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Hom-Jacobi-Jordan and Hom-antiassociative algebras with symmetric invariant nondegenerate bilinear forms

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Abstract. The aim of this paper is first to introduce and study quadratic Hom-Jacobi-Jordan algebras, which are Hom-Jacobi-Jordan algebras with symmetric invariant nondegenerate bilinear forms. We provide several constructions leading to examples. We reduce the case where the twist map is invertible to the study of involutive quadratic Jacobi-Jordan algebras. Also elements of a representation theory for Hom-Jacobi-Jordan algebras, including adjoint and coadjoint representations are supplied with application to quadratic Hom-Jacobi-Jordan algebras.

Secondly, introduce a hom-antiassociative algebra built as a direct sum of a given hom- anti-associative algebra $(\mathcal{A}, \cdot, \alpha)$ and its dual $(\mathcal{A}^*, \circ, \alpha^*)$, endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$, where $\cdot$ and $\circ$ are the products defined on $\mathcal{A}$ and $\mathcal{A}^*$, respectively, α and α^* stand for the corresponding algebra homomorphisms.

1. Introduction

The Hom-algebra structures arose first in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures in which the Jacobi condition is twisted. The first examples of q-deformations, in which the derivations are replaced by σ-derivations, concerned the Witt and Virasoro algebras, see for example [2, 9, 10, 11, 12, 14, 16]. A general study and construction of Hom-Lie algebras are considered in [13, 17, 18] and a more general framework bordering color and super Lie algebras was introduced in [13, 17, 18, 19]. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

In [21] and [22], the theory of Hom-coalgebras and related structures are developed. Further development could be found in [3, 4, 15].

The quadratic Lie algebras, also called metrizable or orthogonal, are intensively studied, one of the fundamental results of constructing and characterizing quadratic Lie algebras is due to Medina and Revoy (see [23]) using double extension, while the concept of T^*-extension is due to Bordemann (see [7]). The T^*-

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extension concerns nonassociative algebras with nondegenerate associative symmetric bilinear form, such algebras are called metrizable algebras. In [7], the metrizable nilpotent associative algebras and metrizable solvable Lie algebras are described. The study of graded quadratic Lie algebras could be found in [5]. Jacobi-Jordan algebras (JJ algebras for short) were introduced in [8] in 2014 as vector spaces $\mathcal{A}$ over a field k, equipped with a bilinear map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras the commutativity $x \cdot y = y \cdot x$, for all $x, y \in \mathcal{A}$ is imposed. This class of algebras appear under different names in the literature reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other's results (see [27] for more details). Wörz-Busekros in [26] relates these type of algebras with Bernstein algebras. One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [1, 24] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [8] the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. There's two entertaining facts on Jacobi-Jordan algebras. The first one is that in [1] prove that a finite dimensional JJ algebras is Frobenius if and only if there exists an invariant non degenerate bilinear form (Proposition 1.8). The other entertaining fact (noted in [25]) is that Jacobi-Jordan algebras can be produced from antiassociative algebras the same way as they are produced from associative ones. Hence there's a strong link in between antiassociative algebras and Jacobi-Jordan algebras. By antiassociative algebras, we mean algebras subject to operation $(a, b) \to ab$ satisfying $(ab)c + a(bc) = 0$ for each a, b and c. This class of algebras first arise in the literature specially in [25] where the authors gave their main properties. The purpose of this paper on the first hand is to study and construct quadratic Hom-Jacobi-Jordan algebras as S. Benayadi and A. Makhlouf did for the case of Lie algebra structures in [6]. On the other hands to establish a double construction of hom-antiassociative algebra equipped with a non degenerate symmetric invariant bilinear form.

In the first Section, we define the notions of Hom-Jacobi-Jordan algebras, Hom-antiassociative algebras and their related propreties. Some key constructions of Hom-Jacobi-Jordan algebras are derived. Section 2 is dedicated to a theory of representations of Hom-Jacobi-Jordan algebras including adjoint and coadjoint representations. In Section 3, we introduce the notion of quadratic Hom-Jacobi-Jordan algebra and give some properties. Several procedures of construction leading to some examples are provided in Section 4. We show in Section 5 that there exists biunivoque correspondence between some classes of Jacobi-Jordan algebras and classes of Hom-Jacobi-Jordan algebras. In Section 6, we introduce the concepts of matched pairs of hom-antiassociative algebras and establish some properties. In Section 7, we give and discuss of double constructions of multiplicative hom-antiassociative algebras. In section 8, we end with some concluding remarks.

2. Preliminaries

In the following we give the definitions of Hom-Jacobi-Jordan and Hom- anti-associative algebraic structures generalizing the well known Jacobi-Jordan and antiassociative algebras. Also we define in this case the notion of modules over Hom-algebras.

Throughout the article we let $\mathbb{K}$ be an algebraically closed field of characteristic 0. We mean by a Hom-algebra a triple (A, μ, α) consisting of a vector space A, a bilinear map μ and a linear map α. In all the examples involving the unspecified products are either symmetric or zero.

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [13, 17, 18] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. In this article, we follow notations from [20]. In this part, we analogously define the Hom-Jacobi-Jordan algebras which is a kind of deformation of Jacobi-Jordan algebras. But first let's recall the notions antiassociative and Jacobi-Jordan algebras.

Definition 2.1. [25] Let "$\cdot$" be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z). \tag{2.1}$$

Then, we call the pair $(\mathcal{A}, \cdot)$ an *antiassociative algebra.*

Definition 2.2. [27] An algebra $(\mathfrak{g}, [\ ,\ ])$ over K is called *Jacobi-Jordan* if it is commutative:

$$[x, y] = [y, x], \tag{2.2}$$

and satisfies the *Jacobi identity*:

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0 \tag{2.3}$$

for any x,y,$z \in \mathfrak{g}$.

Theorem 2.3. [27] *Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^\dagger$ with multiplication give by the "anticommutator"*

$$[a, b] = \frac{1}{2}\left(a \cdot b + b \cdot a\right), \tag{2.4}$$

is a Jacobi-Jordan algebra.

Since Jacobi-Jordan algebras are commutative, the left and right actions of an algebra coincide, so we can speak about just modules.

Definition 2.4. [27] A vector space V is a module over a Jacobi-Jordan algebra $\mathfrak{g}$, if there is a linear map (a representation) $\rho : \mathfrak{g} \to End(V)$ such that

$$\rho([x, y])(v) = -\rho(x)(\rho(y)v) - \rho(y)(\rho(x)v) \tag{2.5}$$

for any $x, y \in \mathfrak{g}$ and $v \in V$.

Definition 2.5. A *Hom-Jacobi-Jordan algebra* is a triple $(\mathfrak{g}, [\ ,\ ], \alpha)$ consisting of a linear space $\mathfrak{g}$ on which $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ a linear map satisfying

$$[x, y] = [y, x], \quad \text{(symmetry)} \tag{2.6}$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi condition)} \tag{2.7}$$

for all x, y, z from $\mathfrak{g}$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

We recover classical Jacobi-Jordan algebra when $\alpha = id_{\mathfrak{g}}$ and the identity (2.7) is the Jacobi identity in this case.

Proposition 2.6. *Every symmetric bilinear map on a 2-dimensional linear space defines a Hom-Jacobi-Jordan algebra.*

Proof. The Hom-Jacobi identity (2.7) is satisfied for any triple (x, x, y). □

Let $(\mathfrak{g}, \mu, \alpha)$ and $\mathfrak{g}' = (\mathfrak{g}', \mu', \alpha')$ be two Hom-Jacobi-Jordan algebras. A linear map $f\ : \mathfrak{g} \to \mathfrak{g}'$ is a *morphism of Hom-Jacobi-Jordan algebras* if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

In particular, Hom-Jacobi-Jordan algebras $(\mathfrak{g}, \mu, \alpha)$ and $(\mathfrak{g}, \mu', \alpha')$ are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1} \circ \mu' \circ (f \otimes f) \quad \text{and} \quad \alpha = f^{-1} \circ \alpha' \circ f.$$

A subspace I of $\mathfrak{g}$ is said to be an ideal if for $x \in I$ and $y \in \mathfrak{g}$ we have $[x, y] \in I$ and $\alpha(x) \in I$. A Hom-Jacobi-Jordan algebra in which the anticommutator is not identically zero and which has no proper ideals is called simple.

