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Menger algebras of terms induced by transformations with restricted range

Sarawut Phuapong and Thodsaporn Kumduang

Abstract. In this paper, a special kind of n-ary terms of type τ_n, which are called $T(\bar{n}, Y)$-full terms, are introduced. They are derived by applying transformations on the set $\bar{n} = \{1, 2, \ldots, n\}$ with restricted range. Under the superposition operation S^n, the algebra of such terms called the clone of $T(\bar{n}, Y)$-full terms is constructed. We prove that the superassociative law is satisfied in the clone of $T(\bar{n}, Y)$-full terms and the freeness is investigated using a generating set and a suitable homomorphism. Based on the theory of hypervariety, we study $T(\bar{n}, Y)$-full hypersubstitutions which are maps taking all operation symbols to our obtained terms. These lead us to provide the classes of $T(\bar{n}, Y)$-full hyperidentities and $T(\bar{n}, Y)$-full solid varieties. A connection between identities in $clone_{T(\bar{n},Y)}(\tau_n)$ and $T(\bar{n}, Y)$-full hyperidentities is established.

1. Introduction

It is commonly known that the idea of terms is one of fundamental tools in study of universal algebra. It is also connect with various fields of science, for instance, graph theory and automata theory. Normally, terms are formal expression defined from variables and operation symbols. Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as letters to X as an alphabet, and also refer to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an *n_i-ary operation symbol*, where $1 \leqslant n_i \leqslant n$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The sequence of the values of function τ, written as $(n_i)_{i\in I}$, is called a *type*. An *n-ary term of type τ* is defined inductively as follows: (i) Every variable $x_j \in X_n$ is an n-ary term of type τ. (ii) $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ where $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol. The set of all n-ary terms of type τ, closed under finite number of applications of (ii), is denoted by $W_\tau(X_n)$. The symbol $W_\tau(X) := \bigcup\limits_{n=1}^{\infty} W_\tau(X_n)$ stands for the set of all terms of type τ. See [13, 14, 15, 21, 22, 24] for example of current trands in the study of terms.

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The set of all terms of type τ can be used as the universe of an algebra of type τ. For every $i \in I$, an n_i-ary operation $\bar{f}_i : W_\tau(X)^{n_i} \longrightarrow W_\tau(X)$ is defined by

$$\bar{f}_i\,(t_1, \ldots, t_{n_i}) := f_i\,(t_1, \ldots, t_{n_i})\,.$$

The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i \in I})$ is called the *absolutely free algebra* of type τ over the set X.

There is another way to consider the operation on the set of terms. Now, we recall the concept of superposition operation of terms. For each natural numbers $m, n \geqslant 1$, the superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

defined by

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$, if $x_j \in X_n$,

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra can be defined by

$$\text{clone}\,\tau := ((W_\tau(X_n))_{n \in \mathbb{N}^+}; (S^n_m)_{n,m \in \mathbb{N}^+}, (x_i)_{i \leqslant n \in \mathbb{N}^+}),$$

which is called *the clone of all terms of type* τ. For recent developments in this way, see [3].

Let $\tau_n = (n, n, \ldots, n)$ be a type consisting of the same values equal to n, i.e. $\tau_n = (n_i)$ with $n_i = n$ for all $i \in I$. The concept of full terms is used in [6] to study the depth of terms and full hypersubstitutions, and solid varieties. The composed full terms are derived by operation symbols and terms in which all input variables occur. Thus the resulting subterms in each step of composition, content whole set of the input variables, which can be permuted, only.

In 2004, Denecke and Jampachon [5] inductively defined n-ary full terms of type τ_n, based on the full transformations (mappings) instead of the permutations, as follows:

(i) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n if f_i is an n-ary operation symbol and $\alpha \in T_n$ where T_n is the set of all full transformation on $\{1, 2, \ldots, n\}$;

(ii) $f_i\,(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n if f_i is an n-ary operation symbol and $t_1, \ldots, t_n$ are n-ary full terms of type τ_n.

The set of all n-ary full terms of type τ_n, closed under finite application of (ii), is denoted by $W^F_{\tau_n}(X_n)$. If T_n is replaced by the submonoid $\{1_n\}$, then $W^F_{\tau_n}(X_n)$ is denoted by $W^{SF}_{\tau_n}(X_n)$ called the set of all *strongly full terms* of type τ_n [4]. Actually, there are many generalizations of full terms as in [4, 18, 19, 27, 28].

