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Some applications of the independence to the semigroup of all binary systems

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Abstract. We extend the notions of *right (left) independency* and *absorbent* from groupoids to $Bin(X)$ as a semigroup of all the groupoids on a set X and study and investigate many of their properties. We show that these new concepts are different by presenting several examples. In general, the concept of right (left) independence is a generalization and alternative of classical concept of the converse of *injective function*.

1. Introduction

Bruck [2] published a book, *A survey of binary systems* discussed in the theory of groupoids, loops and quasigroups, and several algebraic structures. Borůvka [3] stated the theory of decompositions of sets and its application to binary systems. Nebeský [12] introduced the notion of a travel groupoid by adding two axioms to a groupoid, and he described an algebraic interpretation of the graph theory. Allen et al. [1] introduced the concept of several types of groupoids related to semigroups, viz., twisted semigroups for which twisted versions of the associative law hold. Kim et al. [7] showed that every selective groupoid induced by a fuzzy subset is a pogroupoid, and they discussed several properties in quasi ordered sets by introducing the notion of a framework. Liu et al. [11] extended the theory of groupoids already developed for semigroups $(Bin(X), \square)$ in a growing number of research papers with X a set and $Bin(X)$ the set of groupoids defined on X to the generalizations: fuzzy (sub)groupoids and hyperfuzzy (sub)groupoids. Hwang et al. [8] generalized the notion of an implicativity discussed in BCK-algebras, and applied it to some groupoids and BCK-algebras. Also, they discussed the notion of the locally finiteness and convolution products in groupoids [9]. Fayoumi introduced the notions of locally zero groupoids and the center of $Bin(X)$ of all binary systems on a set X [4]. Also, she introduced two methods of factorization for this binary system under the binary groupoid product in the semigroup $(Bin(X), \square)$ and showed that a strong non-idempotent groupoid can be represented as a product of its similar- and signature- derived factors. Moreover, she showed that a groupoid with the orientation property is a product of its orient- and skew-factors [5]. Feng et al. discussed on some relations among axioms in groupoids, and

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obtained some useful properties [6].

The motivation of this study came from the idea of the converse of "injective function". We applied this concepts to $Bin(X)$, and obtained several properties. Moreover, we discuss the right (left) absorbent subsets of $Bin(X)$. We provide several (counter-) examples to describe the concepts.

2. Preliminaries

A groupoid $(X, *)$ is said to be a *right zero semigroup* if $x * y = y$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$ for any $x, y \in X$. A groupoid $(X, *)$ is said to be a *right oid* for $f : X \to X$ if $x * y = f(y)$ for any $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a *leftoid* for $f : X \to X$ if $x * y = f(x)$ for any $x, y \in X$. Note that a right (left, resp.) zero semigroup is a special case of a right oid(leftoid, resp.) (see [10]). A groupoid $(X, *)$ is said to be *right cancellative* (or *left cancellative*, resp.) if $y * x = z * x$ ($x * y = x * z$, resp.) implies $y = z$. A groupoid $(X, *)$ is said to be *locally zero* [4] if

(i) $x * x = x$ for all $x \in X$,

(ii) for any $x \neq y \in X$, $(\{x, y\}, *)$ is either a left zero semigroup or a right zero semigroup.

Given a groupoid $(X, *)$ (i.e., $(X, *) \in Bin(X)$), a non-empty subset E of X is said to be *right independence* if $x \neq y \in E$, then $x * u \neq y * u$ for all $u \in X$. Also E is said to be *left independence* if $x \neq y \in E$, then $u * x \neq u * y$ for all $u \in X$. E is said to be *independence* if it both right and left independence [13].

The notion of the semigroup $(Bin(X), \Box)$ was introduced by Kim and Neggers [10]. Given binary operations "$*$" and "$\bullet$" on a set X, they defined a product binary operation "$\Box$" as follows: $x \Box y = (x * y) \bullet (y * x)$. This in turn yields a binary operation on $Bin(X)$, the set of all groupoids, defined on X turning $(Bin(X), \Box)$ into a semigroup with identity $(x * y = x)$, the left zero semigroup, and an analog of negative one in the right zero semigroup [10].

Example 2.1. Let $X := \{a, b\}$ be a set. Then we have 16 groupoids $(X, *_i)$ for $i \in \{1, \ldots, 16\}$ with the following tables.

$*_1$	a	b
a	a	a
b	a	a

$*_2$	a	b
a	b	a
b	a	a

$*_3$	a	b
a	a	b
b	b	a

$*_4$	a	b
a	b	b
b	a	a

$*_5$	a	b
a	a	a
b	b	a

$*_6$	a	b
a	b	a
b	b	a

$*_7$	a	b
a	a	b
b	a	b

$*_8$	a	b
a	b	b
b	b	a

$*_9$	a	b
a	a	a
b	b	b

$*_{10}$	a	b
a	b	a
b	a	b

$*_{11}$	a	b
a	a	b
b	b	b

$*_{12}$	a	b
a	b	b
b	a	b

$*_{13}$	a	b
a	a	a
b	a	b

$*_{14}$	a	b
a	b	a
b	b	b

$*_{15}$	a	b
a	a	b
b	a	a

$*_{16}$	a	b
a	b	b
b	b	b

It follows that $Bin(X) = \{(X, *_i)\}_{i \in \{1, \ldots, 16\}}$. We see that $(Bin(X), \Box)$, where $\Box$ is defined by $x \Box y = (x *_i y) *_j (y *_i x)$ for all $i, j \in \{1, \ldots, 16\}$, forms a semigroup.

For example, $(X,*_1)\Box(X,*_2)$ and $(X,*_2)\Box(X,*_1)$ are groupoids with the following tables:

$\Box$	a	b
a	b	b
b	b	b

$\Box$	a	b
a	a	a
b	a	a

It is seen that $(X,*_1)\Box(X,*_2)=(X,*_{16})\neq(X,*_2)\Box(X,*_1)=(X,*_1)$. Also, for example, in $(X,*_6)\Box(X,*_7)$, we have $a\Box b=(a *_6 b) *_7 (b *_6 a)=a *_7 b=b$, but $b\Box a=(b *_6 a) *_7 (a *_6 b)=b *_7 a=a$, and so $a\Box b\neq b\Box a$. Further, $(Bin(X),\Box)$ it is not a left cancellative semigroup, since $(X,*_2)\Box(X,*_3)=(X,*_2)\Box(X,*_5)=(X,*_1)$, but $(X,*_3)\neq(X,*_5)$. Also, it is not a right cancellative semigroup, since $(X,*_{13})\Box(X,*_{14})=(X,*_1)\Box(X,*_{14})=(X,*_{16})$, but $(X,*_{13})\neq(X,*_1)$.

3. right (left) independence in $Bin(X)$

Definition 3.1. A non-empty subset $\mathbb{A}\subseteq Bin(X)$ is said to be *right independence* if $(X,*)\neq(X,\bullet)$ in $\mathbb{A}$, then $(X,*)\Box(X,\diamond)\neq(X,\bullet)\Box(X,\diamond)$ for all $(X,\diamond)\in Bin(X)$. Also $\mathbb{A}$ is said to be *left independence* if $(X,*)\neq(X,\bullet)\in\mathbb{A}$, then $(X,\diamond)\Box(X,*)\neq(X,\diamond)\Box(X,\bullet)$ for all $(X,\diamond)\in Bin(X)$. $\mathbb{A}$ is said to be *independence* if it both right and left independence.

