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On quasi-cancellative AG-groupoids

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Abstract. We proved the analog of the Burmistrovich's theorem for semigroups: a cyclic-associative AG-groupoid is quasi-cancellative if and only if it is a semilattice of cancellative cyclic-associative AG-subgroupoids. We also proved that an AG-groupoid in which all elements are 3-potent is quasi-cancellative.

1. Introduction

A magma is a fundamental type of an algebraic structure, consist of a non-empty set together with one binary operation. Abel-Grassmann's groupoids (abbreviated as AG-groupoids) [9] (also known as left almost semigroups (LA-semigroups) [5]) can be considered as the non-empty set H with the binary operation satisfying the identity $xy \cdot z = zy \cdot x$. This structures was introduced by Kazim and Naseeruddin in [5].

Protić and Stevanović introduced in [10] the concept of 3-potent elements, AG-3-bands, AG-bands and anti-rectangular AG-bands. The notion of cyclic-associative AG-groupoids (AC-AG-groupoids) was introduced by Iqbal et al. in [4]. Dudek and Gigoń [2, 3] studied some fundamental properties of completely inverse AG**-groupoids and determine certain fundamental congruences on it. Mushtaq and Yusuf proved in [7] that a left cancellative AG-groupoid is right cancellative. Shah et al. proved in [12] that in AG-monoids the set of all cancellative elements is an AG-subgroupoid. They further proved that a finite AG-monoid has at least one non-cancellative element and the set of non-cancellative elements form a maximal ideal.

In this note we will prove the Burmistrovich theorem for AG-groupoids: a cyclic-associative AG-groupoid is quasi-cancellative if and only if it is a semilattice of cancellative cyclic-associative AG-subgroupoids. Also we will prove that any AG-groupoid H in which $xx \cdot x = x \cdot xx = x$ for all $x \in H$ is quasi-cancellative.

2. Results

A groupoid $(H, \cdot)$, or simply H, satisfying the identity $xy \cdot z = zy \cdot x$ (known as the left invertive law (L.I.Law) [5]) is called an *AG-groupoid*. Every AG-groupoid satisfies the *medial law* (M.Law): $xy \cdot zt = xz \cdot yt$. An AG-groupoid contains at most one left identity [7]. An AG-groupoid having a left identity satisfies the *paramedial law* (P.Law): $xy \cdot zt = ty \cdot zx$.

An element $h \in H$ is called an *idempotent* if $h^2 = h$. The set of all idempotent elements of H is denoted by $E(H)$. An AG-groupoid containing only idempotent elements is called an *AG-band* [13]. A commutative AG-band is called a *semilattice*. An element $h \in H$ is *3-potent* if $(hh)h = h(hh) = h$. If all elements of an AG-groupoid H are 3-potents, then H is called an *AG-3-band*. An AG-groupoid H is called an *AG*-groupoid* [6] if $xy \cdot z = y \cdot xz$ for all $x, y, z \in H$ (known as a weak associative law); an *AG**-groupoid* [8] if $x \cdot yz = y \cdot xz$ and a *cyclic-associative AG-groupoid* (CA-AG-groupoid) if $x \cdot yz = z \cdot xy$ [4]. Every CA-AG-groupoid is paramedial [4]. An element h of an AG-groupoid H is *right* (*left*) *cancellative* if for all $x, y \in H$, $xh = yh$ ($hx = hy$) implies $x = y$. The element h is *cancellative* if it is simultaneously right and left

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cancellative. H is (right/left) cancellative if all elements of H are (right/left) cancellative. H is *quasi-cancellative* [11] if for all $x, y \in H$: (i) $x^2 = xy$ and $y^2 = yx$ imply $x = y$, (ii) $x^2 = yx$ and $y^2 = xy$ imply $x = y$.

Lemma 1. *If a quasi-cancellative AG-groupoid is cyclic-associative, then*
(A) $xa = xb \Longleftrightarrow ax = bx,$
(B) $x^2a = x^2b \Rightarrow ax = bx,$
(C) $x^2a = x^2b \Rightarrow xa = xb,$
(D) $xy \cdot a = xy \cdot b \Rightarrow a \cdot yx = b \cdot yx,$
(E) $xy \cdot a = xy \cdot b \Rightarrow yx \cdot a = yx \cdot b,$
(F) $a \cdot xy = b \cdot xy \Rightarrow a \cdot yx = b \cdot yx,$
(G) $a \cdot xy = b \cdot xy \Rightarrow yx \cdot a = yx \cdot b,$
(H) $xy \cdot a = xy \cdot b \Longleftrightarrow a \cdot yx = b \cdot yx.$

Proof. (A). Assume $xa = xb$, then $xa \cdot xa = xb \cdot xa$ and $xa \cdot xb = xb \cdot xb$. Now by the cyclic-associativity and M.Law we get

$$xa \cdot xa = a(xa \cdot x) = x(a \cdot xa) = x(a \cdot ax) = x(x \cdot aa) = aa \cdot xx = ax \cdot ax = (ax)^2.$$

Analogously,

$$\begin{aligned} xb \cdot xa &= a(xb \cdot x) = x(a \cdot xb) = x(b \cdot ax) = x(x \cdot ba) = ba \cdot xx = bx \cdot ax = x(bx \cdot a) \\ &= x(ax \cdot b) = b(x \cdot ax) = ax \cdot bx. \end{aligned}$$

Thus $(ax)^2 = ax \cdot bx$. Similarly, we obtain $xa \cdot xb = ba \cdot xx = bx \cdot ax$. Thus $(bx)^2 = bx \cdot ax$.
By quasi-cancellativity, from $(ax)^2 = ax \cdot bx$ and $(bx)^2 = bx \cdot ax$, we have $ax = bx$.
The converse implication follows by symmetry.