Beginning with the notions of terms, we define $T(\bar{n}, Y)$-full terms through transformations with restricted range. The Menger algebea of $T(\bar{n}, Y)$-full terms is presented. In Section 3, we construct the monoid of $T(\bar{n}, Y)$-full hypersubstitution of type τ_n which consists of a mapping from the set of operation sysmbols to the set of all $T(\bar{n}, Y)$-full terms. These mappings preserve the arity of operation symbols and the arity of $T(\bar{n}, Y)$-full terms, together with one binary associative operation and the identity element. Finally, the $T(\bar{n}, Y)$-full solid varieties of type τ_n are charaterized.

2. The algebra of $T(\bar{n}, Y)$-full terms

The first aim of our main results is to propose the new concept of a specific term, based on full transformation mappings and the original notions of terms. For this, we recall the concept of the full transformations.

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under composition of mappings and let Y be a nonempty subset of X. Then $T(X, Y)$ was introduced by Symons [26] to be the set of all transformations from X to Y called the *full transformation semigroup with restricted range*, that means

$$T(X, Y) := \{\alpha \in T(X) \mid X\alpha \subseteq Y\}.$$

Clearly, $T(X, Y)$ is a subsemigroup of $T(X)$ and if $X = Y$ then $T(X, Y) = T(X)$. For more information about $T(X, Y)$, we refer to[1, 11, 25].

Let $\tau_n = (n_i)_{i \in I}$ be a type and let $(f_i)_{i \in I}$ be an indexed set of operation symbols of type τ. The *full transformation semigroup* T_n consists of the set of all maps $\alpha : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ and the usual composition of mappings. Indeed, T_n is a monoid and identity map 1_n acts as its identity. Let $\bar{n} := \{1, 2, \ldots, n\}$. For a fixed nonempty subset Y of $\bar{n}$, it is well-known that the set

$$T(\bar{n}, Y) := \{\alpha \in T_n \mid \operatorname{Im} \alpha \subseteq Y\} \cup \{1_n\}$$

is a submonoid of T_n.

Then we introduce the definition of n-ary $T(\bar{n}, Y)$-full term of type τ_n.

Definition 2.1. Let f_i be an n-ary operation symbol and $\alpha \in T(\bar{n}, Y)$. An n-ary $T(\bar{n}, Y)$-full term of type τ_n is defined in the following way:

(i) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary $T(\bar{n}, Y)$-full term of type τ_n;

(ii) if $t_1, \ldots, t_n$ are n-ary $T(\bar{n}, Y)$-terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary $T(\bar{n}, Y)$-full term of type τ_n.

Let $W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ be the set of all n-ary $T(\bar{n}, Y)$-full terms of type τ_n.

Now we give an example of Definition 2.1.

Example 2.2. Let $\tau_n = (n)$ be a type with one operation symbol f and let us consider the following examples:

(i) Let $n = 2$, and $Y = \{2\}$, then
$$f(x_1, x_2), f(x_2, x_2), f(f_1(x_2, x_2), f(x_2, x_2)) \in W_{\tau_2}^{T(\bar{2},Y)}(X_2).$$

(ii) Let $n = 3$, and $Y = \{1, 3\}$, then
$$f(x_1,x_2, x_3), f(x_3, x_3, x_3), f(f_2(x_3, x_3, x_1), f(x_1, x_1, x_1), f(x_1,x_3, x_3)) \in W_{\tau_3}^{T(\bar{3},Y)}(X_3).$$

(iii) Let $n = 4$, and $Y = \{2, 3, 4\}$, then
$$f(x_1, x_2, x_3, x_4), f(x_2, x_2, x_4, x_2), f(x_2, x_4, x_2, x_4) \in W_{\tau_4}^{T(\bar{4},Y)}(X_4).$$

Let us note that if $Y = \bar{n}$ then the set $W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ of all $T(\bar{n}, Y)$-full terms is equal to the set $W_{\tau_n}^{F}(X_n)$ of all n-ary full terms of type τ_n, as defined in [5]. This means that $T(\bar{n}, Y)$-full terms of type τ_n are natural generalization of the full terms of type τ_n, discussed in [5] and [6]. By the definition of $T(\bar{n}, Y)$-full terms of type τ_n we have that $\left(W_{\tau_n}^{T(\bar{n},Y)}(X_n); (\bar{f}_i)_{i\in I}\right)$ is a subalgebra of $\left(W_\tau(X); (\bar{f}_i)_{i\in I}\right)$.

Normally, terms have many measures of their complexity, see [23]. As a result, there is a possibility to measure a complexity of $T(\bar{n}, Y)$-full terms. The depth of a $T(\bar{n}, Y)$-full term t, denoted by $Depth(t)$, is the longest distance from a first operation symbol that appears in a term (from the left) to variables. It can be inductively defined by

(i) $Depth(t) = 1$ if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ and $\alpha \in T(\bar{n}, Y)$;

(ii) $Depth(t) = 1 + max\{Depth(t_j) \mid 1 \leqslant j \leqslant n\}$ if $t = f_i(t_1, \ldots, t_n)$.