Example 2.7. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional linear space $\mathfrak{g}$ over $\mathbb{K}$. The following bracket and linear map α on $\mathfrak{g} = \mathbb{K}^3$ define a Hom-Jacobi-Jordan algebra over $\mathbb{K}$:

$$\begin{array}{lll} [x_1, x_1] = -bx_3, & [x_1, x_2] = b(-x_1 + \frac{1}{2}x_3), & \alpha(x_1) = x_1, \\ [x_2, x_2] = ax_3, & [x_1, x_3] = \frac{b}{2}x_2, & \alpha(x_2) = 2x_2, \\ [x_3, x_3] = ax_3, & [x_2, x_3] = 2(ax_1 + bx_3), & \alpha(x_3) = 2x_3 \end{array}$$

with $[x_2, x_1]$, $[x_3, x_1]$ and $[x_3, x_2]$ defined via symmetry. It's a Jacobi-Jordan algebra only in case $b = 0$ and $a = 0$ or $b = 0$ and $a \neq 0$, since

$$[x_1, [x_2, x_3]] + [x_3, [x_1, x_2]] + [x_2, [x_3, x_1]] = \frac{b^2}{2}x_2 + abx_3.$$

For simplicity we will use in the sequel the following terminology and notations.

Definition 2.8. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Jacobi-Jordan algebra. The Hom-algebra is called

- a *multiplicative Hom-Jacobi-Jordan algebra* if for all $x, y \in \mathfrak{g}$ we have
$$\alpha([x,y]) = [\alpha(x), \alpha(y)];$$
- a *regular Hom-Jacobi-Jordan algebra* if α is an automorphism;
- an *involutive Hom-Jacobi-Jordan algebra* if α is an involution, that is $\alpha^2 = id$.

The *center* of the Hom-Jacobi-Jordan algebra is denoted $\mathcal{Z}(\mathfrak{g})$ and defined by

$$\mathcal{Z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x,y] = 0 \ \forall y \in \mathfrak{g}\}.$$

We give in the following the definition of Hom-antiassociative algebra which provide a different way for constructing Hom-Jacobi-Jordan algebras by extending the fundamental construction of Jacobi-Jordan algebras from antiassociative algebras via anticommutator bracket multiplication.

Definition 2.9. A Hom-antiassociative algebra is a triple (A, μ, α) consisting of a linear space A, $\mu : A \times A \to A$ is a bilinear map and $\alpha : A \to A$ is a linear map, satisfying
$$\mu(\alpha(x), \mu(y,z)) = -\mu(\mu(x,y), \alpha(z)). \tag{2.8}$$

We can talk about functor from the category of Hom-antiassociative algebras in the category of Hom-Jacobi-Jordan algebras.

Proposition 2.10. *Let (A, μ, α) be a Hom-antiassociative algebra defined on the linear space A by the multiplication μ and a homomorphism α. Then the triple $(A, [\ ,\ ], \alpha)$, where the bracket is defined for $x, y \in A$ by $[x,y] = \mu(x,y) + \mu(y,x)$, is a Hom-Jacobi-Jordan algebra.*

Proof. The bracket is obviously symmetric and with a direct computation we have

$$\begin{aligned}
&[\alpha(x),[y,z]] + [\alpha(z),[x,y]] + [\alpha(y),[z,x]] \\
&\quad = \mu(\alpha(x),\mu(y,z)) + \mu(\alpha(x),\mu(z,y)) + \mu(\mu(y,z),\alpha(x)) + \mu(\mu(z,y),\alpha(x)) \\
&\quad + \mu(\alpha(z),\mu(x,y)) + \mu(\alpha(z),\mu(y,x)) + \mu(\mu(x,y),\alpha(z)) + \mu(\mu(y,x),\alpha(z)) \\
&\quad + \mu(\alpha(y),\mu(z,x)) + \mu(\alpha(y),\mu(x,z)) + \mu(\mu(z,x),\alpha(y)) + \mu(\mu(x,z),\alpha(y)) = 0. \ \square
\end{aligned}$$

A structure of module over Hom-associative algebras is defined in [21] and [22]. Here we define the analogous notion over Hom-antiassociative algebras as follows.

Definition 2.11. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-antiassociative algebra. A (left) $\mathcal{A}$-module is a triple (M, f, γ) where M is a $\mathbb{K}$-vector space and f, γ are $\mathbb{K}$-linear maps, $f : M \to M$ and $\gamma : \mathcal{A} \otimes M \to M$, such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes M & \xrightarrow{\mu \otimes f} & \mathcal{A} \otimes M \\
\downarrow{\scriptstyle \alpha \otimes \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{A} \otimes M & \xrightarrow{\gamma} & M
\end{array}$$

Remark 2.12. A Hom-antiassociative algebra $(\mathcal{A}, \mu, \alpha)$ is a left $\mathcal{A}$-module with $M = \mathcal{A}$, $f = \alpha$ and $\gamma = \mu$.

The following result shows that Jacobi-Jordan algebras deform into Hom-Jacobi-Jordan algebras via endomorphisms.

Theorem 2.13. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Jacobi-Jordan algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Jacobi-Jordan algebra endomorphism. Then $\mathfrak{g}_\alpha = (\mathfrak{g}, [\ ,\ ]_\alpha, \alpha)$ is a Hom-Jacobi-Jordan algebra, where $[\ ,\ ]_\alpha = \alpha \circ [\ ,\ ]$. Moreover, suppose that $(\mathfrak{g}', [\ ,\ ]')$ is another Jacobi-Jordan algebra and $\alpha' : \mathfrak{g}' \to \mathfrak{g}'$ is a Jacobi-Jordan algebra endomorphism. If $f : \mathfrak{g} \to \mathfrak{g}'$ is a Jacobi-Jordan algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then*

$$f : (\mathfrak{g}, [\ ,\ ]_\alpha, \alpha) \longrightarrow (\mathfrak{g}', [\ ,\ ]'_{\alpha'}, \alpha')$$

is a morphism of Hom-Jacobi-Jordan algebras.

Proof. Observe that $[\alpha(x), [y, z]_\alpha]_\alpha = \alpha[\alpha(x), \alpha[y, z]] = \alpha^2[x, [y, z]]$. Therefore the Hom-Jacobi identity for $\mathfrak{g}_\alpha = (\mathfrak{g}, [\,,\,]_\alpha, \alpha)$ follows obviously from the Jacobi identity of $(\mathfrak{g}, [\,,\,])$. The symmetry and the second assertion are proved similarly. □

In the sequel we denote by $\mathfrak{g}_\alpha$ the Hom-Jacobi-Jordan algebra $(\mathfrak{g}, \alpha \circ [\ ,\ ], \alpha)$ corresponding to a given Jacobi-Jordan algebra $(\mathfrak{g}, [\ ,\ ])$ and an endomorphism α. We say that the Hom-Jacobi-Jordan algebra is obtained by composition.

Proposition 2.14. *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a regular Hom-Jacobi-Jordan algebra. Then $(\mathfrak{g}, [\ ,\ ]_{\alpha^{-1}} = \alpha^{-1} \circ [\ ,\ ])$ is a Jacobi-Jordan algebra.*

Proof. It follows from
$\circlearrowleft_{x,y,z}[x, [y, z]_{\alpha^{-1}}]_{\alpha^{-1}} = \circlearrowleft_{x,y,z}\alpha^{-1}([x, \alpha^{-1}([y, z])]) = \circlearrowleft_{x,y,z}\alpha^{-2}[\alpha(x), [y, z]] = 0.$ □

Remark 2.15. In particular the proposition is valid when α is an involution.

We may also derive new Hom-Jacobi-Jordan algebras from a given multiplicative Hom-Jacobi-Jordan algebra using the following procedure.

Definition 2.16. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a multiplicative Hom-Jacobi-Jordan algebra and $n \geqslant 0$. The nth derived Hom-algebra of $\mathfrak{g}$ is defined by

$$\mathfrak{g}_{(n)} = \left(\mathfrak{g}, [\ ,\ ]^{(n)} = \alpha^n \circ [\ ,\ ], \alpha^{n+1}\right). \tag{2.9}$$

Note that $\mathfrak{g}_{(0)} = \mathfrak{g}$ and $\mathfrak{g}_{(1)} = \big(\mathfrak{g}, [\ ,\ ]^{(1)} = \alpha \circ [\ ,\ ], \alpha^2\big)$.

Observe that for $n \geqslant 1$ and $x, y, z \in \mathfrak{g}$ we have

$$[[x, y]^{(n)}, \alpha^{n+1}(z)]^{(n)} = \alpha^n([\alpha^n([x, y]), \alpha^{n+1}(z)]) = \alpha^{2n}([[x, y], \alpha(z)]).$$

Hence, one obtains the following result.

