_{3t} \cap (M_1 \cup M_2) \neq \mathfrak{I}$, then the graph $\Gamma_{S,M}$ is a subdivision of the graph in Figure 4 and clearly it is planar.

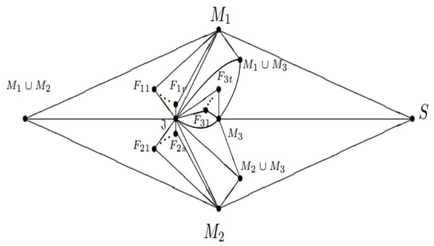

Figure 4. □

In the sequel of this section, we deal with the outerplanarity of $\Gamma_{S,M}$. By [7, Lemma 2.9], we know that every outerplanar graph is a ring graph and every ring graph is a planar graph. Let M be a subset of ideals of $\mathfrak{I}^*(S)$ which has the property $(*)$, $|M| = 3$ and $\Gamma_{S,M}$ is a planar graph. By Proposition 4.2, for all i, j, we have $|S_{ij}| = 0$, and even if for all i, $i = 1, 2, 3$, $S_i = \varnothing$, then $\Gamma_{S,M}$ has an induced subgraph H that is satisfied in the conditions of Lemma 1.1. Therefore $\Gamma_{S,M}$ has a subdivision isomorphic to K_4, as it is shown in Figure 5. Hence it is not a ring graph.

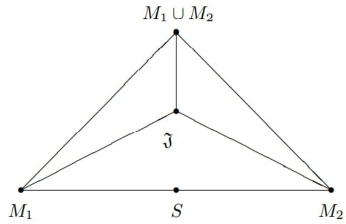

Figure 5.

By [7, Lemma 2.9], for $n \geqslant 3$, $K_{2,n}$ is not a ring graph. Assume that $|M| = 2$. If $S_{12} \neq \varnothing$, then $\Gamma_{S,M}$ has an induced subgraph isomorphic to $K_{2,3}$, which is not a ring graph. Now let $S_{12} = \varnothing$ and $|S_i| > 1$, for $i = 1, 2$. Then, similar to the above case, $\Gamma_{S,M}$ has an induced subgraph isomorphic to $K_{2,3}$.

By the above discussion we have the following theorem.

Theorem 4.3. *Let M be a subset of $\mathfrak{I}^*(S)$ which has the property $(*)$. Then $\Gamma_{S,M}$ is a ring graph if and only if $|M| = 2$, $S_{12} = \varnothing$ and $|S_1| = |S_2| = 1$.*

Acknowledgments. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

References

[1] **M. Aigner**, *Combinatorial Theory*, Springer-verlag, New York, (1997).

[2] **M. Afkhami, Z. Barati and K. Khashyarmanesh**, *Cayley graphs of partially ordered sets*, J. Algebras and its Appl., **12** (2013), 1250184 − 1250197.

[3] **M. Afkhami, Z. Barati, K. Khashyarmanesh and N. Paknejad**, *Cayley sum graphs of ideals of a commutative ring*, J. Aust. Math. Soc., **96** (2014), 289 − 304.

[4] **J.A. Bondy and U.S.R. Murty,** *Graph Theory with Applications,* Elsevier, New York, (1976).

[5] **A.H. Clifford and G.B. Preston**, *The Algebraic Theory of Semigroups*, Math. Surveys No. **7**, Amer. Math. Sot., Providence, RI, Vol. I, 1961; Vol. II, (1967).

[6] **G. Cooperman, L. Finkelstein and N. Sarawagi**, *Applications of Cayley graphs*, Appl. Algebra and Error-Correcting Codes, (1990), 367 − 378.

[7] **I. Gitler, E. Reyes and R.H. Villarreal**, *Ring graphs and complete intersection toric ideals*, Discrete Math., **310** (2010), 430 − 441.

[8] **P.A. Grillet**, *Semigroups: An introduction to the structure theory*, Marcel Dekker, (1995).

[9] **T. Harju**, *Lecture Notes on semigroups*, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland, (1996).

[10] **K. Iseki**, *A characterization of regular semigroup*, Proc. Japan Acad., **32** (1956), 676 − 677.

[11] **A.V. Kelarev,** *Labelled Cayley graphs and minimal automata,* Australasian J. Combinatorics, **30** (2004), 95 − 101.

[12] **A.V. Kelarev**, *On Cayley Graphs of Inverse Semigroups*, Semigroup Forum, **72** (2006), 411 − 418.

[13] **A.V. Kelarev and C.E. Praeger**, *On transitive Cayley graphs of groups and semigroups*, European J. Combin., **24** (2003), 59 − 72.

[14] **A.V. Kelarev and S.J. Quinn**, *A combinatorial property and Cayley graphs of semigroups*, Semigroup Forum, **66** (2003), 89 − 96.

[15] **A.V. Kelarev, J. Ryan and J. Yearwood,** *Cayley graphs as classifiers for data mining: the influence of asymmetries,* Discrete Math., **309** (2009), 5360 − 5369.

[16] **S. Schwarz**, *prime ideals and maximal ideals in semigroups*, Czechoslovak Math. J., **19** (1969), 72 − 79.

[17] **S. Zhou**, *A class of arc-transitive Cayley graphs as models for interconnection networks*, SIAM J. Discrete Math., **23** (2009), 694 − 714.

Received April 11, 2021

M. Afkhami
Department of Mathematics, University of Neyshabur, P.O.Box 91136-899, Neyshabur, Iran
e-mail: mojgan.afkhami@yahoo.com

M. Hassankhani
Department of Mathematics, Chalous Branch, Islamic Azad University, P.O.Box 46315-397, Chalous, Iran

e-mail: hassankhani@iauc.ac.ir

K. Khashyarmanesh
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran
e-mail: khashyar@ipm.ir