On the set $W_{\tau_n}^{T(\bar{n},Y)}(X_n)$, we define an $(n + 1)$-ary operation S^n,
$$S^n : \left(W_{\tau_n}^{T(\bar{n},Y)}(X_n)\right)^{n+1} \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$$
for all $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ by

(i) $S^n\left(f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right), t_1, \ldots, t_n\right) := f_i\left(t_{\alpha(1)}, \ldots, t_{\alpha(n)}\right)$;

(ii) $S^n(f_i(t_1, \ldots, t_n), s_1, \ldots, s_n) := f_i\left(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)\right)$.

Then we form the algebra
$$clone_{T(\bar{n},Y)}(\tau_n) := \left(W_{\tau_n}^{T(\bar{n},Y)}(X_n), S^n\right)$$
which is called *the clone of all $T(\bar{n}, Y)$-full terms of type τ_n*. Theorem 2.3, presented below, shows that the algebra $\left(W_{\tau_n}^{T(\bar{n},Y)}(X_n), S^n\right)$ satisfies the superassociative law (SASS):
$$\begin{aligned} &S^n(X_0, S^n(Y_1, Z_1, \ldots, Z_n), \ldots, S^n(Y_n, Z_1, \ldots, Z_n)) \\ &\quad\approx S^n(S^n(X_0, Y_1, \ldots, Y_n), Z_1, \ldots, Z_n) \end{aligned} \tag{1}$$

where S^n is an $(n+1)$-ary operation symbol and X_0, Y_j, Z_j are variables for all $1 \leqslant j \leqslant n$.

Next, we shall show that the superassociative law is satisfied in the clone of all $T(\bar{n}, Y)$-full terms.

Theorem 2.3. *The algebra $clone_{T(\bar{n},Y)}(\tau_n)$ satisfies the superassociative law.*

Proof. We give a proof by induction on the depth of an n-ary $T(\bar{n}, Y)$-full term t which is substituted for X_0 from (1). If we substitute for X_0 from (1) by a $T(\bar{n}, Y)$-full term $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ where $\alpha \in T(\bar{n}, Y)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
&S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\\
&\qquad = f_i(S^n(x_{\alpha(1)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots,\\
&\qquad\quad S^n(x_{\alpha(n)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)))\\
&\qquad = f_i(S^n(t_{\alpha(1)}, s_1, \dots, s_n), \dots, S^n(t_{\alpha(n)}, s_1, \dots, s_n))\\
&\qquad = S^n(f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)}), s_1, \dots, s_n)\\
&\qquad = S^n(S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), t_1, \dots, t_n), s_1, \dots, s_n).
\end{aligned}$$

If we substitute for X_0 from (1) by a $T(\bar{n}, Y)$-full term $t = f_i(r_1, \dots, r_n)$ where $r_1, \dots, r_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ and assume that

$$S^n(r_k, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)) = S^n(S^n(r_k, t_1, \dots, t_n), s_1, \dots, s_n)$$

for all $1 \leqslant k \leqslant n$, and $max_{1 \leqslant k \leqslant n} Depth(r_k) = m$, then $Depth(t) = m + 1$ and we have

$$\begin{aligned}
&S^n(f_i(r_1, \dots, r_n), S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\\
&\quad = f_i(S^n(r_1, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots,\\
&\quad\quad S^n(r_n, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)))\\
&\quad = f_i\left(S^n\left(S^n(r_1, t_1, \dots, t_n), s_1, \dots, s_n\right), \dots, \left(S^n(r_n, t_1, \dots, t_n), s_1, \dots, s_n\right)\right)\\
&\quad = S^n(f_i(S^n(r_1, t_1, \dots, t_n), \dots, S^{n_i}(r_n, t_1, \dots, t_n)), s_1, \dots, s_n)\\
&\quad = S^n(S^n(f_i(r_1, \dots, r_n), t_1, \dots, t_n), s_1, \dots, s_n). \qquad \square
\end{aligned}$$

An algebra $\mathcal{M} := (M, S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if $\mathcal{M}$ satisfies the condition (SASS) [2]. It follows immediately from Theorem 2.3 that $clone_{T(\bar{n},Y)}(\tau_n)$ is a Menger algebra of rank n. For basics and some advanced developments of Menger algebras can be found in the works of W.A. Dudek and V.S. Trokhimenko, for example, see [8, 9, 10].