Theorem 2.17. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a multiplicative Hom-Jacobi-Jordan algebra. Then its nth derived Hom-algebra is a Hom-Jacobi-Jordan algebra.*

In the following we construct Hom-Jacobi-Jordan algebras involving elements of the centroid of Jacobi-Jordan algebras. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Jacobi-Jordan algebra. An endomorphism $\theta \in End(\mathfrak{g})$ is said to be an element of the centroid if $\theta[x,y] = [\theta(x), y]$ for any $x, y \in \mathfrak{g}$. The centroid is defined by

$$Cent(\mathfrak{g}) = \{\theta \in End(\mathfrak{g}) : \theta[x,y] = [\theta(x), y],\ \forall x, y \in \mathfrak{g}\}.$$

The same definition is assumed for Hom-Jacobi-Jordan algebra.

Proposition 2.18. *Let* $(\mathfrak{g}, [\cdot,\cdot])$ *be a Jacobi-Jordan algebra and* $\theta \in Cent(\mathfrak{g})$. *Set for* $x, y \in \mathfrak{g}$

$$\{x, y\} = [\theta(x), y],$$
$$[x, y]_\theta = [\theta(x), \theta(y)].$$

Then $(\mathfrak{g}, \{\cdot,\cdot\}, \theta)$ *and* $(\mathfrak{g}, [\cdot,\cdot]_\theta, \theta)$ *are Hom-Jacobi-Jordan algebras.*

Proof. For $\theta \in Cent(\mathfrak{g})$ we have $[\theta(x), y] = \theta([x,y]) = \theta([y,x]) = [\theta(y), x] = [x, \theta(y)]$. Then

$$\{x, y\} = [\theta(x), y] = [\theta(y), x] = \theta[y, x] = \{y, x\}.$$

Also we have

$$\begin{aligned}\{\theta(x), \{y, z\}\} &= [\theta^2(x), \{y, z\}] = [\theta^2(x), [\theta(y), z]] \\ &= \theta([\theta(x), [\theta(y), z]]) = [\theta(x), \theta([\theta(y), z])] \\ &= [\theta(x), [\theta(y), \theta(z)]].\end{aligned}$$

It follows $\circlearrowleft_{x,y,z} \{\theta(x), \{y, z\}\} = \circlearrowleft_{x,y,z} [\theta(x), [\theta(y), \theta(z)]] = 0$ since $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra. Therefore the Hom-Jacobi is satisfied. Thus $(\mathfrak{g}, \{\cdot,\cdot\}, \theta)$ is a Hom-Jacobi-Jordan algebra.

Similarly we have the symmetry and the Hom-Jacobi identity satisfied for $(\mathfrak{g}, [\cdot,\cdot]_\theta, \theta)$. Indeed

$$[x, y]_\theta = [\theta(x), \theta(y)] = [\theta(y), \theta(x)] = [y, x]_\theta,$$

and

$$[\theta(x), [y, z]_\theta]_\theta = [\theta^2(x), \theta([y, z]_\theta)] = [\theta^2(x), \theta([\theta(y), \theta(z)]] = \theta^2([\theta(x), [\theta(y), \theta(z)]],$$

which leads to $\circlearrowleft_{x,y,z} [\theta(x), [y, z]_\theta]_\theta = \theta^2(\circlearrowleft_{x,y,z} [\theta(x), [\theta(y), \theta(z)]]) = 0.$ □

3. Representations of Hom-Jacobi-Jordan Algebras

In this section we introduce a representation theory of Hom-Jacobi-Jordan algebras and discuss the cases of adjoint and coadjoint representations for Hom-Jacobi-Jordan algebras.

Definition 3.1. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Jacobi-Jordan algebra. A *representation* of $\mathfrak{g}$ is a triple (V, ρ, β), where V is a $\mathbb{K}$-vector space, $\beta \in End(V)$ and $\rho : \mathfrak{g} \to End(V)$ is a linear map satisfying

$$\rho([x, y]) \circ \beta = -\rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x) \quad \forall x, y \in \mathfrak{g} \tag{3.1}$$

One recovers the definition of a representation in the case of Jacobi-Jordan algebras by setting $\alpha = Id_{\mathfrak{g}}$ and $\beta = Id_V$.

Definition 3.2. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Jacobi-Jordan algebra. Two representations (V, ρ, β) and (V', ρ', β') of $\mathfrak{g}$ are said to be *isomorphic* if there exists a linear map $\phi\ : V \to V'$ such that

$$\forall x \in \mathfrak{g} \quad \rho'(x) \circ \phi = \phi \circ \rho(x) \qquad \text{and} \qquad \phi \circ \beta = \beta' \circ \phi.$$

Proposition 3.3. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \alpha)$ be a Hom-Jacobi-Jordan algebra and (V, ρ, β) be a representation of $\mathfrak{g}$. The direct summand $\mathfrak{g} \oplus V$ with a bracket defined by*

$$[x + u, y + w] := [x, y]_{\mathfrak{g}} + \rho(x)(w) + \rho(y)(u) \quad \forall x, y \in \mathfrak{g}\ \forall u, w \in V \tag{3.2}$$

and the twisted map $\gamma : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ defined by

$$\gamma(x + w) = \alpha(x) + \beta(u) \quad \forall x \in \mathfrak{g}\ \forall u \in V. \tag{3.3}$$

is a Hom-Jacobi-Jordan algebra.

Proof. The symmetry of the bracket is obvious. We show that the Hom-Jacobi identity is satisfied:

Let $x, y, z \in \mathfrak{g}$ and $\forall u, v, w \in V$.

$$\begin{aligned}
& \circlearrowleft_{(x,u),(y,v),(z,w)} [\gamma(x + u), [y + v, z + w]] \\
&= \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x) + \beta(u), [y, z]_{\mathfrak{g}} + \rho(y)(w) - \rho(z)(v)] \\
&= \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} + \rho(\alpha(x)(\rho(y)(w) - \rho(z)(v)) + \rho([y, z]_{\mathfrak{g}})(\beta(u)) \\
&= \circlearrowleft_{(x,u),(y,v),(z,w)} \rho(\alpha(x)(\rho(y)(w)) + \rho(\alpha(x)(\rho(z)(v)) + \rho(\alpha(y)(\rho(z)(u)) \\
&\quad +\rho(\alpha(z)(\rho(y)(u)) \\
&= \rho(\alpha(x)(\rho(y)(w)) + \rho(\alpha(x)(\rho(z)(v)) + \rho(\alpha(y)(\rho(z)(u))\rho(\alpha(z)(\rho(y)(u)) \\
&\quad +\rho(\alpha(y)(\rho(z)(u)) + \rho(\alpha(y)(\rho(x)(w)) + \rho(\alpha(z)(\rho(x)(v)) + \rho(\alpha(x)(\rho(z)(v)) \\
&\quad +\rho(\alpha(z)(\rho(x)(v)) + \rho(\alpha(z)(\rho(y)(u)) + \rho(\alpha(x)(\rho(y)(w)) + \rho(\alpha(y)(\rho(x)(w)) \\
&= 0,
\end{aligned}$$

where $\circlearrowleft_{(x,u),(y,v),(z,w)}$ denotes summation over the cyclic permutation on $(x, u), (y, v), (z, w)$. □

Now, we discuss the adjoint representations of a Hom-Jacobi-Jordan algebra.

Proposition 3.4. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a Hom-Jacobi-Jordan algebra and* $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$ *be an operator defined for* $x \in \mathfrak{g}$ *by* $\mathrm{ad}(x)(y) = [x, y]$. *Then* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *is a representation of* $\mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is a Hom-Jacobi-Jordan algebra, the Hom-Jacobi condition on $x, y, z \in \mathfrak{g}$ is

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$$

and may be written as

$$ad[x, y](\alpha(z)) = -ad(\alpha(x))(ad(y)(z)) - ad(\alpha(y))(ad(x)(z)).$$

Then the operator ad satisfies

$$ad[x, y] \circ \alpha = -ad(\alpha(x)) \circ ad(y) - ad(\alpha(y)) \circ (ad(x).$$

Therefore, it determines a representation of the Hom-Jacobi-Jordan algebra $\mathfrak{g}$. $\square$

We call the representation defined in the previous proposition the *adjoint representation* of the Hom-Jacobi-Jordan algebra.

In the following, we explore the dual representations and coadjoint representations of Hom-Jacobi-Jordan algebras.

Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Jacobi-Jordan algebra and (V, ρ, β) be a representation of $\mathfrak{g}$. Let V^* be the dual vector space of V.

We define a linear map $\widetilde{\rho} : \mathfrak{g} \to End(V^*)$ by $\widetilde{\rho}(x) = -{}^t\rho(x)$.