Example 3.2. (a). Let $(R,+,\cdot,0,1)$ be a commutative ring with identity 1, and let $L(R)$ denote the collection of all groupoids $(R,*)$ such that, for all $x,y\in R$,

$$x*y=ax+by+c,$$

where $a,b,c\in R$. Such a groupoid is said to be a *linear groupoid.* Notice that $a=1$, $b=c=0$ yields $x*y=1\cdot x=x$, and thus the left zero semigroup on R is a linear groupoid. Now, suppose that $(R,*)$ and $(R,\bullet)$ are linear groupoids where $x*y=ax+by+c$ and $x\bullet y=dx+ey+f$. Then

$$x\Box y=d(ax+by+c)+e(ay+bx+c)+f=(da+cb)x+(db+ca)y+(d+e)c+f,$$

whence $(R,\Box)=(R,*)\Box(R,\bullet)$ is also a linear groupoid (i.e., $(L(R),\Box)$ is a semigroup with identity (cf. [5])).

Let $L(A)$ denote the collection of all groupoids $(R,*)$ such that for all $x,y\in R$,

$$x*y=ax,$$

where $a\in R$. Now, suppose that $(R,*)\neq(R,\bullet)\in L(A)$ where $x*y=a_1x$ and $x\bullet y=a_2x$, for some $a_1\neq a_2\in R$. Let $(R,\diamond)\in L(R)$, where $x\diamond y:=ax+by+c$ for some $a,b,c\in R$ with $abc\neq 0$. Hence

$x\Box y=(x*y)\diamond(y*x)=a_1x\diamond a_1y=aa_1x+ba_1y+c$ in $(R,*)\Box(R,\diamond)$ and

$x\Box y=(x\bullet y)\diamond(y\bullet x)=a_2x\diamond a_2y=aa_2x+ba_2y+c$ in $(R,\bullet)\Box(R,\diamond)$.

Assume $(R,*)\Box(R,\diamond)=(R,\bullet)\Box(R,\diamond)$. Then $aa_1x+ba_1y+c=aa_2x+ba_2y+c$ and hence $a(a_1-a_2)x+b(a_1-a_2)y=0$. Since $a_1\neq a_2$, we obtain $a=b=0$, a contradiction. Thus, $(R,*)\Box(R,\diamond) \neq (R,\bullet)\Box(R,\diamond)$, and hence $L(A)$ is a right independence subset of $L(R)$. Moreover, $x\Box y = (x\diamond y)*(y\diamond x) = (ax+by+c)*(ay+bx+c) = a_1(ax+by+c) = a_1ax+a_1by+a_1c$ in $(R,\diamond)\Box(R,*)$ and $x\Box y = (x\diamond y)\bullet(y\diamond x) = (ax+by+c)\bullet(ay+bx+c) = a_2(ax+by+c) = a_2ax+a_2by+a_2c$ in $(R,\diamond)\Box(R,\bullet)$. It is easy to see that $a_1ax+a_1by+a_1c \neq a_2ax+a_2by+a_2c$. Thus, $(R,\diamond)\Box(R,*) \neq (R,\diamond)\Box(R,\bullet)$, and so $L(A)$ is a left independence subset of $L(R)$. Therefore $L(A)$ is an independence subset of $L(R)$.

(b). Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$, and let $L(\mathbb{R}^*)$ denote the collection of all groupoids on $\mathbb{R}^*$ (e.g., $(\mathbb{R}^*,\cdot)$, $(\mathbb{R}^*,+)$, $(\mathbb{R}^*,-)$, $(\mathbb{R}^*,\div)$ and $(\mathbb{R}^*,\bullet)$ where $\bullet : \mathbb{R}^*\times\mathbb{R}^* \longrightarrow \mathbb{R}^*$ is an arbitrary binary relation on $\mathbb{R}^*$. Take $A = \{(\mathbb{R}^*,+),(\mathbb{R}^*,\cdot)\}$. Then A is not a right independence subset of $L(\mathbb{R}^*)$. Since $(\mathbb{R}^*,+) \neq (\mathbb{R}^*,\cdot) \in A$ and $(\mathbb{R}^*,\div) \in L(\mathbb{R}^*)$, for all $x,y \in \mathbb{R}^*$, we get $x\Box y = (x+y)\div(y+x) = 1$ in $(\mathbb{R}^*,+)\Box(\mathbb{R}^*,\div)$ and $x\Box y = (x\cdot y)\div(y\cdot x) = 1$ in $(\mathbb{R}^*,\cdot)\Box(\mathbb{R}^*,\div)$. Thus, $(\mathbb{R}^*,+)\Box(\mathbb{R}^*,\div) = (\mathbb{R}^*,\cdot)\Box(\mathbb{R}^*,\div)$.

Note that the singleton set $\{(X,*)\} \subseteq Bin(X)$ is right (left) independence, since $\{(X,*)\}$ has no element $(X,\bullet) \in Bin(X)$ such that $(X,*) \neq (X,\bullet)$. Also, if $(Bin(X),\Box)$ is a group, then every subset of $Bin(X)$ is both right and left independence, and so it is an independence subset of $Bin(X)$. By routine calculation we can see that if $A_i \subseteq Bin(X)$ for $i \in \Lambda$ are right (left) independence, then $\bigcap_{i\in\Lambda} A_i$ and $\bigcup_{i\in\Lambda} A_i$ are right (left) independence. Note that if $\mathbb{B}$ and $\mathbb{D}$ are not right (left) independence subsets of $Bin(X)$, then $\mathbb{B}\cap\mathbb{D}$, $\mathbb{B}\cup\mathbb{D}$, $\mathbb{D}\setminus\mathbb{B}$ and $\mathbb{B}\triangle\mathbb{D}$ are not right (left) independence subsets of $Bin(X)$.

The following example shows that there exists a right (left) independence subset $\mathbb{A}$ of $Bin(X)$ such that $\mathbb{A}' = Bin(X)\setminus\mathbb{A}$ is not a right (left) independence subset of $Bin(X)$.

Example 3.3. Consider groupoid $(X,*_1)$ at Example 2.1. Then $\mathbb{A} = \{(X,*_1)\}$ is a right independence subset of $Bin(X)$ and

$$\mathbb{A}' = Bin(X)\setminus\{(X,*_1)\} = \{(X,*_i)\}_{i\in\{2,\ldots,16\}}.$$

The subset $\mathbb{A}'$ is not a right independence subset of $Bin(X)$, since $(X,*_{11}) \neq (X,*_{12}) \in \mathbb{A}'$, but $(X,*_{11})\Box(X,*_{16}) = (X,*_{12})\Box(X,*_{16})$. Moreover, it is not a left independence subset of $Bin(X)$, since $(X,*_{16})\Box(X,*_{11}) = (X,*_{16})\Box(X,*_{12}) = \{b\}$. Thus, $\mathbb{A}'$ is not an independence subset of $Bin(X)$.