It is clear that $clone_{T(\bar{n},Y)}(\tau_n)$ is generated by

$$F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)} := \left\{f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right) \mid i \in I, \alpha \in T(\bar{n}, Y)\right\}.$$

Let $V^{T(\bar{n},Y)}$ be the variety of type $\tau = (n+1)$ generated by the superassociative law (SASS). Now let $\mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ be the free algebra with respect to $V^{T(\bar{n},Y)}$, freely generated by an alphabet $\{Y_l \mid l \in J\}$ where $J = \{(i,\alpha) \mid i \in I\ , \alpha \in T(\bar{n},Y)\}$. The operation of $\mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ is denoted by $\tilde{S}^n$. Next, we are going to prove that the clone of all $T(\bar{n},Y)$-full terms is a free algebra with respect to the variety $V^{T(\bar{n},Y)}$.

Theorem 2.4. *The algebra* $clone_{T(\bar{n},Y)}(\tau_n)$ *is isomorphic to* $\mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ *and therefore it is free with respect to the variety* $V^{T(\bar{n},Y)}$*, and freely generated by the set*

$$\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}) \mid i \in I, \alpha \in T(\bar{n},Y)\}.$$

Proof. We define the mapping $\varphi : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow \mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ inductively as follows:

(i) $\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}) = y_{(i,\alpha)}$;

(ii) $\varphi(f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})) = \tilde{S}^n(y_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n))$.

Since φ maps the generating system of $clone_{T(\bar{n},Y)}(\tau_n)$ onto the generating system of $\mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$, it is surjective. We prove the homomorphism property

$$\varphi(S^n(t_0,t_1,\ldots,t_n)) = \tilde{S}^n(\varphi(t_0),\varphi(t_1),\ldots,\varphi(t_n))$$

by induction on the depth of an n-ary $T(\bar{n},Y)$-full term t_0.If $t_0 = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in T(\bar{n},Y)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
&\varphi(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n))\\
&\qquad\qquad = \varphi(f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)}))\\
&\qquad\qquad = \tilde{S}^n(y_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n))\\
&\qquad\qquad = \tilde{S}^n(\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

If $t_0 = f_i(r_1,\ldots,r_n)$ and assume that

$$\varphi(S^n(r_k,t_1,\ldots,t_n)) = \tilde{S}^n(\varphi(r_k),\varphi(t_1),\ldots,\varphi(t_n))$$

for all $1 \leqslant k \leq n$ and $max_{1\leqslant k\leqslant n}Depth(r_k) = m$, then $Depth(t) = m+1$ and we have

$$\begin{aligned}
&\varphi(S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n))\\
&\quad = \varphi(f_i(S^n(r_1,t_1,\ldots,t_n),\ldots,S^n(r_n,t_1,\ldots,t_n)))\\
&\quad = \tilde{S}^n(y_{(i,1_n)},\varphi(S^n(r_1,t_1,\ldots,t_n)),\ldots,\varphi(S^n(r_n,t_1,\ldots,t_n)))\\
&\quad = \tilde{S}^n(y_{(i,1_n)},\tilde{S}^n(\varphi(r_1),\varphi(t_1),\ldots,\varphi(t_n)),\ldots,\\
&\qquad \tilde{S}^n(\varphi(r_n),\varphi(t_1),\ldots,\varphi(t_n)))\\
&\quad = \tilde{S}^n(\tilde{S}(y_{(i,1_n)},\varphi(r_1),\ldots,\varphi(r_n)),\varphi(t_1),\ldots,\varphi(t_n))\\
&\quad = \tilde{S}^n(\varphi(f_i(r_1,\ldots,r_n)),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

Thus φ is a homomorphism. The mapping φ is clearly bijective since the set $\{y_{(i,\alpha)} \mid i \in I,\ \alpha \in T(\bar{n},Y)\}$ is free independent. Therefore we have

$$y_{(i,\alpha)} = y_{(j,\beta)} \Longrightarrow (i,\alpha) = (j,\beta) \Longrightarrow i = j,\ \alpha = \beta.$$

So $f_i(x_{\alpha 1}), \ldots, x_{\alpha(n)}) = f_j(x_{\beta(1)}, \ldots, x_{\beta(n)})$. Thus φ is a bijection between the generating sets of $clone_{T(\bar{n},Y)}(\tau_n)$ and $\mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ and therefore φ is an isomorphism. $\square$

3. $T(\bar{n}, Y)$-full hypersubstitutions

The concept of a hypersubstitution is the main tool used to study hyperidentities and hypervarieties, see, for instance, in [7, 16, 17, 20] for more background. In this section, the monoid of hypersubstitution will be studied. First, we recall the definition and notation of hypersubstitutions.