Let $f \in V^*$, $x, y \in \mathfrak{g}$ and $u \in V$. We compute the right hand side of the identity (3.1)

$$\begin{aligned}-(\widetilde{\rho}(\alpha(x))\circ\widetilde{\rho}(y)-\widetilde{\rho}(\alpha(y))\circ\widetilde{\rho}(x))(f)(u) &= -(\widetilde{\rho}(\alpha(x))(\widetilde{\rho}(y)(f))-\widetilde{\rho}(\alpha(y))(\widetilde{\rho}(x)(f)))(u)\\ &= \widetilde{\rho}(y)(f)(\rho(\alpha(x))(u))+\widetilde{\rho}(x)(f)(\rho(\alpha(y))(u))\\ &= -f(\rho(y)\rho(\alpha(x))(u)) - f(\rho(x)\rho(\alpha(y))(u))\\ &= -f(\rho(y)\rho(\alpha(x)) - \rho(x)\rho(\alpha(y))(u)).\end{aligned}$$

On the other hand, we set that the twisted map for $\widetilde{\rho}$ is $\widetilde{\beta} ={}^t \beta$, then the left hand side of (3.1) writes

$$((\widetilde{\rho}([x, y])\widetilde{\beta})(f))(u) = (\widetilde{\rho}([x, y])(f \circ \beta)(u) = -f \circ \beta(\rho([x, y])(u)).$$

Therefore, we have the following proposition:

Proposition 3.5. *Let* $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ *be a Hom-Jacobi-Jordan algebra and* (V, ρ, β) *be a representation of* $\mathfrak{g}$. *The triple* $(V^*, \widetilde{\rho}, \widetilde{\beta})$, *where* $\widetilde{\rho} : \mathfrak{g} \to End(V^*)$ *is given by* $\widetilde{\rho}(x) = -{}^t\rho(x)$, *defines a representation of the Hom-Jacobi-Jordan algebra* $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ *if and only if*

$$\beta \circ \rho([x, y]) = -\rho(x)\rho(\alpha(y)) - \rho(y)\rho(\alpha(x)). \tag{3.4}$$

We obtain the following characterization in the case of adjoint representation.

Corollary 3.6. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a Hom-Jacobi-Jordan algebra and* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *be the adjoint representation of* $\mathfrak{g}$*, where* $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$*. We set* $\widetilde{\mathrm{ad}} : \mathfrak{g} \to End(\mathfrak{g}^*)$ *and* $\widetilde{\mathrm{ad}}(x)(f) = -f \circ \mathrm{ad}(x)$*. Then* $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ *is a representation of* $\mathfrak{g}$ *if and only if*

$$\alpha([[x, y], z]) = [x, [\alpha(y), z]] + [y, [\alpha(x), z]] \quad \forall x, y, z \in \mathfrak{g}. \tag{3.5}$$

4. Quadratic Hom-Jacobi-Jordan Algebras

In this section we extend the notion of quadratic Jacobi-Jordan algebra to Hom-Jacobi-Jordan algebras and provide some properties. But let's first define quadratic Jacobi-Jordan algebra.

Definition 4.1. Let $(\mathfrak{g}, [\ ,\ ])$ be a Jacobi-Jordan algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a symmetric nondegenerate bilinear form satisfying

$$B([x, y], z) = B(x, [y, z]) \quad \forall x, y, z \in \mathfrak{g}. \tag{4.1}$$

The identity (4.1) may be written $B([x, y], z) = -B(y, [x, z])$ and is called an *invariance* of B. The triple $(\mathfrak{g}, [\ ,\ ], B)$ is called the *quadratic Jacobi-Jordan algebra.*

More generally, for nonassociative algebras $(A, \cdot)$, a triple $(A, \cdot, B)$ where B is a symmetric nondegenerate bilinear form satisfying

$$B(x \cdot y, z) = B(x, y \cdot z) \quad \forall x, y, z \in A \tag{4.2}$$

defines a quadratic algebra, called also metrizable algebra. A bilinear form B satisfying (4.2) is said to be invariant form.

Definition 4.2. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Jacobi-Jordan algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be an invariant symmetric nondegenerate bilinear form satisfying

$$B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in \mathfrak{g}. \tag{4.3}$$

The quadruple $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ is called a *quadratic Hom-Jacobi-Jordan algebra.*

If α is an involution (resp. invertible), the quadratic Hom-Jacobi-Jordan algebra is said to be involutive (resp. regular) quadratic Hom-Jacobi-Jordan algebra and we write for shortness IQH-Jacobi-Jordan algebra (resp. RQH-Jacobi-Jordan algebra).

We recover the notion of quadratic Jacobi-Jordan algebra when α is the identity map. One may consider a larger class with a definition without condition (4.3). We may also introduce in the following a generalized quadratic Hom-Jacobi-Jordan algebra notion where the invariance is twisted by a linear map.

Definition 4.3. A Hom-Jacobi-Jordan algebra $(\mathfrak{g}, [\ ,\ ], \alpha)$ is called *Hom-quadratic* if there exist a pair (B, γ) where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is a symmetric nondegenerate bilinear form and $\gamma : \mathfrak{g} \to \mathfrak{g}$ is a linear map satisfying

$$B([x, y], \gamma(z)) = -B(\gamma(y), [x, z]) \quad \forall x, y, z \in \mathfrak{g}. \tag{4.4}$$

We call the identity (4.4) the γ-invariance of B. We recover the quadratic Hom-Jacobi-Jordan algebras when $\gamma = id$.

Proposition 4.4. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Jacobi-Jordan algebra. If there exists $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form such that the quadruple $(\mathfrak{g}, [\cdot,\cdot], \alpha, B)$ is a quadratic Hom-Jacobi-Jordan algebra then*

1. $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ *is a representation of* $\mathfrak{g}$.
2. *The representations* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *and* $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ *are isomorphic.*

Proof. To prove the first assertion, we should show that for any z we have

$$\alpha \circ ad([x, y])(z) + \rho(x)ad(\alpha(y))(z) + ad(y)ad(\alpha(x))(z) = 0, \tag{4.5}$$

that is

$$\alpha[[x, y], z] + [x, [\alpha(y), z]] + [y, [\alpha(x), z]] = 0.$$

Let $u \in \mathfrak{g}$

$$\begin{aligned}
&B(\alpha[[x, y], z] + [x, [\alpha(y), z]] + [y, [\alpha(x), z]], u)\\
&\qquad = B(\alpha[[x, y], z], u) + B([x, [\alpha(y), z]], u) + B([y, [\alpha(x), z]], u)\\
&\qquad = B([[x, y], z], \alpha(u)) - B([\alpha(y), z], [x, u]) - B([\alpha(x), z], [y, u])\\
&\qquad = -(B(z, [[x, y], \alpha(u)]) + B(z, [\alpha(y), [x, u]]) + B(z, [\alpha(x), [y, u]]))\\
&\qquad = -(B(z, [[x, y], \alpha(u)] + [\alpha(y), [x, u]]) + [\alpha(x), [y, u]])\\
&\qquad = -(B(z, [\alpha(u), [y, x]]) + [\alpha(y), [x, u]]) + [\alpha(x), [u, y]]))\\
&\qquad = 0.
\end{aligned}$$

This proves (4.5) since B is nondegenerate.

For the second assertion let's consider the map $\phi : \mathfrak{g} \to \mathfrak{g}^\star$ defined by $x \to B(x, \cdot)$ which is bijective since B is nondegenerate. It's obvious to prove that ϕ is also a module morphism. □

Definition 4.5. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra.

1. An ideal I of $\mathfrak{g}$ is said to be *nondegenerate* if $B_{|I\times I}$ is nondegenerate.
2. The quadratic Hom-Jacobi-Jordan algebra is said to be *irreducible* (or *B-irreducible*) if $\mathfrak{g}$ doesn't contain any nondegenerate ideal I such that $I \neq \{0\}$ and $I \neq \mathfrak{g}$.
3. Let I be an ideal of $\mathfrak{g}$. The *orthogonal* $I^\perp$ of I with respect to B is defined by $\{x \in \mathfrak{g} : B(x, y) = 0\ \forall y \in I\}$.

Remark 4.6. Let I be a nondegenerate ideal of a quadratic Hom-Jacobi-Jordan algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha, B)$. Then $(I, [\ ,\ ]_{|I\times I}, \alpha_{|}I, B_{|I\times I})$ is a quadratic Hom-Jacobi-Jordan algebra.