(B). Let $x^2a = x^2b$. Then $x^2a \cdot a = x^2b \cdot a \Rightarrow aa \cdot xx = ab \cdot xx \Rightarrow ax \cdot ax = ax \cdot bx \Rightarrow (ax)^2 = ax \cdot bx$.
Similarly from $x^2a = x^2b$ we have $x^2a \cdot b = x^2b \cdot b$, which gives $(bx)^2 = bx \cdot ax$. This together with $(ax)^2 = ax \cdot bx$ implies $ax = bx$.

(C). Follows from (A) and (B).

(D). Assume $xy \cdot a = xy \cdot b$. Then $a^2 \cdot xy = (xy \cdot a)a = (xy \cdot b)a = ab \cdot xy$. So, $a^2 \cdot xy = ab \cdot xy$. Thus, $(a^2 \cdot xy) \cdot xy = (ab \cdot xy) \cdot xy$. But $(xy \cdot xy)a^2 = (yy \cdot xx)a^2 = (yx \cdot yx)a^2 = (a \cdot yx)(a \cdot yx) = (a \cdot yx)^2$. Similarly, $(ab \cdot xy) \cdot xy = (xy \cdot xy) \cdot ab = (yy \cdot xx) \cdot ab = (yx \cdot yx) \cdot ab = (b \cdot yx)(a \cdot yx) = (b \cdot yx)(a \cdot yx)$. Therefore $(a \cdot yx)^2 = (b \cdot yx)(a \cdot yx)$.
In the same way from $xy \cdot a = xy \cdot b$ we obtain $(a \cdot yx)(b \cdot yx) = (b \cdot yx)^2$, which together with the previous equality implese $a \cdot yx = b \cdot yx$.

(E). Follows from (D) and (A); (F) – from (A) and (D); (G) – from (F) and (A); (H) – from (D) and (G). ☐

The following theorem is an analog of the Burmistrovich's theorem for semigroups from [1].

Theorem 1. *A cyclic-associative AG-groupoid is quasi-cancellative if and only if it is a semilattice of cancellative cyclic-associative AG-subgroupoids.*

Proof. Necessity. Let a cyclic-associative AG-groupoid be quasi-cancellative. Let σ by the relation on H such that $x \,\sigma\, y$ if for any $p, q \in H$, $xp = xq \Longleftrightarrow yp = yq$. It is an equivalence relation. To prove that σ is a congruence, let $x \,\sigma\, y$ and $z \in H$. If $xz \cdot p = xz \cdot q$, then $pz \cdot x = qz \cdot x$. Thus, $x \cdot pz = x \cdot qz$, by Lemma 1 (A). Hence $z \cdot xp = z \cdot xq$, which by our assumption gives $z \cdot yp = z \cdot yq$. So, $p \cdot zy = q \cdot zy$, i.e. $y \cdot pz = y \cdot qz$. The last, by Lemma 1 (A), gives $pz \cdot y = qz \cdot y$. Consequently, $yz \cdot p = yz \cdot q$. By symmetry $yz \cdot p = yz \cdot q$ implies $xz \cdot p = xz \cdot q$. Hence $xz \,\sigma\, yz$. Therefore, σ is right compatible.

Now if $zx \cdot p = zx \cdot q$, then $xz \cdot p = xz \cdot q$, by Lemma 1 (E). So, as it is proved above, $yz \cdot p = yz \cdot q$. This, by Lemma 1 (E), implies $zy \cdot p = zy \cdot q$. By symmetry $zy \cdot p = zy \cdot q$ implies $zx \cdot p = zx \cdot q$. Hence, $zx \,\sigma\, zy$, therefore σ is left compatible. Consequently, σ is a congruence.

Then H/σ, by Lemma 1 (A) and (B), is an AG-band, By Lemma 1 (E), it is commutative. Consequently, σ is a semilattice congruence.

Suppose $zx = zy$, $x\,\sigma\,z$ and $y\,\sigma\,z$. Since $x\,\sigma\,z$, $zx = zy$ implies that $x^2 = xy$ and since $y\sigma z$, thus $yx = y^2$. This, by quasi-cancellativity, gives $x = y$. If $xz = yz$ with $x\,\sigma\,z$ and $y\,\sigma\,z$, then $zx = zy$, by Lemma 1 (A), and this reduces to the case just considered before. Hence, each σ-class is cancellative.