Proposition 3.2. *Let $\mathbb{A},\mathbb{B} \subseteq Bin(X)$ and $\mathbb{A}$ be a right (left) independence subset of $Bin(X)$. Then $\mathbb{A}\cap\mathbb{B}$ a right (left) independence subset of $Bin(X)$.*

Proof. Assume $\mathbb{A}$ is a right (left) independence subset of $Bin(X)$ and $\mathbb{B}$ is an arbitrary subset of $Bin(X)$. Let $(X,*) \neq (X,\bullet)$ in $\mathbb{A}\cap\mathbb{B}$. Since $\mathbb{A}\cap\mathbb{B} \subseteq \mathbb{A}$, we get $(X,*) \neq (X,\bullet)$ in $\mathbb{A}$. Since $\mathbb{A}$ is a right (left) independence subset of $Bin(X)$, for all $(X,\diamond) \in Bin(X)$, we have $(X,*)\Box(X,\diamond) \neq (X,\bullet)\Box(X,\diamond)$, and hence $\mathbb{A}\cap\mathbb{B}$ is a right (left) independence subset of $Bin(X)$. $\Box$

Corollary 3.3. *Let* $\mathbb{A}, \mathbb{B} \subseteq Bin(X)$ *and* $\mathbb{A}$ *be a right* (*left*) *independence subset of* $Bin(X)$*. Then* $\mathbb{A} \setminus \mathbb{B}$ *a right* (*left*) *independence subset of* $Bin(X)$*.*

Proof. Since $\mathbb{A} \setminus \mathbb{B} = \mathbb{A} \cap \mathbb{B}'$, using Proposition 3.2, we obtain that $\mathbb{A} \setminus \mathbb{B}$ is a right (left) independence subset of $Bin(X)$. $\square$

Corollary 3.4. *Let* $\mathbb{A}, \mathbb{B} \subseteq Bin(X)$ *and* $\mathbb{A}$ *be a right* (*left*) *independence subset of* $Bin(X)$*. If* $\mathbb{B} \subseteq \mathbb{A}$*, then* $\mathbb{B}$ *is a right* (*left*) *independence subset of* $Bin(X)$*.*

Corollary 3.5. *Let* $Bin(X)$ *be right* (*left*) *independence and let* $\mathbb{A} \subseteq Bin(X)$*. Then* $\mathbb{A}$ *is a right* (*left*) *independence subset of* $Bin(X)$*.*

Proof. It follows immediately from Corollary 3.4. $\square$

The following example shows that there exists a right (left) independence subset $\mathbb{A}$ of $Bin(X)$ such that $\mathbb{A} \cup \mathbb{B}$ is not a right (left) independence subset of $Bin(X)$ for some $\mathbb{B} \subseteq Bin(X)$.

Example 3.4. Consider Example 3.3, and take $\mathbb{B} := \mathbb{A}'$, the complement of $\mathbb{A}$ in $Bin(X)$. Then $\mathbb{B}$ is not an independence subset of $Bin(X)$. Then $\mathbb{A} \cup \mathbb{B} = \mathbb{A} \cup \mathbb{A}' = Bin(X)$, which is not a right (left) independence subset of $Bin(X)$, since $(X, *_{11}) \neq (X, *_{12}) \in Bin(X)$, but $(X, *_{11}) \square (X, *_{16}) = (X, *_{12}) \square (X, *_{16})$. Moreover, it is not a left independence subset of $Bin(X)$, since $(X, *_{16}) \square (X, *_{11}) = (X, *_{16}) \square (X, *_{12}) = \{b\}$. Thus, $Bin(X)$ itself is not an independence subset of $Bin(X)$. Also, $\mathbb{A} \triangle \mathbb{B} = \mathbb{A} \triangle \mathbb{A}' = (\mathbb{A} \cup \mathbb{A}') \setminus (\mathbb{A} \cap \mathbb{A}') = Bin(X) \setminus \emptyset = Bin(X)$, which is not a right (left) independence subset of $Bin(X)$.

Theorem 3.6. *Let* $Bin(X) := \mathbb{A} \cup \mathbb{B}$*, where* $\mathbb{B} \subseteq Bin(X)$ *is a non-trivial group and* A *be a right* (*left*) *independence subset of* $Bin(X)$*. Then* $Bin(X)$ *is independence.*

Proof. Assume $\mathbb{B}$ is a non-trivial group and $\mathbb{A}$ is a right independence subset of $Bin(X)$ satisfying $Bin(X) = \mathbb{A} \cup \mathbb{B}$. Let $(X, *) \neq (X, \bullet)$ in $Bin(X)$.

Case 1. if $(X, *) \neq (X, \bullet)$ in $Bin(X) \cap \mathbb{A}$, since $\mathbb{A}$ is a right independence subset of $Bin(X)$, we get $(X, *) \square (X, \diamond) \neq (X, \bullet) \square (X, \diamond)$ for all $(X, \diamond) \in Bin(X)$.

Case 2. if $(X, *) \neq (X, \bullet)$ in $Bin(X) \cap \mathbb{B}$. We claim that

$$(X, *) \square (X, \diamond) \neq (X, \bullet) \square (X, \diamond) \text{ for all } (X, \diamond) \in Bin(X).$$

Assume $(X, *) \square (X, \diamond) = (X, \bullet) \square (X, \diamond)$ for some $(X, \diamond) \in Bin(X)$. Since $\mathbb{B}$ is a non-trivial group, we have $|\mathbb{B}| \geqslant 2$. Hence there is at least one element $(X, \circ) \in \mathbb{B}$, and so there is $(X, \circ)^{-1} \in \mathbb{B}$ as an inverse of $(X, \circ)$ (i.e., $(X, \circ) \square (X, \circ)^{-1} = (X, \star)$ and $(X, \star)$ is the left zero semigroup). Thus,

$$((X, *) \square (X, \circ)) \square (X, \circ)^{-1} = (X, *) \square ((X, \circ) \square (X, \circ)^{-1}) = (X, *) \square (X, \star) = (X, *)$$

and

$$((X, \bullet) \square (X, \circ)) \square (X, \circ)^{-1} = (X, \bullet) \square ((X, \circ) \square (X, \circ)^{-1}) = (X, \bullet) \square (X, \star) = (X, \bullet).$$

Therefore, $(X,*) = (X,\bullet)$, which is a contradiction.

Case 3. Let $(X,*) \in \mathbb{A}$ and $(X,\bullet) \in \mathbb{B}$ such that $(X,*) \neq (X,\bullet)$. We claim that $(X,*)\Box(X,\diamond) \neq (X,\bullet)\Box(X,\diamond)$ for all $(X,\diamond) \in Bin(X)$.

Assume $(X,*)\Box(X,\diamond) = (X,\bullet)\Box(X,\diamond)$ for some $(X,\diamond) \in Bin(X)$. Since $(X,\bullet) \in \mathbb{B}$ and $\mathbb{B}$ is a non-trivial group, there is $(X,\bullet)^{-1} \in \mathbb{B}$ as an inverse of $(X,\bullet)$ (i.e., $(X,\bullet)\Box(X,\bullet)^{-1} = (X,\star)$ and $(X,\star)$ is the left zero semigroup). Thus,

$$\begin{aligned}((X,*)\Box(X,\bullet)^{-1})\Box(X,\bullet) &= ((X,\bullet)\Box(X,\bullet)^{-1})\Box(X,\bullet)\\ &= (X,\star)\Box(X,\bullet) = (X,\bullet) \in \mathbb{B}.\end{aligned}$$

Since $(X,\star)$ is a left zero semigroup, we get $(X,*)\Box(X,\bullet)^{-1} = (X,\star)$, and so $(X,*) = (X,\bullet)$, which is a contraction.