Lemma 4.7. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Jacobi-Jordan algebra. Then the center $\mathcal{Z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.*

Proof. We have $[\mathfrak{g}, \mathcal{Z}(\mathfrak{g})] = \{0\} \subseteq \mathcal{Z}(\mathfrak{g})$. Let $x \in \mathcal{Z}(\mathfrak{g})$ and $y \in \mathfrak{g}$. For any $z \in \mathfrak{g}$ the invariance and the symmetry of B leads to $B([\alpha(x), y], z) = B(\alpha(x), [y, z]) = B(x, \alpha([y, z])) = B(x, [\alpha(y), \alpha(z)]) = B([x, \alpha(y)], \alpha(z)]) = 0$ (since $x \in \mathcal{Z}(\mathfrak{g})$).

Then for any $y \in \mathfrak{g}$ we have $[\alpha(x), y] = 0$ since B is nondegenerate. Thus $\alpha(x) \in \mathcal{Z}(\mathfrak{g})$. □

Lemma 4.8. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra and I be an ideal of $\mathfrak{g}$. Then the orthogonal $I^\perp$ of I with respect to B is an ideal of $\mathfrak{g}$.*

Proof. It is clear that $[\mathfrak{g}, I^\perp] \subseteq I^\perp$. Let $y \in I$ and $z \in I^\perp$, then $B(\alpha(y), z) = B(y, \alpha(z)) = 0$ since $\alpha(I) \subseteq I$. We conclude that $I^\perp$ is an ideal of $\mathfrak{g}$. □

Proposition 4.9. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ such that*

1. *$\mathfrak{g}_i$ is an irreducible ideal of $\mathfrak{g}$, for any $i \in \{1, \cdots, n\}$,*
2. *$B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$, for any $i, j \in \{1, \cdots, n\}$ such that $i \neq j$,*
3. *$(\mathfrak{g}_i, [\cdot,\cdot]_{|\mathfrak{g}_i\times\mathfrak{g}_i}, \alpha_{|\mathfrak{g}_i}, B_{|\mathfrak{g}_i\times\mathfrak{g}_i})$ is an irreducible quadratic Hom-Jacobi-Jordan algebra.*

Proof. By induction on the dimension of $\mathfrak{g}$. □

Now, let $\mathfrak{g} = (\mathfrak{g}, [\ ,\ ], \alpha, B)$ be a quadratic multiplicative Hom-Jacobi-Jordan algebra. We provide in the following some observations.

Proposition 4.10. *If the linear map α is an automorphism and the center $\mathcal{Z}(\mathfrak{g}) = \{0\}$ then α is an involution i.e. $\alpha^2 = id$.*

Proof. For $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned} B([\alpha(x), y], z) &= B(\alpha(x), [y, z]) = B(x, \alpha([y, z]) \\ &= B(x, [\alpha(y), \alpha(z)]) = B([x, \alpha(y)], \alpha(z)) \\ &= B(\alpha([x, \alpha(y)]), z) = B([\alpha(x), \alpha^2(y)], z). \end{aligned}$$

Then $B([\alpha(x), y] - [\alpha(x), \alpha^2(y)], z) = 0$ which may be written $B([\alpha(x), y - \alpha^2(y)], z) = 0$. Hence, for any $x, y \in \mathfrak{g}$ we have $[\alpha(x), (id - \alpha^2)(y)] = 0$. Since α is bijective and $\mathcal{Z}(\mathfrak{g}) = \{0\}$ then $\alpha^2 = id$. □

Proposition 4.11. *There exist two nondegenerate ideals I, J of $\mathfrak{g} = (\mathfrak{g}, [\ ,\ ], \alpha, B)$ such that*

1. $B(I, J) = \{0\}$,
2. $\mathfrak{g} = I \oplus J$,
3. $\alpha_{|I}$ *is nilpotent and* $\alpha_{|J}$ *is invertible.*

Proof. The fitting decomposition with respect to the linear map α leads to the existence of an integer n such that $\mathfrak{g} = I \oplus J$, where $I = Ker(\alpha^n)$ and $J = Im(\alpha^n)$, such that $\alpha(I) \subseteq I$, $\alpha(J) \subseteq I$, $\alpha_{|I}$ is nilpotent and $\alpha_{|J}$ is invertible.

Let $x \in \mathfrak{g}$, $y \in I$. We have $\alpha^n([x, y]) = [\alpha^n(x), \alpha^n(y)] = 0$ since $\alpha^n(y) = 0$, and $[x, y] \in I$. Then $[\mathfrak{g}, I] \subseteq I$. In addition $\alpha^n(\alpha(y)) = \alpha^{n+1}(y) = 0$ which implies that $\alpha(y) \in Ker(\alpha^n)$. Therefore I is an ideal of $\mathfrak{g}$.

Let $x, y \in J$ then there exist $x', y' \in \mathfrak{g}$ such that $x = \alpha^n(x')$ and $y = \alpha^n(y')$. We have $[x, y] = [\alpha^n(x'), \alpha^n(y')] = \alpha^n([x', y']) \in J$. In addition $\alpha(J) \subseteq J$. Therefore J is a subalgebra.

Let $x \in I$ and $y \in J$. There exists $y' \in \mathfrak{g}$ such that $y = \alpha^n(y')$. For any $z \in \mathfrak{g}$, we have $B([x, y], z) = -B([y, x], z) = -B(y, [x, z]) = -B(\alpha^n(y'), [x, z]) = -B(y', \alpha^n([x, z]) = -B(y', [\alpha^n(x), \alpha^n(z)]) = 0$. Then $[x, y] = 0$, since B is a nondegenerate bilinear form. We conclude that $I = Im(\alpha^n)$ is an ideal of $\mathfrak{g}$ and $[I, J] = 0$.

Now let $x \in I$ and $y = \alpha^n(y') \in J$, where $y' \in \mathfrak{g}$. We have $B(x, y) = B(x, \alpha^n(y')) = B(\alpha^n(x), y') = 0$ since $\alpha^n(x) = 0$. Therefore $B(I, J) = 0$. □

Corollary 4.12. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Jacobi-Jordan algebra which is B-irreducible. Then either α is nilpotent or α is an automorphism of $\mathfrak{g}$.*

5. Constructions and Examples

We show in the following some constructions leading to some examples of quadratic Hom-Jacobi-Jordan algebras. We use Theorem 2.13 and Theorem 2.17 to provide some classes of quadratic Hom-Jacobi-Jordan algebras starting from an ordinary quadratic Jacobi-Jordan algebras, respectively from any multiplicative quadratic Hom-Jacobi-Jordan algebra. Also we provide constructions using elements in the centroid of a Jacobi-Jordan algebras and constructions of T^*-extension type.

Let $(\mathfrak{g}, [\ ,\ ], B)$ be a quadratic Jacobi-Jordan algebra. We denote by $Aut_S(\mathfrak{g}, B)$ the set of symmetric automorphisms of $\mathfrak{g}$ with respect of B, that is automorphisms $f : \mathfrak{g} \to \mathfrak{g}$ such that $B(f(x), y) = B(x, f(y))$, $\forall x, y \in \mathfrak{g}$.

Proposition 5.1. *Let $(\mathfrak{g}, [\ ,\ ], B)$ be a quadratic Jacobi-Jordan algebra and $\alpha \in Aut_S(\mathfrak{g}, B)$. Then $\mathfrak{g}_\alpha = (\mathfrak{g}, [\ ,\ ]_\alpha, \alpha, B_\alpha)$, where for any $x, y \in \mathfrak{g}$*

$$[x, y]_\alpha = [\alpha(x), \alpha(y)] \tag{5.1}$$

$$B_\alpha(x, y) = B(\alpha(x), y), \tag{5.2}$$

is a quadratic Hom-Jacobi-Jordan algebra.

Proof. The triple $(\mathfrak{g}, [\ ,\ ]_\alpha, \alpha)$ is a Hom-Jacobi-Jordan algebra by Theorem 2.13.

The linear form B_α is nondegenerate since B is nondegenerate and α bijective.

We show that the identity (4.1) is satisfied by $\mathfrak{g}_\alpha = (\mathfrak{g}, [\ ,\ ]_\alpha, \alpha, B_\alpha)$. Let $x, y, z \in \mathfrak{g}$, then

$$\begin{aligned} B_\alpha([x,y]_\alpha, z) &= B(\alpha([\alpha(x), \alpha(y)]), z) = B([\alpha(x), \alpha(y)], \alpha(z)) \\ &= B(\alpha(x), [\alpha(y), \alpha(z)]) \qquad \text{(Invariance of } B) \\ &= B(\alpha(x), [y,z]_\alpha) = B_\alpha(x, [y,z]_\alpha). \end{aligned}$$

Therefore B_α is invariant.