A hypersubstitution of type τ is a mapping $\sigma : \{f_i \,|\, i \in I\} \longrightarrow W_\tau(X)$ which maps each operation symbol f_i to an n_i-ary term $\sigma(f_i)$ of type τ. Any hypersubstitution $\sigma : \{f_i \,|\, i \in I\} \longrightarrow W_\tau(X)$ can be uniquely extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$; and

(ii) $\hat{\sigma}[t] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_{n_i})$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the binary operation $\circ_h$, defined by

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma_1} \circ \sigma_2$$

where $\circ$ denotes the usual composition of mappings.

The identity is $\sigma_{id} : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ such that $\sigma_{id}(f_i) = f_i(x_1, ..., x_{n_i})$.

Now, we call mapping

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n).$$

$T(\bar{n}, Y)$-*full hypersubstitution* of type τ_n.

For a $T(\bar{n}, Y)$-full term t we need the $T(\bar{n}, Y)$-full term t_β derived from t by replacement a variable $x_{\alpha(j)}$ in t by a variable $x_{\beta(\alpha(j))}$ for a mapping $\beta \in T(\bar{n}, Y)$. This can be defined as follows.

Let $t, t_1, \ldots, t_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ and $\alpha, \beta \in T(\bar{n}, Y)$. Then we define the $T(\bar{n}, Y)$-full term t_β in the following steps:

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then $t_\beta := f_i(x_{\beta\alpha(1)}, \ldots, x_{\beta\alpha(n)})$.

(ii) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta := f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

It is observed that if t is an $T(\bar{n}, Y)$-full term of type τ_n, then t_β is an $T(\bar{n}, Y)$-full term of type τ_n for all $\beta \in T(\bar{n}, Y)$. Then an $T(\bar{n}, Y)$-full hypersubstitution $\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ of type τ_n can be extended to a mapping

$$\hat{\sigma} : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$$

as follows:

(i) $\hat{\sigma}[f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})] := (\sigma(f_i))_{\alpha}$,

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_n)] := S^n\left(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}]\right)$.

The set of all $T(\bar{n},Y)$-full hypersubstitutions of type τ_n will be denoted by $Hyp^{T(\bar{n},Y)}(\tau_n)$. It is easy to see that $\left(Hyp^{T(\bar{n},Y)}(\tau_n);\circ_h,\sigma_{id}\right)$ is a submonoid of $(Hyp(\tau_n);\circ_h,\sigma_{id})$.

The following lemma shows the property of a term t_α and the extension $\hat{\sigma}$.

Lemma 3.1. *Let* $t,t_1,...,t_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$. *Then*

$$S^n(t,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = S^n(t_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$$

for all $\alpha \in T(\bar{n},Y)$.

Proof. We begin with the case when $t = f_i(x_{\alpha(1)},x_{\alpha(2)},\ldots,x_{\alpha(n)})$, which is the first claim of the first step of the induction $Depth(t) = 1$. In fact, we have $S^n(f_i(x_1,x_2,\ldots,x_n),\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = f_i(\hat{\sigma}[t_{\alpha(1)}],\hat{\sigma}[t_{\alpha(2)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]) = S^n(f_i(x_1,x_2,\ldots,x_n)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$.

If $t = f_i(s_1,\ldots,s_n)$ and assume that

$$S^n(s_k,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]) = S^n((s_k)_{\alpha_i},\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}])$$

for all $1 \leqslant k \leqslant n$ and $\alpha \in T(\bar{n},Y)$ then

$$\begin{aligned}
&S^n(t,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}])\\
&\quad= S^n(f_i(s_1,\ldots,s_n),\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}])\\
&\quad= f_i(S^n(s_1,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]),\ldots,S^n(s_n,\hat{\sigma}[t_{\alpha(1)}],\ldots,\hat{\sigma}[t_{\alpha(n)}]))\\
&\quad= f_i(S^n((s_1)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]),\ldots,S^n((s_n)_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]))\\
&\quad= S^n(f_i((s_1)_\alpha,\ldots,(s_n)_\alpha),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])\\
&\quad= S^n(t_\alpha,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]).
\end{aligned}$$

□

Using Lemma 3.1 we show that the extension $\hat{\sigma}$ of each $T(\bar{n},Y)$-full hypersubstitution σ preserves the operation S^n on the set $W_{\tau_n}^{T(\bar{n},Y)}(X_n)$.

Theorem 3.2. *For* $\sigma \in Hyp^{T(\bar{n},Y)}(\tau_n)$, *the extension*

$$\hat{\sigma} : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$$

is an endomorphism on the algebra $clone_{T(\bar{n},Y)}(\tau_n)$.