Similarly, we prove the theorem for the case of a left independence subset in $Bin(X)$. $\Box$

Corollary 3.7. *If $Bin(X) = \bigcup_{i\in\Lambda} \mathbb{A}_i$ is a right (left) independence, $\mathbb{A}_i \neq \emptyset$ for all $i \in \Lambda$, and $\mathbb{A}_j$ is a non-trivial group for some $j \in \Lambda$. Then every $\mathbb{A}_i$ $(i \neq j \in \Lambda)$ is a right (left) independence subset of $Bin(X)$.*

Proposition 3.8. *Let $(\mathbb{A},\Box_1)$ and $(\mathbb{B},\Box_2)$ be right (left, respectively) independence subsets of $(Bin(X),\Box_1)$ and $(Bin(Y),\Box_2)$ respectively. Then $\mathbb{A}\times\mathbb{B}$ is a right (left, respectively) independence subset of $(Bin(X)\times Bin(Y),\Box)$, where $\Box$ is defined by $(x,u)\Box(y,v) := (x\Box_1 y, u\Box_2 v)$.*

Proof. Assume $(\mathbb{A},\Box_1)$ and $(\mathbb{B},\Box_2)$ are right independence subsets of $Bin(X)$ and $Bin(Y)$ respectively. Let $(X,*_1)\times(Y,\circ_1) \neq (X,*_2)\times(Y,\circ_2)$, where $(X,*_i) \in \mathbb{A}$ and $(Y,\circ_i) \in \mathbb{B}$ for $i \in \{1,2\}$. Then either $(X,*_1) \neq (X,*_2)$ or $(Y,\circ_1) \neq (Y,\circ_2)$. Since $\mathbb{A}$ and $\mathbb{B}$ are right independence subsets of $Bin(X)$ and $Bin(Y)$ respectively, we obtain either $(X,*_1)\Box_1(X,\bullet) \neq (X,*_2)\Box_1(X,\bullet)$ or $(Y,\circ_1)\Box_2(Y,\diamond) \neq (Y,\circ_2)\Box_2(Y,\diamond)$ for all $(X,\bullet) \in Bin(X)$ and $(Y,\diamond) \in Bin(Y)$. It follows that

$$((X,*_1)\times(Y,\circ_1))\Box((X,\bullet)\times(Y,\diamond)) \neq ((X,*_2)\times(Y,\circ_2))\Box((X,\bullet)\times(Y,\diamond))$$

for all $(X,\bullet)\times(Y,\diamond) \in \mathbb{A}\times\mathbb{B}$. Therefore, $\mathbb{A}\times\mathbb{B}$ is a right independence subset of $Bin(X)\times Bin(Y)$. Similarly, we can prove the case of the left independence, and we omit it. $\Box$

Let $\emptyset \neq \mathbb{A} \subseteq Bin(X)$, and let $(X,*) \in Bin(X)$. Define two sets $(X,*)\Box\mathbb{A}$ and $\mathbb{A}\Box(X,*)$ as follows:

$$(X,*)\Box\mathbb{A} = \{(X,*)\Box(X,\circ) : (X,\circ) \in \mathbb{A}\}$$

and

$$\mathbb{A}\Box(X,*) = \{(X,\circ)\Box(X,*) : (X,\circ) \in \mathbb{A}\}.$$

Note that if $\mathbb{A} = \{(X,\diamond)\}$ (i.e., $|\mathbb{A}| = 1$), then $\{(X,*)\Box(X,\diamond)\}$ and $\{(X,\diamond)\Box(X,*)\}$ are also singleton sets, and so these are independence subsets of $Bin(X)$.

Proposition 3.9. *Let $Bin(X)$ be a right (left) zero semigroup, and $(X,*) \in Bin(X)$. Then $A\Box(X,*)$ (resp., $(X,*)\Box A$) is an independence subset of $Bin(X)$.*

Proof. Assume $Bin(X)$ is a right (left) zero semigroup. Then $A\Box(X,*) = \{(X,*)\}$ (resp., $(X,*)\Box A = \{(X,*)\}$). Thus, the proof is complete. $\Box$

Proposition 3.10. *If $Bin(X)$ is a right (left) zero semigroup, $A \subseteq Bin(X)$ is a right (left) independence subset, and $(X,*) \in Bin(X)$, then $(X,*)\Box A$ (resp., $A\Box(X,*)$) is a right (left) independence subset of $Bin(X)$.*

Proof. Assume $Bin(X)$ is a right (left) zero semigroup, $A \subseteq Bin(X)$ is a right (left) independence and $(X,*) \in Bin(X)$. Then $A\Box(X,*) \subseteq A$ (resp., $(X,*)\Box A \subseteq A$). Using Proposition 3.2, we get $(X,*)\Box A \subseteq A$ (resp., $A\Box(X,*) \subseteq A$) is a right (left) independence subset of $Bin(X)$. $\Box$

Proposition 3.11. *If $Bin(X)$ is a right cancellative, and $A \subseteq Bin(X)$ (right (left) independence or not), where $|A| > 1$ and $(X,*) \in Bin(X)$, then $(X,*)\Box A$ and $(X,*)\Box A$ are independence subsets of $Bin(X)$.*

Proof. Assume $(X,*)\Box(X,*_1) \neq (X,*)\Box(X,*_2) \in (X,*)\Box A$ for some $(X,*_i) \in A$ for $i \in \{1,2\}$, and let $(X,\diamond) \in Bin(X)$.

On the contrary, if $((X,*)\Box(X,*_1))\Box(X,\diamond) = ((X,*)\Box(X,*_2))\Box(X,\diamond)$ for some $(X,\diamond) \in Bin(X)$, then using cancellative laws we get $(X,*)\Box(X,*_1) = (X,*)\Box(X,*_2)$, which is a contradiction. Thus, $(X,*)\Box A$ is an independence subset of $Bin(X)$.

Similarly, if $Bin(X)$ is a left cancellative, then $(X,*)\Box A$ is an independence subset of $Bin(X)$.

By a similar argument for the set $A\Box(X,*)$ the result is valid. $\Box$

Let $\mathbb{E} \subseteq Bin(X)$, and $(X,*) \in Bin(X)$. Define

$$(X,*)\mathbb{E} := \{(X,\bullet) \in \mathbb{E} : (X,*)\Box(X,\bullet) = (X,\bullet)\},$$

$$\mathbb{E}(X,*) := \{(X,\bullet) \in \mathbb{E} : (X,\bullet)\Box(X,*) = (X,\bullet)\}$$

and

$$(X,*)\mathbb{E}(X,*) := \{(X,\bullet) \in \mathbb{E} : (X,*)\Box(X,\bullet) = (X,\bullet)\Box(X,*) = (X,\bullet)\}.$$

(a) If $\mathbb{E} = \emptyset$, then $(X,*)\mathbb{E} = \mathbb{E}(X,*) = (X,*)\mathbb{E}(X,*) = \emptyset$, for all $(X,*) \in Bin(X)$.