We have $\alpha \in Aut_S(\mathfrak{g}_\alpha, B_\alpha)$. Indeed

$$\alpha([x,y]_\alpha) = \alpha([\alpha(x), \alpha(y)]) = [\alpha^2(x), \alpha^2(y)] = [\alpha(x), \alpha(y)]_\alpha,$$

and

$$B_\alpha(\alpha(x), y) = B(\alpha(\alpha(x)), y) = B(\alpha(x), \alpha(y)) = B_\alpha(x, \alpha(y)). \qquad \square$$

The following theorem permits to obtain new quadratic Hom-Jacobi-Jordan algebras starting from a multiplicative quadratic Hom-Jacobi-Jordan algebra.

Proposition 5.2. *Let $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ be a multiplicative quadratic Hom-Jacobi-Jordan algebra. For any $n \geqslant 0$, the quadruple*

$$\mathfrak{g}_{(n)} = \Big(\mathfrak{g}, [\ ,\ ]^{(n)} = \alpha^n \circ [\ ,\ ], \alpha^{n+1}, B_{\alpha^n}\Big), \tag{5.3}$$

where B_{α^n} is defined for $x, y \in \mathfrak{g}$ by $B_{\alpha^n}(x,y) = B(\alpha^n(x), y)$, determine a multiplicative quadratic Hom-Jacobi-Jordan algebra.

Proof. The triple $\mathfrak{g}_{(n)} = \big(\mathfrak{g}, [\ ,\ ]^{(n)} = \alpha^n \circ [\ ,\ ], \alpha^{n+1}\big)$ is a Hom-Jacobi-Jordan algebra by Theorem 2.17.

Since $\alpha \in Aut(\mathfrak{g})$ by induction we have $\alpha^n \in Aut(\mathfrak{g})$. The bilinear form B_{α^n} is nondegenerate because B is nondegenerate and α^n is bijective. It is is symmetric. Indeed

$$B_{\alpha^n}(x,y) = B(\alpha^n(x), y) = B(x, \alpha^n(y)) = B(\alpha^n(y), x) = B_{\alpha^n}(y,x).$$

The invariance of B_{α^n} is given by

$$\begin{aligned} B_{\alpha^n}([x,y]^n, z) &= B(\alpha^n \circ \alpha^n([x,y]), z) = B(\alpha^n([x,y]), \alpha^n(z)) = B([\alpha^n(x), \alpha^n(y)], \alpha^n(z)) \\ &= B(\alpha^n(x), [\alpha^n(y), \alpha^n(z)]) = B(\alpha^n(x), \alpha^n([y,z])) = B_{\alpha^n}(x, [y,z]^n). \end{aligned}$$

We have also $B_{\alpha^n}(\alpha^n(x), y) = B_{\alpha^n}(x, \alpha^n(y))$, indeed

$$B_{\alpha^n}(\alpha^n(x), y) = B(\alpha^{2n}(x), y) = B(\alpha^n(x), \alpha^n(y)) = B_{\alpha^n}(x, \alpha^n(y)). \qquad \square$$

We provide here a construction a Hom-Jacobi-Jordan algebra $\mathcal{L}$ and also the double extension of $\{0\}$ by $\mathcal{L}$ see [23].

Proposition 5.3. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a Jacobi-Jordan algebra and $\mathfrak{g}^*$ be the underlying dual vector space. The vector space $\mathcal{L} = \mathfrak{g} \oplus \mathfrak{g}^*$ equipped with the following product*

$$[\ ,\ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \qquad (x + f, y + h) \mapsto [x, y]_{\mathfrak{g}} + f \circ ady + h \circ adx \tag{5.4}$$

and a bilinear form

$$B : \mathcal{L} \times \mathcal{L} \to \mathbb{K}, \qquad (x + f, y + h) \mapsto f(y) + h(x) \tag{5.5}$$

is a quadratic Jacobi-Jordan algebra, which we denote by $\mathcal{L}$.

In the sequel we denote $\mathcal{L}$ by $T^*(\mathfrak{g})$ and B by B_0.

Theorem 5.4. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Jacobi-Jordan algebra and $\alpha \in Aut(\mathfrak{g})$. Then the endomorphism $\Omega := \alpha +\ {}^t\alpha$ of $T^*(\mathfrak{g})$ is a symmetric automorphism of $T^*(\mathfrak{g})$ with respect to B_0 if and only if $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$, where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Hence, if $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$ then $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, B_\Omega)$ is a RQH-Jacobi-Jordan algebra where $\Omega = \alpha +\ {}^t\alpha$.*

Proof. Let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^*$.

$$\begin{aligned}\Omega([x + f, y + h]) &= \Omega([x, y]_{\mathfrak{g}} + f \circ ady + h \circ adx) \\ &= \alpha([x, y]_{\mathfrak{g}}) + f \circ ady \circ \alpha + h \circ adx \circ \alpha,\end{aligned}$$

and

$$\begin{aligned}[\Omega(x + f), \Omega(y + h)] &= [\alpha(x) + f \circ \alpha, \alpha(y) + h \circ \alpha] \\ &= [\alpha(x), \alpha(y)]_{\mathfrak{g}} + f \circ \alpha \circ ad\alpha(y) + h \circ \alpha \circ ad\alpha(x).\end{aligned}$$

Then $\Omega([x + f, y + h]) = [\Omega(x + f), \Omega(y + h)]$ if and only if

$$\forall x, y \in \mathfrak{g}, \quad f \circ ady \circ \alpha + h \circ adx \circ \alpha = f \circ \alpha \circ ad\alpha(y) + h \circ \alpha \circ ad\alpha(x).$$

That is for all $z \in \mathfrak{g}$

$$f([y, \alpha(z)]) + h([x, \alpha(z)]) = f(\alpha[\alpha(y), z]) + h(\alpha[\alpha(x), z]).$$

Hence, Ω is an automorphism of $T^*(\mathfrak{g})$ if and only if $f([x, \alpha(y)]) = f(\alpha[\alpha(x), y])$, $\forall f \in \mathfrak{g}^*$ $\forall x, y \in \mathfrak{g}$, which is equivalent to $[x, \alpha(y)] = \alpha[\alpha(x), y]$ $\forall x, y \in \mathfrak{g}$.

As a consequence, $\Omega \in Aut(T_0^*(\mathfrak{g}))$ if and only if $[\alpha^2(x) - x, \alpha(y)]_{\mathfrak{g}} = 0$ $\forall x, y \in \mathfrak{g}$, i.e. $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$, since $\alpha \in Aut(\mathfrak{g})$.

In the following we show that Ω is symmetric with respect to B_0. Indeed, let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^*$

$$\begin{aligned}B_0(\Omega(x + f), y + h) &= B_0(\alpha(x) + f \circ \alpha, y + h) = f \circ \alpha(y) + h(\alpha(x)) \\ &= f \circ \alpha(y) + h \circ \alpha(x) = B_0(x + f, \alpha(y) + h \circ \alpha) \\ &= B_0(x + f, \Omega(y + h)).\end{aligned}$$

The last assertion is a consequence of the previous calculations and Proposition 3.3. $\square$

In the following we provide examples which show that the class of Jacobi-Jordan algebras with automorphisms satisfying the condition $Im(\alpha^2(x) - x) \in \mathcal{Z}(\mathfrak{g})$ is large. We consider first Jacobi-Jordan algebras with involutions.

Corollary 5.5. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a Jacobi-Jordan algebra and $\theta \in Aut(\mathfrak{g})$ such that $\theta^2 = id$ (θ is an involution), then $\theta^2(x) - x = 0 \in \mathcal{Z}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. Thus $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, B_\Omega)$ is a RQH-Jacobi-Jordan algebra where $\Omega = \alpha +\ {}^t\alpha$.*

Example 5.6. Considering an involution on a Jacobi-Jordan algebra $\mathfrak{g}$ is equivalent to have a $\mathbb{Z}_2$-graduation on $\mathfrak{g}$. From the above he Jacobi-Jordan algebras with involutions are symmetric.

Starting from a Jacobi-Jordan algebra one may construct a symmetric Jacobi-Jordan algebra in the following way :

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Jacobi-Jordan algebra, we consider the Jacobi-Jordan algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$ where $\mathfrak{L} = \mathfrak{g} \times \mathfrak{g}$ and the bracket defined by for all $x, y, x', y' \in \mathfrak{g}$ by $[(x, y), (x', y')]_{\mathfrak{L}} := ([x, x'], [y, y'])$.

It is easy to check that the map $\theta : \mathfrak{L} \to \mathfrak{L}, \quad (x, y) \mapsto (y, x)$ is an automorphism of $\mathfrak{L}$. Then the trivial T^*-extension of $\mathfrak{L}$ has $\Omega = \theta +\ {}^t\theta$ as a symmetric automorphism with respect to B_0. Moreover, Ω is an involution. According to Corollary 5.5, we have $(T_0^*(\mathfrak{L})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$ is a quadratic Hom-Jacobi-Jordan algebra.

