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Generalized Moufang G–loops

Alexander S. Basarab

Abstract

In this note some relations among generalized Moufang loops and G-loops are considered.

A loop $\mathcal{Q}(\cdot)$ is called:

(a) a *G-loop* [1], if every loop which is isotopic to $\mathcal{Q}(\cdot)$ is also isomorphic to it;

(b) a *generalized Moufang loop* [2], if one of the following identities holds:

$$x \cdot (yz \cdot x) = I(I^{-1}y \cdot I^{-1}x) \cdot zx\,, \qquad (x \cdot yz) \cdot x = xy \cdot I^{-1}(Ix \cdot Iz)\,;$$

(c) an *Osborn loop* [2], if the identity

$$xy \cdot (\Theta_x z \cdot x) = (x \cdot yz) \cdot x$$

holds, where Θ_x is a permutation which depends on x;

(d) a *K-loop* [3], if the following identities hold:

$$(x \cdot yIx) \cdot xz = x \cdot yz \quad \text{and} \quad (y \cdot x) \cdot ((I^{-1}xz) \cdot x) = yz \cdot x\,, \qquad (1)$$

where $Ix = x^{-1}$ and $I^{-1}x = {}^{-1}x$;

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(e) a *VD-loop*, if we have the equalities

$$(\cdot)_x = (\cdot)_x^{L_x^{-1}R_x} \quad \text{and} \quad {}_x(\cdot) = (\cdot)^{R_x^{-1}L_x}, \tag{2}$$

which are true for any $x \in \mathcal{Q}$, where

$$(\cdot)_x = (\cdot)^{(L_x,\, 1,\, L_x)}\,, \qquad {}_x(\cdot) = (\cdot)^{(1,\, R_x,\, R_x)}\,.$$

Any K-loop (any VD-loop) is a G-loop and any VD-loop is an Osborn loop.

Theorem 1. *A generalized Moufang loop $\mathcal{Q}(\cdot)$ is a K-loop if and only if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

Proof. For a generalized Moufang loop $\mathcal{Q}(\cdot)$ the property *WIP* is universal [2]. Therefore, by a result from [4], one has the autotopy $T = (R_y^{-1}L_z, R_x^{-1}L_y, L_zR_x^{-1})$, where $z = I^{-1}(y \cdot x)$. By identifying x and y in the autotopy T, one obtains

$$T_1 = (R_x^{-1}L_{I^{-1}(x\cdot x)}\,, R_x^{-1}L_x\,, L_{I^{-1}(x\cdot x)}R_x^{-1}\,.$$

In any generalized Moufang loop $I^{-1}(x\cdot x) = I(x \cdot x)$ holds, hence T_1 provides the equality

$$R_x^{-1}L_{I(x\cdot x)}u \cdot R_x^{-1}L_x v = L_{I(x\cdot x)}R_x^{-1}(u\cdot v). \tag{3}$$

Let $v = 1$ in (3), then

$$R_x^{-1}L_{I(x\cdot x)} = L_{I(x\cdot x}R_x^{-1}. \tag{4}$$

Identity (4) implies $T_1 = (L_{I(x\cdot x)}R_x^{-1}\,, R_x^{-1}L_x\,, L_{I(x\cdot x}R_x^{-1})$. Hence T_1^{-1} implies $L_x^{-1}R_x$ is a pseudoautomorphism with the right companion $\alpha_1 = R_xL_{I(x\cdot x)}^{-1}1 = (x\cdot x)\cdot x$. Thus

$$(\cdot)_{x^2\cdot x} = (\cdot)^{R_x^{-1}L_x} \tag{5}$$

In any generalized Moufang loop the equalities

$${}_{I_x}(\cdot) = (\cdot)_x \tag{6}$$

$$R_x L_{I^{-1}x} = L_x^{-1} R_x \,, \qquad L_x R_{I_x} = R_x^{-1} L_x \tag{7}$$

hold.