Proof. It is clear that $\hat{\sigma} : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$. Let $t_0,t_1,\ldots,t_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$. We will show by induction on the depth of t_0 that

$$\hat{\sigma}[S^n(t_0,t_1,\ldots,t_n)] = S^n(\hat{\sigma}[t_0],\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]).$$

If $t_0 = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in T(\bar{n},Y)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
\hat{\sigma}[S^n(t_0,t_1,\ldots,t_n)] &= \hat{\sigma}[S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n)]\\
&= \hat{\sigma}[f_i(t_{\alpha 1)},\ldots,t_{\alpha(n)})]
\end{aligned}$$

$$
\begin{aligned}
&= S^n(\sigma(f_i), \hat{\sigma}[t_{\alpha(1)}], \dots, \hat{\sigma}[t_{\alpha(n)}]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]).
\end{aligned}
$$

If $t_0 = f_i(r_1, \dots, r_n)$ and we assume that

$$\hat{\sigma}[S^n(r_k, t_1, \dots, t_n)] = S^n(\hat{\sigma}[r_k], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])$$

for all $1 \leqslant k \leqslant n$ and $max_{1\leqslant k\leqslant n}Depth(r_k) = m$, then $Depth(t) = m + 1$ and we have

$$
\begin{aligned}
&\hat{\sigma}[S^n(t_0, t_1, \dots, t_n)] \\
&\quad = \hat{\sigma}[S^n(f_i(r_1, \dots, r_n), t_1, \dots, t_n)] \\
&\quad = \hat{\sigma}[f_i(S^n(r_1, t_1, \dots, t_n), \dots, S^{n_i}(r_n, t_1, \dots, t_n))] \\
&\quad = S^n(\sigma(f_i), \hat{\sigma}[S^n(r_1, t_1, \dots, t_n)], \dots, \hat{\sigma}[S^n(r_{n_i}, t_1, \dots, t_n)]) \\
&\quad = S^n(\sigma(f_i), S^n(\hat{\sigma}[r_1], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]), \dots, S^n(\hat{\sigma}[r_n], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])) \\
&\quad = S^n(S^n(\sigma(f_i), \hat{\sigma}[r_1], \dots, \hat{\sigma}[r_n]), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\
&\quad = S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]). \qquad \square
\end{aligned}
$$

We complete this section by studying the connection between $T(\bar{n}, Y)$-full terms and the extension of a mapping which maps fundamental term to any $T(\bar{n}, Y)$-full terms.

As mentioned, the algebra $clone_{T(\bar{n},Y)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)} := \left\{ f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right) \mid i \in I, \alpha \in T(\bar{n}, Y) \right\}.$$

Thus, any mapping

$$\eta : F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)} \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$$

called $T(\bar{n}, Y)$-*full clone substitution*, can be uniquely extended to endomorphism

$$\bar{\eta} : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n).$$

Let $Subst_{T(\bar{n},Y)}(\tau_n)$ be the set of all $T(\bar{n}, Y)$-full clone substitutions. On the set $Subst_{T(\bar{n},Y)}(\tau_n)$, a binary operation $\odot$ can be defined by

$$\eta_1 \odot \eta_2 := \bar{\eta_1} \circ \eta_2$$

where $\circ$ denotes the usual composition of mappings. Furthermore, the identity mapping with respect to $\odot$ is denoted by $id_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}$.

Then clearly, $\left(Subst_{T(\bar{n},Y)}(\tau); \odot, id_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}\right)$ forms a monoid.

Consider $\sigma \in Hyp^{T(\bar{n},Y)}(\tau_n)$ and by Theorem 3.2,

$$\hat{\sigma} \colon W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow W_{\tau_n}^{T(\bar{n},Y)}(X_n)$$

is an endomorphism. Since $F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}$ generates $clone_{T(\bar{n},Y)}(\tau_n)$, $\hat{\sigma}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}$ is an $T(\bar{n}, Y)$-full clone substitution with

$$\overline{\hat{\sigma}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}} = \hat{\sigma}.$$

Define a mapping $\psi : Hyp^{T(\bar{n},Y)}(\tau_n) \longrightarrow Subst_{T(\bar{n},Y)}(\tau_n)$ by

$$\psi(\sigma) = \hat{\sigma}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}.$$

We have that ψ is a homomorphism. In fact: Let $\sigma_1, \sigma_2 \in Hyp^{T(\bar{n},Y)}(\tau_n)$. Then

$$\begin{aligned}
\psi(\sigma_1 \circ_h \sigma_2) &= (\sigma_1 \circ_h \sigma_2)^{\hat{}}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}} = (\hat{\sigma_1} \circ \hat{\sigma_2})\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}} \\
&= \overline{\hat{\sigma_1}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}}} \circ \hat{\sigma_2}\big|_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}} = \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\
&= \psi(\sigma_1) \odot \psi(\sigma_2).
\end{aligned}$$

Clearly, ψ is an injection. Hence we have proved, the following corollary.