(b) For all $(X,*) \in Bin(X)$, $(X,*)\mathbb{E}$, $\mathbb{E}(X,*)$ and $(X,*)\mathbb{E}(X,*)$ are subsets of $Bin(X)$ and we have:

(i) $(X,*)\mathbb{E} \cap \mathbb{F} = (X,*)\mathbb{E} \cap (X,*)\mathbb{F}$,
$\mathbb{E} \cap \mathbb{F}(X,*) = \mathbb{E}(X,*) \cap \mathbb{F}(X,*)$,
$(X,*)\mathbb{E} \cap \mathbb{F}(X,*) = (X,*)\mathbb{E}(X,*) \cap (X,*)\mathbb{F}(X,*)$.

(ii) $(X,*)\mathbb{E}\cup\mathbb{F}\subseteq(X,*)\mathbb{E}\cap(X,*)\mathbb{F}$,
$\mathbb{E}\cap\mathbb{F}(X,\cup)\subseteq\mathbb{E}(X,*)\cup\mathbb{F}(X,*)$,
$(X,*)\mathbb{E}\cup\mathbb{F}(X,*)\subseteq(X,*)\mathbb{E}(X,*)\cup(X,*)\mathbb{F}(X,*)$.

(iii) $(X,*)\mathbb{E}\cap Bin(X)=(X,*)\mathbb{E}$.

(iv) $(X,*)\mathbb{E}\cup Bin(X)=(X,*)Bin(X)$.

(v) if $\mathbb{E}\subseteq\mathbb{F}$, then $(X,*)\mathbb{E}\subseteq(X,*)\mathbb{F}$, $\mathbb{E}(X,*)\subseteq\mathbb{F}(X,*)$, and so $(X,*)\mathbb{E}(X,*)\subseteq(X,*)\mathbb{F}(X,*)$.

(vi) $(X,*)\mathbb{E}(X,*)=(X,*)\mathbb{E}\cap\mathbb{E}(X,*)$,

(vii) $(X,*)(\mathbb{E}\setminus\mathbb{F})=(X,*)\mathbb{E}\setminus(X,*)\mathbb{F}$,
$(\mathbb{E}\setminus\mathbb{F})(X,*)=\mathbb{E}(X,*)\setminus\mathbb{F}(X,*)$,
$(X,*)(\mathbb{E}\setminus\mathbb{F})(X,*)=(X,*)\mathbb{E}(X,*)\setminus(X,*)\mathbb{F}(X,*)$.

(viii) If $\mathbb{E}$ is a group in $Bin(X)$, then for all $(X,\bullet)\in(X,*)\mathbb{E}$ (resp., $(X,\bullet)\in\mathbb{E}(X,*)$ or $(X,\bullet)\in(X,*)\mathbb{E}(X,*)$) we have $(X,*)=(X,\star)$, as a zero element.

(ix) If $(X,*)\in(X,*)\mathbb{E}$ (resp., $(X,*)\in\mathbb{E}(X,*)$ or $(X,*)\in(X,*)\mathbb{E}(X,*)$), then $(X,*)\Box(X,*)=(X,*)$, and so $(X,*)$ is an idempotent element in $Bin(X)$,

(x) If $Bin(X)$ is commutative, then $(X,*)\mathbb{E}=\mathbb{E}(X,*)=(X,*)\mathbb{E}(X,*)$,

(c) If $(X,*)\mathbb{E}\neq\emptyset$, then it is a closed subset. Let $(X,\bullet)$ and $(X,\diamond)$ be elements in $(X,*)\mathbb{E}$, we get $(X,*)\Box(X,\bullet)=(X,\bullet)$ and $(X,*)\Box(X,\diamond)=(X,\diamond)$. Hence

$$(X,*)\Box((X,\bullet)\Box(X,\diamond))=((X,*)\Box(X,\bullet))\Box(X,\diamond)=(X,\bullet)\Box(X,\diamond).$$

Thus, $(X,\bullet)\Box(X,\diamond)\in(X,*)\mathbb{E}$, and so $(X,*)\mathbb{E}$ is a subsemigroup of $Bin(X)$.

If $\mathbb{E}(X,*)\neq\emptyset$, then it is a closed subset. Let $(X,\bullet)$ and $(X,\diamond)$ be elements in $\mathbb{E}(X,*)$. So $(X,\bullet)\Box(X,*)=(X,\bullet)$ and $(X,\diamond)\Box(X,*)=(X,\diamond)$. Hence

$$((X,\bullet)\Box(X,\diamond))\Box(X,*)=(X,\bullet)\Box((X,\diamond)\Box(X,*))=(X,\bullet)\Box(X,\diamond).$$

Thus, $(X,\bullet)\Box(X,\diamond)\in\mathbb{E}(X,*)$, and so $\mathbb{E}(X,*)$ is a subsemigroup of $Bin(X)$.

Similarly, $(X,*)\mathbb{E}(X,*)$ is a closed set.

(d) If $Bin(X)$ is a monoid or group and $(X,\star)$ is a unique right (left) zero semigroup, then $(X,\star)Bin(X)=Bin(X)(X,\star)=(X,\star)Bin(X)(X,\star)=Bin(X)$, and so the cancellation law is valid.

(e) Let $\mathbb{E}$ be the set of all right zero semigroups. Then $(X,*)Bin(X)=Bin(X)$ for all $(X,*)\in\mathbb{E}$, and so the left cancellation law is valid in $\mathbb{E}$.

(f) Let $\mathbb{E}$ be the set of all left zero semigroups. Then $Bin(X)(X,*)=Bin(X)$ for all $(X,*)\in\mathbb{E}$, and so the right cancellation law is valid in $\mathbb{E}$.

(g) If for all $(X,*) \in \mathbb{E}$ the set $(X,*)\mathbb{E}(X,*) = \{(X,\bullet)\}$ for some $(X,\bullet) \in Bin(X)$ (i.e., $(X,*)\mathbb{E}(X,*)$ is a singleton set), then $\mathbb{E}$ is a group in semigroup $Bin(X)$.

(h) If there exists $(X,*) \in Bin(X)$ such that $(X,*)\mathbb{E} \cap \mathbb{E}(X,*) = \emptyset$, then $\mathbb{E}$ is not a group.

(i) If there exists $(X,*) \in Bin(X)$ such that $((X,*)\mathbb{E})' = \mathbb{E}(X,*)$, then $Bin(X) = (X,*)\mathbb{E} \cup \mathbb{E}(X,*)$ and $\mathbb{E}$ is not a group.

(j) If $(X,*) \in Bin(X)$ is an idempotent element (i.e., $(X,*)\Box(X,*) = (X,*)$), then $(X,*) \in (X,*)Bin(X)(X,*)$.

(k) Let $(X,*) \in Bin(X)$. If there exists $\emptyset \neq \mathbb{E} \subseteq Bin(X)$, where $(X,*) \in (X,*)\mathbb{E} \cup \mathbb{E}(X,*)$, then $(X,*)$ is an idempotent element.