Sufficiency. Let H is a semilattice of cancellative cyclic-associative AG-subgroupoids and x, y are elements such that $x^2 = yx$ and $y^2 = xy$. Suppose η be the component of H that contains yx. As H is semilattice, consequently H is commutative, thus $xy \in \eta$ as well. Hence, $x^2, y^2 \in \eta$. As η is a cyclic-associative AG-groupoid, thus by the closure property in η we have $x, y \in \eta$. But η is cancellative and therefore the equality $xx = xy$ implies $x = y$. By similar argument if $x^2 = xy$ and $y^2 = yx$, then $x = y$. Hence, H is quasi-cancellative. □

The following example illustrate Theorem 1.

Example 1. The Cayley table given below defines a quasi-cancellative cyclic-associative AG-groupoid H that is a semilattice of cancellative cyclic-associative AG-subgroupoids $I = \{1\}$ and $J = \{2, 3, 4, 5\}$ such thst I, J commute and $I^2 = I$, $J^2 = J$.

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	2	5	4
4	1	4	5	2	3
5	1	5	4	3	2

Theorem 2. *Every AG-3-band is quasi-cancellative.*

Proof. Suppose H is AG-3-band and $x, y \in H$.

To prove that $x^2 = xy$ and $y^2 = yx$ imply $x = y$ suppose $x^2 = xy$ and $y^2 = yx$. then, by the definition of AG-3-band, supposition, L.I.Law and M.Law we obtain

$$\begin{aligned}
x &= x^2x = xy \cdot x = ((xx \cdot x)y)x = (yx \cdot xx)x = (x \cdot xx) \cdot yx = x \cdot yx = xy^2 \\
&= (xx \cdot x) \cdot yy = (xx \cdot y) \cdot xy = (yx \cdot x) \cdot xy = (y^2x) \cdot xy = (yy \cdot x) \cdot xy \\
&= (xy \cdot y) \cdot xy = (xy \cdot x) \cdot yy = (((xx \cdot x)y)x) \cdot yy = ((yx \cdot xx)x) \cdot yy \\
&= ((x \cdot xx) \cdot yx) \cdot yy = (x \cdot yx) \cdot yy = xy^2 \cdot yy = xy \cdot y^2y = xy \cdot y \\
&= yy \cdot x = yy \cdot (x \cdot xx) = yx \cdot (y \cdot xx) = y^2 \cdot yx^2 = y^2\left((yy \cdot y) \cdot xx\right) \\
&= y^2\left((yy \cdot x) \cdot yx\right) = y^2\left((xy \cdot y) \cdot yx\right) = y^2\left(x^2y \cdot yx\right) = y^2\left((xx \cdot y) \cdot yx\right) \\
&= y^2\left((yx \cdot x) \cdot yx\right) = y^2\left((yx \cdot y) \cdot xx\right) = y^2\left((((yy \cdot y)x)y) \cdot xx\right) \\
&= y^2\left(((xy \cdot yy)y) \cdot xx\right) = y^2\left(((y \cdot yy) \cdot xy) \cdot xx\right) = y^2\left((y \cdot xy) \cdot xx\right) \\
&= y^2\left(yx^2 \cdot xx\right) = y^2\left(yx \cdot x^2x\right) = y^2\left(yx \cdot x\right) = y^2(y^2x) = y^2\left(yy \cdot x\right) \\
&= y^2\left(xy \cdot y\right) = y^2(xy \cdot y^2y) = y^2\left(xy^2 \cdot yy\right) = y^2\left((x \cdot yx) \cdot yy\right) \\
&= y^2(((x \cdot xx) \cdot yx) \cdot yy) = y^2\left(((yx \cdot xx)x) \cdot yy\right) = y^2\left((((xx \cdot x)y)x) \cdot yy\right) \\
&= y^2\left((xy \cdot x) \cdot yy\right) = y^2\left((xy \cdot y) \cdot xy\right) = y^2\left((yy \cdot x) \cdot xy\right) = y^2\left(y^2x \cdot xy\right) \\
&= y^2\left((yx \cdot x) \cdot xy\right) = y^2\left((xx \cdot y) \cdot xy\right) = y^2\left((xx \cdot x) \cdot yy\right) = y^2 \cdot xy^2 \\
&= yy \cdot xy^2 = yx \cdot yy^2 = y^2y = y.
\end{aligned}$$

This shows that $x^2 = xy$ and $y^2 = yx$ imply $x = y$.

To prove that $x^2 = yx$ and $y^2 = xy$ imply $x = y$ suppose $x^2 = yx$ and $y^2 = xy$. Then, as in the previous case,

$$\begin{aligned} x &= x^2x = yx \cdot x = xx \cdot y = x^2y = yx \cdot y = (y^2y \cdot x)y \\ &= ((xy \cdot y)x)y = ((yy \cdot x)x)y = (xx \cdot yy)y = (xy \cdot xy)y \\ &= (y^2 \cdot y^2)y = (yy \cdot yy)y = ((yy \cdot y)y)y = yy \cdot y = y. \end{aligned}$$

Hence $x = y$. This completes the proof. □

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