Corollary 3.3. *The monoid* $\left(Hyp^{T(\bar{n},Y)}(\tau_n); \circ_h, \sigma_{id}\right)$ *can be embedded into* $(Subst_{T(\bar{n},Y)}(\tau_n); \odot, id_{F_{W_{\tau_n}^{T(\bar{n},Y)}(X_n)}})$.

4. $T(\bar{n}, Y)$-full hyperidentities and clone identities

In this section we examine the relationship between a variety V of type τ_n and the identity in the $clone_{T(\bar{n},Y)}(\tau_n)$.

Let V be a variety of type τ_n and let IdV be the set of all identities of V. Let $Id^{T(\bar{n},Y)}V$ be the set of all $s \approx t$ of V such that s and t are both $T(\bar{n}, Y)$-full term of type τ_n; that is

$$Id^{T(\bar{n},Y)}V := \left(W_{\tau_n}^{T(\bar{n},Y)}(X_n)\right)^2 \cap IdV.$$

It is well-known that IdV is a congruence on the free algebra $\mathcal{F}_\tau(X)$. However, in general this is not true for $Id^{T(\bar{n},Y)}V$. The following theorem shows that $Id^{T(\bar{n},Y)}V$ is a congruence on $clone_{T(\bar{n},Y)}(\tau_n)$.

Theorem 4.1. *Let* V *be a variety of type* τ_n*. Then* $Id^{T(\bar{n},Y)}V$ *is a congruence on the algebra* $clone_{T(\bar{n},Y)}(\tau_n)$*.*

Proof. We will prove that if $t \approx r$, $t_k \approx r_k \in Id^{T(\bar{n},Y)}V$, $k = 1, 2, \ldots, n$, then $S^n(t, t_1, \ldots, t_n) \approx S^n(r, r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V$. Firstly, we give a proof by induction on the depth of a term $t \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ that for every $i \in I$ from $t_k \approx r_k \in Id^{T(\bar{n},Y)}V$, $k = 1, 2, \ldots, n$, there follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V$. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, where $\alpha \in T(\bar{n}, Y)$, and $Depth(t) = 1$, then we have

$$\begin{aligned}
S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n) &= f_i(t_{\alpha(1)}, \ldots, \overline{t_{\alpha(n)})} \\
&\approx f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) = \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\
&= S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V,
\end{aligned}$$

since IdV is compatible with the operation $\overline{f_i}$ of the absolutely free algebra $\mathcal{F}_\tau(X)$ and by the definition of $T(\bar{n}, Y)$-full terms.

If $t = f_i(l_1, \ldots, l_n) \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ and assume that

$$S^n(l_k, t_1, ..., t_n) \approx S^n(l_k, r_1, ..., r_n) \in Id^{T(\bar{n},Y)}V.$$

for all $1 \leqslant k \leqslant n$ and $max_{1 \leqslant k \leqslant n} Depth(r_k) = m$, then $Depth(t) = m + 1$ and we obtain

$$\begin{aligned} S^n(f_i(l_1, \ldots, l_n), t_1, \ldots, t_n) &= f_i(S^n(l_1, t_1, \ldots, t_n), \ldots, S^n(l_n, t_1, \ldots, t_n)) \\ &\approx f_i(S^n(l_1, r_1, \ldots, r_n), \ldots, S^{n_i}(l_n, r_1, \ldots, r_n)) \\ &= S^n(f_i(l_1, \ldots, l_n), r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V. \end{aligned}$$

This means

$$S^n(t, t_1, \ldots, t_n) \approx S^n(t, r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V.$$

This is a consequence of the fact that IdV is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$. Assume now that $t \approx r, t_k \approx r_k \in Id^{T(\bar{n},Y)}V$. Then

$$S^n(t, t_1, \ldots, t_n) \approx S^n(r, t_1, \ldots, t_n) \approx S^n(r, r_1, \ldots, r_n) \in Id^{T(\bar{n},Y)}V. \qquad \square$$

By using the concepts of $T(\bar{n}, Y)$-full hypersubstitution as we presented in Section 3. We shall define $T(\bar{n}, Y)$-full hyperidentities in a variety of typer τ_n.

Let V be a variety of type τ_n. An identity $s \approx t \in Id^{T(\bar{n},Y)}V$ is called a $T(\bar{n}, Y)$*-full hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all $\sigma \in Hyp^{T(\bar{n},Y)}(\tau_n)$. Moreover, the variety V is called $T(\bar{n}, Y)$*-full solid* if the following holds:

$$\forall s \approx t \in Id^{T(\bar{n},Y)}V \; \forall \sigma \in Hyp^{T(\bar{n},Y)}(\tau_n) \; \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV.$$

Next theorem characterizes the $T(\bar{n}, Y)$-full solid variety.