Let $x^2 \in N$, where N is the nucleus of the generalized Moufang loop $\mathcal{Q}(\cdot)$. Since $(\cdot)_{x^2} = (\cdot)$ (5) implies $(\cdot)_x = (\cdot)^{R_x^{-1}L_x}$, or, in other words,

$$T_2 = (R_x \,, L_x^{-1} R_x \,, R_x) = (R_x \,, R_x L_{I^{-1}x} \,, R_x)$$

(by (7)) is an autotopy, that yields $yx \cdot (I^{-1}xz \cdot x) = yz \cdot x$, which coincides with the second equality from (1). By using (6) in the equality $(\cdot)_x = (\cdot)^{R_x^{-1}L_x}$ we obtain $I_x(\cdot) = (\cdot)^{R_x^{-1}L_x}$, and consequently, have the autotopy

$$T_3 = (L_x^{-1} R_x \,, R_{Ix} L_x^{-1} R_x \,, R_{Ix} L_x^{-1} R_x) = (L_x^{-1} R_x \,, L_x^{-1}, L_x^{-1})$$

or

$$T_3^{-1} = (R_x^{-1} L_x \, L_x \,, L_x) = (L_x R_{Ix} \,, L_x \,, L_x) \,,$$

hence $(x \cdot yI_x) \cdot xz = x \cdot yz$, meaning that the first equality from (1) is true. Thus, if in the generalized Moufang loop $\mathcal{Q}(\cdot)$, $x^2 \in N$ for any $x \in Q$, then $\mathcal{Q}(\cdot)$ is a K-loop.

Now, let the generalized Moufang loop $\mathcal{Q}(\cdot)$ be a K-loop, then the equality (5) and the equality $(\cdot)_x = (\cdot)^{R_x^{-1}L_x}$ are true and they imply $(\cdot)_{x^2 \cdot x} = (\cdot)_x$, or $(\cdot)_{x^2} = (\cdot)$, or $x^2 \in N$. □

Theorem 2. *A generalized Moufang loop $\mathcal{Q}(\cdot)$ is a VD-loop, if $x^4 \in N$ whichever $x \in \mathcal{Q}$.*

Proof. If $\mathcal{Q}(\cdot)$ is a generalized Moufang loop, then (5) holds, so: $((\cdot)_{x^2 \cdot x})_x = ((\cdot)^{R_x^{-1}L_x})_x$ or $(\cdot)_{x^4} = (\cdot)_x^{R_x^{-1}L_x}$. Suppose $x^4 \in N$, then $(\cdot)_{x^4} = (\cdot)$, hence $(\cdot)_x^{R_x^{-1}L_x} = (\cdot)$, and $(\cdot)_x = (\cdot)^{L_x^{-1}R_x}$. The equalities ${}_{I_x}(\cdot) = (\cdot)_x = (\cdot)^{L_x^{-1}R_x}$ supply ${}_x({}_{I_x}(\cdot)) = {}_x((\cdot)^{L_x^{-1}R_x})$ or $(\cdot) = {}_x(\cdot)^{L_x^{-1}R_x}$. If in the generalized Moufang loop $\mathcal{Q}(\cdot)$ one has $x^4 \in N$ for every $x \in \mathcal{Q}$, then $\mathcal{Q}(\cdot)$ is a *VD*-loop. In each generalized Moufang loop the equality

$$(\cdot)_{x^4} = (\cdot)_x^{R_x^{-1}L_x} \tag{8}$$

holds. If the generalized Moufang loop is a *VD*-loop then (2) implies

$$(\cdot)_x^{R_x^{-1}L_x} = (\cdot) \,. \tag{9}$$

(8) and (9) provide $(\cdot)_{x^4} = (\cdot)$, therefore $x^4 \in N$. $\square$

Theorem 3. *Each VD-loop is an Osborn loop.*

Proof. Let $\mathcal{Q}(\cdot)$ be a *VD*-loop, then the equalities (2) are true, that is $(\cdot)_x = (.)^{L_x^{-1}L_x}$. They imply the autotopies

$$S = (L_x R_x^{-1} L_x \, , R_x^{-1} L_x \, , L_x R_x^{-1} L_x)$$

and

$$S_1 = (L_x^{-1} R_x \, , R_x L_x^{-1} R_x \, , R_x L_x^{-1} R_x) \, .$$

By multiplying the autotopies S and S_1 we obtain

$$SS_1 = (L_x \, , R_x^{-1} L_x R_x L_x^{-1} R_x \, , L_x R_x^{-1} L_x R_x L_x^{-1} R_x) \, ,$$

which carries out the equality

$$L_x u \cdot R_x^{-1} L_x R_x L_x^{-1} R_x v = L_x R_x^{-1} L_x R_x L_x^{-1} R_x (u \cdot v) \, . \tag{10}$$