Theorem 3.12. *Let $\emptyset \neq \mathbb{E} \subseteq Bin(X)$. Then*

(*a*) *if $\mathbb{F} = \bigcap\limits_{(X,*)\in Bin(X)} \mathbb{E}(X,*) \neq \emptyset$, then $\mathbb{F}$ is a right independence subset of $Bin(X)$,*

(*b*) *if $\mathbb{F} = \bigcap\limits_{(X,*)\in Bin(X)} \mathbb{E} \neq \emptyset$, then $\mathbb{F}$ is a left independence subset of $Bin(X)$,*

(*c*) *if $\mathbb{F} = \bigcap\limits_{(X,*)\in Bin(X)} (X,*)\mathbb{E}(X,*) \neq \emptyset$, then $\mathbb{F}$ is an independence subset of $Bin(X)$.*

Proof. (a). Assume $\emptyset \neq \mathbb{E} \subseteq Bin(X)$, $\mathbb{F} = \bigcap\limits_{(X,*)\in Bin(X)} \mathbb{E}(X,*)$ and $(X,\bullet) \neq (X,\circ) \in \mathbb{F}$. Hence $(X,\bullet) \in \bigcap\limits_{(X,*)\in Bin(X)} \mathbb{E}(X,*)$, and so we get $(X,\bullet)\Box(X,*) = (X,\bullet)$.

On the other hand, from $(X,\circ) \in \bigcap\limits_{(X,*)\in Bin(X)} \mathbb{E}(X,*)$, we have $(X,\circ)\Box(X,*) = (X,\circ)$. Thus, $(X,\bullet)\Box(X,*) = (X,\bullet) \neq (X,\circ) = (X,\circ)\Box(X,*)$. Therefore, $\mathbb{F}$ is a right independence subset of $Bin(X)$.

(b). Assume $\emptyset \neq \mathbb{E} \subseteq Bin(X)$, $\mathbb{F} = \bigcap\limits_{(X,*)\in Bin(X)} (X,*)\mathbb{E}$ and $(X,\bullet) \neq (X,\circ) \in \mathbb{F}$. Hence $(X,\bullet) \in \bigcap\limits_{(X,*)\in Bin(X)} (X,*)\mathbb{E}$, and so we get $(X,*)\Box(X,\bullet) = (X,\bullet)$.

On the other hand, from $(X,\circ) \in \bigcap\limits_{(X,*)\in Bin(X)} (X,*)\mathbb{E}$, we have $(X,*)\Box(X,\circ) = (X,\circ)$. Thus, $(X,*)\Box(X,\bullet) = (X,\bullet) \neq (X,\circ) = (X,*)\Box(X,\circ)$. Therefore, $\mathbb{F}$ is a left independence subset of $Bin(X)$.

(c). It follows immediately from (a) and (b). $\Box$

Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two arbitrary subsets of $Bin(X)$. Define $\mathbb{A}\Box\mathbb{B}$ as follows:

$$\begin{aligned}\mathbb{A}\Box\mathbb{B} &= \{(X,*)\Box(X,\circ) : (X,*) \in \mathbb{A} \text{ and } (X,\circ) \in \mathbb{A}\} \\ &= \bigcup_{(X,*)\in\mathbb{A}} ((X,*)\Box\mathbb{B}) = \bigcup_{(X,\circ)\in\mathbb{B}} (\mathbb{A}\Box(X,\circ)).\end{aligned}$$

Note that $\emptyset\Box\mathbb{A} = \mathbb{A}\Box\emptyset = \emptyset\Box\emptyset = \emptyset$, $Bin(X)\Box Bin(X) = Bin(X)$, $\mathbb{A}\Box\mathbb{A} \neq \mathbb{A}$ and $\mathbb{A}\Box\mathbb{B} \neq \mathbb{B}\Box\mathbb{A}$.

Also, let $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ be subsets of $Bin(X)$. Then one can see that:

- if $\mathbb{A} \subseteq \mathbb{B}$, then $\mathbb{A}\Box\mathbb{C} \subseteq \mathbb{B}\Box\mathbb{C}$ and $C\Box\mathbb{A} \subseteq \mathbb{C}\Box\mathbb{B}$,
- $(\mathbb{A} \cap \mathbb{B})\Box\mathbb{C} \subseteq (\mathbb{A}\Box\mathbb{C}) \cap (\mathbb{B}\Box\mathbb{C})$,
- $\mathbb{C}\Box(\mathbb{A} \cap \mathbb{B}) \subseteq (\mathbb{C}\Box\mathbb{A}) \cap (\mathbb{C}\Box\mathbb{B})$,
- $(\mathbb{A} \cup \mathbb{B})\Box\mathbb{C} = (\mathbb{A}\Box\mathbb{C}) \cup (\mathbb{B}\Box\mathbb{C})$,
- $\mathbb{C}\Box(\mathbb{A} \cup \mathbb{B}) = (\mathbb{C}\Box\mathbb{A}) \cup (\mathbb{C}\Box\mathbb{B})$.

Corollary 3.13.
(a) *If $Bin(X)$ is a right (left) zero semigroup and either $\mathbb{A}$ or $\mathbb{B}$ is a right (left) independence subset of $Bin(X)$, then $\mathbb{A}\Box\mathbb{B}$ is also a right (left) independence subset of $Bin(X)$.*
(b) *If $|\mathbb{A}| = 1$ or $|\mathbb{B}| = 1$, then $\mathbb{A}\Box\mathbb{B}$ is a right (left) independence subset of $Bin(X)$.*
(c) *If $Bin(X)$ is a right (left) cancellative semigroup, then $\mathbb{A}\Box\mathbb{B}$ is an independence subset of $Bin(X)$.*

Consider Example 2.1, and put $\mathbb{A} := \{(X,*_1),(X,*_2)\}$. Then $Bin(X)\,\Box\mathbb{A} \neq \mathbb{A}$, since $(X,*_3)\Box(X,*_2) = (X,*_{10}) \notin \mathbb{A}$. Also, $\mathbb{A}\Box Bin(X) \neq \mathbb{A}$, since $(X,*_2)\Box(X,*_5) = (X,*_5) \notin \mathbb{A}$. If take $\mathbb{B} := \{(X,*_{16})\}$, then $Bin(X)\Box\mathbb{B} = \mathbb{B} \neq Bin(X)$. Also, $\mathbb{B}\Box Bin(X) = \{(X,*_1),(X,*_{16})\} \neq \{(X,*_{16})\}$ and $\mathbb{B}\Box Bin(X) \neq Bin(X)\Box\mathbb{B}$.

Now, we can rewrote the definitions of right (left) zero semigruops as the follows:
A semigroup $(Bin(X),\Box)$ is said to be a *right zero semigroup* if

$$Bin(X)\Box(X,*) = \{(X,*)\}$$

and a groupoid $(Bin(X),\Box)$ is said to be a *left zero semigroup* if

$$(X,*)\Box Bin(X) = \{(X,*)\}$$

for any $(X,*) \in Bin(X)$.

4. right (left) absorbent in $Bin(X)$

Definition 4.1. A non-empty subset $\mathbb{A}$ of $Bin(X)$ is said to be *right absorbent* (resp., *left absorbent*) if $Bin(X)\Box\mathbb{A} = \mathbb{A}$ (resp., $\mathbb{A}\Box Bin(X) = \mathbb{A}$). It is *absorbent* if it is both right and left absorbent (i.e., $Bin(X)\Box\mathbb{A} = \mathbb{A}\Box Bin(X) = \mathbb{A}$).