Theorem 4.2. *Let V be a variety of type τ_n. If $Id^{T(\bar{n},Y)}V$ is a fully invariant congruence on $clone_{T(\bar{n},Y)}(\tau_n)$, then V is $T(\bar{n}, Y)$-full solid.*

Proof. Assume that $Id^{T(\bar{n},Y)}V$ is a fully invariant congruence on $clone_{T(\bar{n},Y)}(\tau_n)$. Let $s \approx t \in Id^{T(\bar{n},Y)}V$ and $\sigma \in Hyp^{T(\bar{n},Y)}(\tau_n)$. By Theorem 3.2, $\hat{\sigma}$ is an endomorphism of $clone_{T(\bar{n},Y)}(\tau_n)$. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{T(\bar{n},Y)}V$, which shows that V is $T(\bar{n}, Y)$-full solid. $\square$

For a variety V of type τ_n, $Id^{T(\bar{n},Y)}V$ is a congruence on $clone_{T(\bar{n},Y)}(\tau_n)$ by Theorem 4.1. We can form the quotient algebra

$$clone_{T(\bar{n},Y)}(V) := clone_{T(\bar{n},Y)}(\tau_n)/Id^{T(\bar{n},Y)}V.$$

This quotient algebra belongs to the class of a Menger algebra of rank n. Note that we have a natural homomorphism

$$nat_{Id^{T(\bar{n},Y)}V} : clone_{T(\bar{n},Y)}(\tau_n) \longrightarrow clone_{T(\bar{n},Y)}(V)$$

such that

$$nat_{Id^{T(\bar{n},Y)}V}(t) = [t]_{Id^{T(\bar{n},Y)}V}.$$

Finally, we prove the following connection between $T(\bar{n}, Y)$-full hyperidentities of a variety V and clone identities.

Theorem 4.3. *Let V be a variety of type τ_n. If $s \approx t \in Id^{T(\bar{n},Y)}V$ is an identity in $clone_{T(\bar{n},Y)}(V)$, then $s \approx t$ is $T(\bar{n},Y)$-full hyperidentity of V.*

Proof. Assume that $s \approx t \in Id^{T(\bar{n},Y)}V$ is an identity in $clone_{T(\bar{n},Y)}(V)$. Let $\sigma \in Hyp^{T(\bar{n},Y)}(\tau_n)$. Then $\hat{\sigma} : clone_{T(\bar{n},Y)}(\tau_n) \longrightarrow clone_{T(\bar{n},Y)}(\tau_n)$ is an endomorphism by Theorem 3.2. Thus

$$nat_{Id^{T(\bar{n},Y)}V} \circ \hat{\sigma} : clone_{T(\bar{n},Y)}(\tau_n) \longrightarrow clone_{T(\bar{n},Y)}(V)$$

is a homomorphism. By assumption,

$$\left(nat_{Id^{T(\bar{n},Y)}V} \circ \hat{\sigma}\right)(s) = \left(nat_{Id^{T(\bar{n},Y)}V} \circ \hat{\sigma}\right)(t).$$

That is

$$nat_{Id^{T(\bar{n},Y)}V}(\hat{\sigma}[s]) = nat_{Id^{T(\bar{n},Y)}V}(\hat{\sigma}[t]).$$

Thus

$$[\hat{\sigma}[s]]_{Id^{T(\bar{n},Y)}V} = [\hat{\sigma}[t]]_{Id^{T(\bar{n},Y)}V},$$

and hence

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{T(\bar{n},Y)}V.$$

Therefore, $s \approx t$ is a $T(\bar{n},Y)$-full hyperidentity of V. □

5. Open Problems

Finally, we give three problems and suggestions for the future research in this area.

(1) Determine the semigroup properties of the monoid $\left(Hyp^{T(\bar{n},Y)}(\tau_n); \circ_h, \sigma_{id}\right)$. Find the order of its elements for the particular type. Describe the idempotency and several kinds of regularity of the $T(\bar{n},Y)$-full hypersubstitutions.

(2) Use some difference definions of transformation semigroup, for instance transformations with invariant subset to define new generalizations of full terms. Study the connection between the different kinds of full terms.

(3) Based on [12], define the set of all formulas induced by $T(\bar{n},Y)$-full terms and study this set.

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S. Phuapong
Department of Mathematics, Faculty of Science and Agricultural Technology
Rajamangala University of Technology Lanna, Chiang Mai, Thailand
e-mail: Phuapong.sa@gmail.com

T. Kumduang
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand
e-mail: Kumduang01@gmail.com