Example 5.7. Let $\mathfrak{g} = V \oplus \mathcal{Z}(\mathfrak{g})$, where $V \neq \{0\}$ is a subspace of the vector space $\mathfrak{g}$ with $[V, V] = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathcal{Z}(\mathfrak{g})$. Let $\lambda : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ be a nontrivial linear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ defined by

$$\alpha(v + z) := v + \lambda(v) + z \quad \forall v \in V \quad \forall z \in \mathcal{Z}(\mathfrak{g}).$$

We have $\alpha([v + z, v' + z']) = \alpha([v, v']) = [v, v']$ since $[v, v'] \in \mathcal{Z}(\mathfrak{g})$.

Also $[\alpha(v + z), \alpha(v' + z')]) = [v, v']$. Therefore, the map α is an injective Jacobi-Jordan algebra morphism. Thus α is an automorphism of $\mathfrak{g}$.

Moreover, if $v \in \mathfrak{g}$ and $z \in \mathcal{Z}(\mathfrak{g})$, we have

$$\begin{aligned}(\alpha^2 - id)(v + z) &= \alpha^2(v + z) - (v + z) = \alpha(v + \lambda(v) + z) - (v + z)\\ &= v + 2\lambda(v) + z - v - z = 2\lambda(v).\end{aligned}$$

Then $\alpha^2 - id \neq 0$ and $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$. It follows that $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$, where $\Omega = \alpha +^t \alpha$, is a RQH-Jacobi-Jordan algebra.

It is clear that $T_0^*(\mathfrak{g})_\Omega$ is 2-nilpotente. It 's also a quadratic Jacobi-Jordan algebra.

Proposition 5.8. *Let $\mathcal{A}$ be an anticommutative antiassociative algebra and $\mathfrak{g}$ be a Jacobi-Jordan algebra. If $\mathcal{A}$ has an automorphism θ such that $Im(\theta^2 - id) \subseteq Ann(\mathcal{A})$, where $Ann(\mathcal{A})$ denotes the annihilator of $\mathcal{A}$, then the endomorphism $\widetilde{\theta} := id_{\mathfrak{g}} \otimes \theta$ of $\mathfrak{g} \otimes \mathcal{A}$ is an automorphism of the Jacobi-Jordan algebra $(\mathfrak{g} \otimes \mathcal{A}, [\ ,\ ])$, where $[x \otimes a, y \otimes b] := [x, y]_{\mathfrak{g}} \otimes ab$ for all $x, y \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. In addition, $Im(\widetilde{\theta}^2 - id_{\mathfrak{g}\otimes\mathcal{A}}) \subseteq \mathcal{Z}(\mathfrak{g} \otimes \mathcal{A})$. Then $(T_0^*(\mathfrak{g} \otimes \mathcal{A})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$ is a RQH-Jacobi-Jordan algebra. Moreover, if $\theta^2 \neq id_{\mathcal{A}}$ then $\widetilde{\theta}^2 \neq id_{\mathfrak{g}\otimes\mathcal{A}}$.*

Proof. It follows from direct calculation and Theorem 5.4. $\square$

6. Connection Between Algebras

We establish a connection between some classes of Jacobi-Jordan algebras (resp. quadratic Jacobi-Jordan algebras) and classes of Hom-Jacobi-Jordan algebras (resp. quadratic Hom-Jacobi-Jordan algebras).

Theorem 6.1. *There exists a biunivoque correspondence between the class of Jacobi-Jordan algebras (quadratic Jacobi-Jordan algebras) admitting involutive automorphisms (symmetric involutive automorphisms) and the class of Hom-Jacobi-Jordan algebras (quadratic Hom-Jacobi-Jordan algebras) where twist maps are involutive automorphisms (symmetric involutive automorphisms).*

Proof. Let $(\mathfrak{g}, [\ ,\ ])$ be a symmetric Jacobi-Jordan algebra with θ an involutive automorphism of $\mathfrak{g}$. Then, according to Theorem 2.13, $(g_\theta, [\ ,\ ]_\theta, \theta)$ is a Hom-Jacobi-Jordan algebra where θ is an involutive automorphism of $\mathfrak{g}_\theta$. Moreover, if $\mathfrak{g}$ has an invariant scalar product B such that θ is symmetric with respect to B, we have seen that

$$B_\theta : \mathfrak{g}_\theta \times \mathfrak{g}_\theta \to \mathbb{K}, \quad (x, y) \mapsto B_\theta(x, y) := B(\theta(x), y) \tag{6.1}$$

defines a quadratic structure on $\mathfrak{g}_\theta$.

Conversely, let $(H, [\ ,\ ]_H, \theta)$ be a Hom-Jacobi-Jordan algebra where θ is an involutive automorphism of H.

We will untwist the Hom-Jacobi-Jordan algebra structure by considering the vector space H and the bracket

$$[\ ,\ ] : H \times H \to H \quad (x, y) \mapsto [x, y] := [\theta(x), \theta(y)]_H. \tag{6.2}$$

Obviously the new bracket is bilinear and symmetric. We show that it satisfies the Jacobi identity.

Indeed, for $x, y, z \in H$ we have

$$\begin{aligned}[x, [y, z]] &= [\theta(x), \theta([y, z])]_H = [\theta(x), \theta([\theta(y), \theta(z)]_H)]_H \\ &= [\theta(x), [\theta^2(y), \theta^2(z)]_H)]_H = [\theta(x), [y, z]_H)]_H.\end{aligned}$$

Thus

$$\circlearrowleft_{x,y,z} [x, [y, z]] = \circlearrowleft_{x,y,z} [\theta(x), [y, z]_H)]_H = 0.$$

Thus $(H, [\ ,\ ])$ is a Jacobi-Jordan algebra.

Furthermore, for $x, y \in H$

$$\theta([x, y]) = \theta([\theta(x), \theta(y)]_H) = [\theta^2(x), \theta^2(y)]_H = [x, y]_H$$

and

$$[\theta(x), \theta(y)] = [\theta^2(x), \theta^2(y)]_H = [x, y]_H.$$

Then $\theta([x, y]) = [\theta(x), \theta(y)]$. Therefore θ is an involutive automorphism of the Jacobi-Jordan algebra $(H, [\ ,\ ])$.

Also for $x, y \in H$

$$[x,y]_\theta := [\theta(x),\theta(y)] = [\theta^2(x),\theta^2(y)]_H = [x,y]_H.$$

Then $(H, [\ ,\ ]_\theta, \theta)$ is the Hom-Jacobi-Jordan algebra $(H, [\ ,\ ]_H, \theta)$.

Now, let $(H, [\ ,\ ]_H, \theta, B)$ be a quadratic Hom-Jacobi-Jordan algebra.

The bilinear form

$$T : H \times H \to \mathbb{K}, \quad (x,y) \mapsto T(x,y) = B(\theta(x), y) \tag{6.3}$$

is symmetric and nondegenerate.

Indeed, for Let $x, y, z \in H$, we have

$$\begin{aligned} T([x,y],z) &= B(\theta([x,y]),z) = B(\theta[\theta(x),\theta(y)]_H,z) = B([x,y]_H,z) \\ &= B(x,[y,z]_H) = B(\theta(x),\theta([y,z]_H)) \qquad\qquad \theta \text{ is } B\text{-symmetric} \\ &= B(\theta(x),[\theta(y),\theta(z)]_H)) = B(\theta(x),[y,z])) = T(x,[y,z]). \end{aligned}$$

Then T is invariant. In the other hand,

$$T(\theta(x),y) = B(x,y) = B(\theta(x),\theta(y)) = T(x,\theta(y)).$$

That is θ is symmetric with respect to T. Therefore $(H, [\ ,\ ], T)$ is a quadratic Jacobi-Jordan algebra and $(H, [\ ,\ ]_\theta, \theta, T_\theta)$ is an IQH-Jacobi-Jordan algebra. □

Now we discuss the connection between Hom-Jacobi-Jordan algebras where the twist map is in the centroid and quadratic Jacobi-Jordan algebras. Let $(\mathfrak{g}, [\ ,\ ], B)$ be a quadratic Jacobi-Jordan algebra and $\theta \in Cent(\mathfrak{g})$ such that θ is invertible and symmetric with respect to B. We set

$$Cent_S(\mathfrak{g}) = \{\theta \in Cent(\mathfrak{g}) : \theta \text{ symmetric with respect to } B\}.$$

We consider

$$B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \quad (x,y) \mapsto B_\theta(x,y) := B(\theta(x), y). \tag{6.4}$$

Then B_θ is symmetric, nondegenerate and invariant. Indeed,

$$\begin{aligned} B_\theta(\{x,y\},z) &= B_\theta([\theta(x),y],z) = B(\theta([\theta(x),y]),z) \\ &= B([\theta(x),y],\theta(z)) = B(\theta(x),[y,\theta(z)]) \\ &= B(\theta(x),[\theta(y),z]) = B(\theta(x),\{y,z\}) \\ &= B_\theta(x,\{y,z\}). \end{aligned}$$

Also,

$$B_\theta(\theta(x),y) = B(\theta^2(x),y) = B(\theta(x),\theta(y)) = B_\theta(x,\theta(y)).$$

Then $(\mathfrak{g}, \{\ ,\ \}, \theta, B_\theta)$ is a quadratic Hom-Jacobi-Jordan algebra.