Example 4.5. Consider Example 2.1.
(a) If $\mathbb{C} := \{(X, *_1)\}$, then $Bin(X)\Box\mathbb{C} = \mathbb{C}$, and so $\mathbb{C}$ is a right absorbent of $Bin(X)$, but not a left absorbent, since

$$\mathbb{C}\Box Bin(X) = \{(X, *_1), (X, *_{16})\} \neq \mathbb{C} \neq Bin(X).$$

(b) If $\mathbb{B} := \{(X, *_3\}$, then $\mathbb{B}\Box Bin(X) = \mathbb{B}$, and so $\mathbb{B}$ is a left absorbent of $Bin(X)$, but not a right absorbent, since

$$(X, *_7) = (X, *_6)\Box(X, *_3) \in Bin(X)\Box\mathbb{B}, \text{ but } (X, *_7) \notin \{(X, *_3)\}.$$

(c) If $\mathbb{D} := \{(X, *_1), (X, *_{16})\}$, then $Bin(X)\Box\mathbb{D} = \mathbb{D}$ and $\mathbb{D}\Box Bin(X) = \mathbb{D}$. Thus, $\mathbb{D}$ is an absorbent subset of $Bin(X)$.

Proposition 4.2. *If $Bin(X)$ is a right (left) zero semigroup, then every subset of $Bin(X)$ is a right (left) absorbent subset of $Bin(X)$.*

Proof. Straightforward. $\Box$

The converse of Proposition 4.2, may not be true in general. For this, consider Example 2.1, and take $\mathbb{A} := \{(X, *_1)\}$, so $\mathbb{A}$ is a right absorbent subset, but $Bin(X)$ is neither a right zero semigroup nor a left zero semigroup, since $(X, *_2)\Box(X, *_{14}) = (X, *_{16}) \notin \{(X, *_2), (X, *_{14})\}$.

Proposition 4.3. *Let $\mathbb{A}$ be a right (left) absorbent subset of $Bin(X)$. Then $\mathbb{A}$ is closed under $\Box$ (i.e., $\mathbb{A}$ is a subsemigroup of $Bin(X)$).*

Proof. Assume $\mathbb{A}$ is a right absorbent subset of $Bin(X)$ and $(X, *)$, $(X, \circ) \in \mathbb{A}$. Then $(X, *)\Box(X, \circ) \in \mathbb{A}\Box\mathbb{A} \subseteq Bin(X)\Box\mathbb{A} = \mathbb{A}$. Thus, $(X, *)\Box(X, \circ) \in \mathbb{A}$. Now, suppose that $\mathbb{A}$ is a left absorbent subset of $Bin(X)$, and let $(X, *), (X, \circ) \in \mathbb{A}$. Then $(X, *)\Box(X, \circ) \in \mathbb{A}\Box\mathbb{A} \subseteq \mathbb{A}\ \Box Bin(X) = \mathbb{A}$. Thus, $(X, *)\Box(X, \circ) \in \mathbb{A}$. $\Box$

Proposition 4.4. *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two right (left) absorbent subsets of $Bin(X)$. Then $\mathbb{A}_1 \cup \mathbb{A}_2$ is also a right (left) absorbent subset of $Bin(X)$.*

Proof. Assume $\mathbb{A}_1$ and $\mathbb{A}_2$ are two right absorbent subsets of $Bin(X)$. Then $Bin(X)\Box\mathbb{A} = \mathbb{A}$ and $Bin(X)\Box\mathbb{B} = \mathbb{B}$. It follows that

$$Bin(X)\Box(\mathbb{A} \cup \mathbb{B}) = (Bin(X)\Box\mathbb{A}) \cup (Bin(X)\Box\mathbb{B}) = \mathbb{A} \cup \mathbb{B}.$$

Similarly, the assertion holds for the left absorbent subsets. $\Box$

Corollary 4.5. *Let* $\{\mathbb{A}_i\}_{i\in\Lambda}$ *be a family of right (left) absorbent subsets of* $Bin(X)$. *Then* $\bigcup_{i\in\Lambda}\mathbb{A}_i$ *is a right (left) absorbent subset of* $Bin(X)$.

Let $\mathbb{A}\subseteq Bin(X)$. Define $\mathbb{A}_{(X,*)}$ and ${}_{(X,*)}\mathbb{A}$ as follows:

$$\mathbb{A}_{(X,*)}=\{(X,\bullet)\in Bin(X):(X,*)\square(X,\bullet)\in\mathbb{A}\},$$

$${}_{(X,*)}\mathbb{A}=\{(X,\bullet)\in Bin(X):(X,\bullet)\square(X,*)\in\mathbb{A}\}.$$

Also, we can define:

$${}_{(X,*)}\mathbb{A}_{(X,*)}=\{(X,\bullet)\in Bin(X):(X,\bullet)\square(X,*)\text{ and }(X,*)\square(X,\bullet)\in\mathbb{A}\}.$$

Proposition 4.6. *Let* $\mathbb{A}$ *be a right independence subset of a left cancellative semigroup* $Bin(X)$. *If* $\mathbb{A}_{(X,*)}\neq\emptyset$ *for some* $(X,*)\in Bin(X)$, *then* $\mathbb{A}_{(X,*)}$ *is a right independence subset of* $Bin(X)$.

Proof. Assume $\mathbb{A}$ is a right independence subset of the left cancellative semigroup $Bin(X)$. If $(X,\bullet_1)\neq(X,\bullet_2)$ in $\mathbb{A}_{(X,*)}$, then $(X,*)\square(X,\bullet_1)\in\mathbb{A}$ and $(X,*)\square(X,\bullet_2)\in\mathbb{A}$. We claim $(X,*)\square(X,\bullet_1)\neq(X,*)\square(X,\bullet_2)$. If we assume $(X,*)\square(X,\bullet_1)=(X,*)\square(X,\bullet_2)$, since $Bin(X)$ is left cancellative, we obtain $(X,\bullet_1)=(X,\bullet_2)$, a contradiction. Now, since $\mathbb{A}$ is right independence, we have $[(X,*)\square(X,\bullet_1)]\square(X,\diamond)\neq[(X,*)\ \square(X,\bullet_2)]\square(X,\diamond)$ for all $(X,\diamond)\in Bin(X)$. Since $Bin(X)$ is left cancellative, by the associativity, we obtain $(X,*)\square[(X,\bullet_1)\square(X,\diamond)]$ $\neq(X,*)\square[(X,\bullet_2)\square(X,\diamond)]$, and so $(X,\bullet_1)\square(X,\diamond)\neq(X,\bullet_2)\square(X,\diamond)$ for all $(X,\diamond)\in Bin(X)$. Thus, $\mathbb{A}_{(X,*)}$ is a right independence subset of $Bin(X)$. □

Proposition 4.7. *Let* $\mathbb{A}$ *be a left independence subset of a right cancellative semigroup* $Bin(X)$. *Then* ${}_{(X,*)}\mathbb{A}$ *is a left independence subset of* $Bin(X)$ *for any* $(X,*)\in Bin(X)$.