Let $v = 1$ in (10) (1 is the unit of the loop $\mathcal{Q}(\cdot)$), then

$$R_x L_x = L_x R_x^{-1} L_x R_x L_x^{-1} R_x \tag{11}$$

and

$$L_x^{-1} R_x L_x = R_x^{-1} L_x R_x L_x^{-1} R_x) \, . \tag{12}$$

Applying (11) and (12) to (10), we obtain

$$L_x u \cdot (R_x^{-1} L_x^{-1} R_x L_x v \cdot x) = R_x L_x (u \cdot v) \quad \text{or} \quad \mathrm{xu \cdot \Theta_x vx = (x \cdot uv) \cdot x} \, ,$$

that is $\mathcal{Q}(\cdot)$ is an Osborn loop. $\square$

Theorem 3 implies

Corollary 1. *The three nuclei of each VD-loop $\mathcal{Q}(\cdot)$ coincide, i.e. $N_r = N_m = N_l = N$. Moreover, N is a normal subloop in $\mathcal{Q}(\cdot)$.* $\square$

Proposition 1. *A K-loop $\mathcal{Q}(\cdot)$ is a VD-loop if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

Proof. The following equalities are consequences of the definition of K-loops:

$$(\cdot)_x = (\cdot)^{R_x^{-1}L_x} \quad \text{and} \quad {}_x(\cdot) = (\cdot)^{L_x^{-1}R_x}.$$

By means of the properties of the derived operations, one has

$$((\cdot)_x)_x = ((\cdot)^{R_x^{-1}L_x})_x \quad \text{and} \quad {}_x({}_x(\cdot)) = {}_x((\cdot)^{L_x^{-1}R_x})$$

or

$$(\cdot)_{x^2} = (\cdot)_x^{R_x^{-1}L_x} \quad \text{and} \quad {}_{x^2}(\cdot) = {}_x(\cdot)^{L_x^{-1}R_x} \tag{13}$$

If $x^2 \in N$ then

$$(\cdot)_{x^2} = (\cdot) \quad \text{and} \quad {}_{x^2}(\cdot) = (\cdot)\,. \tag{14}$$

By using (14) in (13), we obtain $(\cdot) = (\cdot)_x^{R_x^{-1}L_x}$ and $(\cdot) = {}_x(\cdot)L_x^{-1}R_x$, or $(\cdot)_x = (\cdot)^{L_x^{-1}R_x}$ and ${}_x(\cdot) = (\cdot)^{R_x^{-1}L_x}$ however, this means $\mathcal{Q}(\cdot)$ is a *VD*-loop. □

Proposition 2. *A VD-loop $\mathcal{Q}(\cdot)$ is a K-loop if $x^2 \in N$ for any $x \in \mathcal{Q}$.*

The proof is analogous to that of Proposition 1. □

References

[1] **V. D. Belousov**: *Foundations of the Theory of Quasigroups and Loops*, (Russian), "Nauka", Moscow, 1967.

[2] **A. S. Basarab**: *A class of WIP-loops*, (Russian), Mat. Issled. **2** (1967), 3 – 24.

[3] **A. S. Basarab**: *K-loops*, Buletinul AS RM, Matematica **1(7)** (1992), 28 – 33.

[4] **J. M. Osborn**: *Loops with weak inverse property,* Pacific J. Math. **10** (1960), 295 – 304.

Department of Physics and Mathematics
Tiraspol State University
5 Iablocikin str.
Kishinau, 277050
MOLDOVA

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