Proof. Assume $\mathbb{A}$ is a left independence subset of the right cancellative semigroup $Bin(X)$. Let $(X,\bullet_1)\neq(X,\bullet_2)$ in $\mathbb{A}$. Then $(X,\bullet_1)\square(X,*)\in\mathbb{A}$ and $(X,\bullet_2)\square(X,*)\in\mathbb{A}$. Since $Bin(X)$ is right cancellative, we obtain $(X,\bullet_1)\square(X,*)\neq(X,\bullet_2)\square(X,*)$. Now, since $\mathbb{A}$ is a left independence subset of $Bin(X)$, we obtain $(X,\diamond)\square[(X,\bullet_1)\square(X,*)]\neq(X,\diamond)\square[(X,\bullet_2)\square(X,*)]$ for all $(X,\diamond)\in Bin(X)$. Since $Bin(X)$ is a right cancellative semigroup, by using the associative laws, we obtain $[(X,\diamond)\square(X,\bullet_1)]\square(X,*)\neq[(X,\diamond)\square(X,\bullet_2)]\square(X,*)$, and hence $(X,\diamond)$ $\square(X,\bullet_1)\neq(X,\diamond)\square(X,\bullet_2)$ for all $(X,\diamond)\in Bin(X)$. Thus, ${}_{(X,*)}\mathbb{A}$ is a left independence subset of $Bin(X)$. □

Corollary 4.8. *Let* $\mathbb{A}$ *be an independence subset of a cancellative semigroup* $Bin(X)$. *Then* ${}_{(X,*)}\mathbb{A}_{(X,*)}$ *is an independence subset of* $Bin(X)$ *for any* $(X,*)\in Bin(X)$.

Proof. It follows immediately from Propositions 4.6 and 4.7. □

Theorem 4.9. *Let $\mathbb{A}$ be a right (left) absorbent subset of $Bin(X)$, and let $(X,*) \in \mathbb{A}$. Then $Bin(X) = \mathbb{A}_{(X,*)}$ (resp., $Bin(X) =_{(X,*)} \mathbb{A}$).*

Proof. Assume $\mathbb{A}$ is a right absorbent subset of $Bin(X)$ and $(X,*) \in \mathbb{A}$. Then $(X,*)\Box(X,\bullet) \in \mathbb{A}\Box Bin(X) = \mathbb{A}$ for all $(X,\bullet) \in Bin(X)$. Thus, $(X,\bullet) \in \mathbb{A}_{(X,*)}$, and so $Bin(X) \subseteq \mathbb{A}_{(X,*)}$. Thus, $Bin(X) = \mathbb{A}_{(X,*)}$.

Assume $\mathbb{A}$ is a left absorbent subset of $Bin(X)$ and $(X,*) \in \mathbb{A}$. Hence $(X,\bullet)\Box(X,*) \in Bin(X)\Box\mathbb{A} = \mathbb{A}$ for all $(X,\bullet) \in Bin(X)$. Thus, $(X,\bullet) \in {}_{(X,*)}\mathbb{A}$, and so $Bin(X) \subseteq \mathbb{A}_{(X,*)}$. Thus, $Bin(X) = {}_{(X,*)}\mathbb{A}$. □

Corollary 4.10. *Let $\mathbb{A}$ be an absorbent subset of $Bin(X)$. Then for $(X,*) \in \mathbb{A}$ we have $Bin(X) =_{(X,*)} \mathbb{A} = \mathbb{A}_{(X,*)}$.*

Theorem 4.11. *Let $\{A_i\}_{i\in\Lambda}$ be a family of disjoint right (left) absorbent subsets, $Bin(X) = \bigcup_{i\in\Lambda} A_i$ and $|A_i| = 1$ for $i \in \Lambda$. Then the following hold:*

(*a*) *$Bin(X)$ is not a commutative semigroup,*

(*b*) *$Bin(X)$ is an independence.*

Proof. (a). Assume $\{\mathbb{A}_i\}_{i\in\Lambda}$ be a partition of right (resp., left) absorbent subsets of $Bin(X)$. Then $Bin(X) = \bigcup_{i\in\Lambda} \mathbb{A}_i$. Let $(X,*) \neq (X,\bullet) \in Bin(X)$. Then there exist $i \neq j \in \Lambda$ such that $(X,*) \in \mathbb{A}_i$ and $(X,\bullet) \in \mathbb{A}_j$. It follows that $(X,*)\Box(X,\bullet) \in Bin(X)\Box\mathbb{A}_j = \mathbb{A}_j$ (resp., $(X,*)\Box(X,\bullet) \in \mathbb{A}_i\Box Bin(X) = \mathbb{A}_i$), since $\mathbb{A}_j$ is a right (resp., $\mathbb{A}_i$ is a left) absorbent subset of $Bin(X)$. On the other hand, since $\mathbb{A}_i$ is a right (resp., $\mathbb{A}_j$ is a left) absorbent subset of $Bin(X)$, $(X,\bullet)\Box(X,*) \in Bin(X)\Box\mathbb{A}_i = \mathbb{A}_i$ (resp., $(X,\bullet)\Box(X,*) \in \mathbb{A}_j\Box Bin(X) = \mathbb{A}_j$), Since $\mathbb{A}_i \cap \mathbb{A}_j = \emptyset$, we get $(X,*)\Box(X,\bullet) \neq (X,\bullet)\Box(X,*)$. This proves (a).

(b). Assume $(X,*) \neq (X,\bullet) \in Bin(X)$. Hence there are $i \neq j \in \Lambda$ such that $(X,*) \in \mathbb{A}_i$ and $(X,\bullet) \in \mathbb{A}_j$. Then for all $(X,\diamond) \in Bin(X)$, since $\mathbb{A}_i$ and $\mathbb{A}_j$ are right absorbent subsets of $Bin(X)$, we get $(X,\diamond)\Box(X,*) \in Bin(X)\Box\mathbb{A}_i = \mathbb{A}_i$ and $(X,\diamond)\Box(X,\bullet) \in Bin(X)\Box\mathbb{A}_j = \mathbb{A}_j$. Since $\mathbb{A}_i \cap \mathbb{A}_j = \emptyset$, we get $(X,\diamond)\Box(X,*) \neq (X,\diamond)\Box(X,\bullet)$, and so $Bin(X)$ is a left independence.

Also, since $\mathbb{A}_i$ and $\mathbb{A}_j$ are left absorbent subsets of $Bin(X)$, $(X,*)\Box(X,\diamond) \in \mathbb{A}_i\Box Bin(X) = \mathbb{A}_i$ and $(X,\bullet)\Box(X,\diamond) \in \mathbb{A}_j\Box Bin(X) = \mathbb{A}_j$. Since $\mathbb{A}_i \cap \mathbb{A}_j = \emptyset$, we get $(X,*)\Box(X,\diamond) \neq (X,\bullet)\Box(X,\diamond)$, and so $Bin(X)$ is a right independence. □

5. Open problem

There is a partition $\{\mathbb{A}_i\}_{i\in\Lambda}$ of right (left) independence subsets of $Bin(X)$ (i.e., $Bin(X) = \bigcup_{i\in\Lambda} \mathbb{A}_i$, $|\mathbb{A}_i| = 1$ and $\mathbb{A}_i \bigcap \mathbb{A}_j = \emptyset$ for $i, j \in \Lambda$).

Is there another partition of $Bin(X)$, where there is at least $i \in \Lambda$ such that $|\mathbb{A}_i| > 1$?

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