Notice that B_θ is an invariant scalar product of the Jacobi-Jordan algebra $\mathfrak{g}$.

We have also that $(\mathfrak{g}, [\ ,\ ]_\theta, \theta, B_\theta)$ is a quadratic Hom-Jacobi-Jordan algebra. Indeed,

$$\begin{aligned} B_\theta([x,y]_\theta, z) &= B_\theta([\theta(x),\theta(y)], z) = B(\theta([\theta(x),\theta(y)]), z) \\ &= B([\theta(x),\theta(y)], \theta(z)) = B(\theta(x), [\theta(y),\theta(z)]) \\ &= B(\theta(x), [y,z]_\theta) = B_\theta(x, [y,z]_\theta). \end{aligned}$$

Observe that

$$\begin{aligned} \theta([x,y]_\theta) = \theta[\theta(x),\theta(y)] &= [\theta^2(x),\theta(y)] = [\theta(x),y]_\theta. \\ \theta(\{x,y\}) = \theta[\theta(x),y] &= [\theta^2(x),y] = \{\theta(x),y\}. \end{aligned}$$

We may say that $\theta \in Cent(\mathfrak{g}, \{\ ,\ \})$ and $\theta \in Cent(\mathfrak{g}, [\ ,\ ]_\theta)$.

Conversely, let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Jacobi-Jordan algebra such that $\alpha \in Cent(\mathfrak{g}, [\ ,\ ], \alpha)$.

We define a new bracket as $\{x,y\} := [\alpha(x), y]$. Then $(\mathfrak{g}, \{\ ,\ \}))$ is a Jacobi-Jordan algebra. Indeed, the bracket is symmetric and

$$\begin{aligned} \{x,\{y,z\}\} &= [\alpha(x),[\alpha(y),z]], \\ \{y,\{z,x\}\} &= [\alpha(y),[\alpha(z),x] = [\alpha^2(y),[z,x]], \\ \{z,\{x,y\}\} &= [\alpha(z),[\alpha(x),y]] = [\alpha(z),[x,\alpha(y)]]. \end{aligned}$$

Then

$$\circlearrowleft_{x,y,z} \{x,\{y,z\}\} = [\alpha(x),[\alpha(y),z]] + [\alpha^2(y),[z,x]] + [\alpha(z),[x,\alpha(y)]] = 0.$$

We may define another bracket which gives rise to also a Jacobi-Jordan algebra by $[x,y]_\alpha := [\alpha(x),\alpha(y)]$. Indeed, the bracket is symmetric and

$$\begin{aligned} [x,[y,z]_\alpha]_\alpha &= [\alpha(x),\alpha([\alpha(y),\alpha(z)])] = [\alpha(x),[\alpha^2(y),\alpha(z)]] = [\alpha^2(x),[\alpha(y),\alpha(z)]], \\ [y,[z,x]_\alpha]_\alpha &= [\alpha(y),\alpha([\alpha(z),\alpha(x)])] = [\alpha(y),[\alpha^2(z),\alpha(x)]] = [\alpha^2(y),[\alpha(z),\alpha(x)]], \\ [z,[x,y]_\alpha]_\alpha &= [\alpha(z),\alpha([\alpha(x),\alpha(y)])] = [\alpha(z),[\alpha^2(x),\alpha(y)]] = [\alpha^2(z),[\alpha(x),\alpha(y)]]. \end{aligned}$$

Therefore

$$[\alpha^2(x),[\alpha(y),\alpha(z)]] + [\alpha^2(y),[\alpha(z),\alpha(x)]] + [\alpha^2(z),[\alpha(x),\alpha(y)]] = 0.$$

Now if there is an invariant scalar product B on $(\mathfrak{g}, [\ ,\ ])$ and assume that α is invertible and symmetric with respect to B. Consider the bilinear form B_α defined by $B_\alpha(x,y) = B(\alpha(x),y)$. We have

$$\begin{aligned} B_\alpha(\{x,y\},z) &= B(\alpha(\{x,y\})]),z) = B(\alpha([\alpha(x),y],z) = B(\alpha(x),[y,\alpha(z)]) \\ &= B(\alpha(x),[\alpha(y),z]) = B_\alpha(x,\{y,z\}). \end{aligned}$$

Similarly we have

$$\begin{aligned} B_\alpha([x,y]_\alpha, z) &= B(\alpha([\alpha(x),\alpha(y)]), z) = B([\alpha(x),\alpha(y)],\alpha(z)) \\ &= B(\alpha(x),[\alpha(y),\alpha(z)]) = B(\alpha(x),[y,z]_\alpha) \\ &= B_\alpha(x,[y,z]_\alpha). \end{aligned}$$

Therefore $(\mathfrak{g}, \{\ ,\ \}, B_\alpha)$ and $(\mathfrak{g}, [\ ,\ ]_\alpha, B_\alpha)$ are quadratic Jacobi-Jordan algebras. Hence, we have the following theorem:

Theorem 6.2. *There exists a biunivoque correspondence between the class of Jacobi-Jordan algebras* (*quadratic Jacobi-Jordan algebras*) *admitting an element in the centroid* (*symmetric invertible element in the centroid*) *and the class of Hom-Jacobi-Jordan algebras* (*quadratic Hom-Jacobi-Jordan algebras*) *where twist map is in the centroid* (*symmetric invertible element in the centroid*).

7. Bimodules of Hom-antiassociative Algebras

Definition 7.1. A hom-antiassociative algebra is said to be *multiplicative* if the triple $(\mathcal{A}, \cdot, \alpha)$ consisting of a linear space $\mathcal{A}$, $\mathcal{K}$-bilinear map $\cdot\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ and a linear space map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfies

$$\alpha(x\cdot y) = \alpha(x)\cdot\alpha(y) \qquad \text{(multiplicativity).} \tag{7.1}$$

Example 7.2. Let $\{e_1, e_2\}$ be a basis of a 2-dimensional vector space $\mathcal{A}$ over $\mathcal{K}$. The following multiplication $\cdot$ and map on $\mathcal{A}$ define a hom-antiassociative algebra:

$$\begin{aligned} &e_1\cdot e_1 = e_2, \\ &\alpha(e_1) = a_1e_1 + a_2e_2, \qquad \alpha(e_2) = 0, \end{aligned} \tag{7.2}$$

where $a_1, a_2 \in \mathcal{K}$.

Definition 7.3. A *hom-module* is a pair (V, β) where V is a $\mathcal{K}$-vector space and $\beta : V \to V$ is a linear map.

Definition 7.4. Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-antiassociative algebra and let (V, β) be a hom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quadruple (l, r, β, V) is called a *bimodule* of $\mathcal{A}$ if

$$\begin{aligned} l(x\cdot y)\beta(v) = -l(\alpha(x))l(y)v, \quad & r(x\cdot y)\beta(v) = -r(\alpha(y))r(x)v, \\ & l(\alpha(x))r(y)v = -r(\alpha(y))l(x)v, \end{aligned} \tag{7.3}$$

$$\beta(l(x)v) = l(\alpha(x))\beta(v), \tag{7.4}$$

$$\beta(r(x)v) = r(\alpha(x))\beta(v), \tag{7.5}$$

for all $x, y \in \mathcal{A}, v \in V$.

Proposition 7.5. *Let (l, r, β, V) a hom-bimodule of a hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Then the direct sum $\mathcal{A} \oplus V$ of vectors spaces is turned into a hom-antiassociative algebra by defining multiplication in $\mathcal{A} \oplus V$ by*

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1),$$
$$(\alpha \oplus \beta)(x_1 + v_1) = \alpha(x_1) + \beta(v_1)$$

for all $x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$.

Proof. Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Set

$$[(x_1+v_1)*(x_2+v_2)]*(\alpha(x_3)+\beta(v_3)) = -(\alpha(x_1)+\beta(v_1))*[(x_2+v_2)*(x_3+v_3)]. \quad (7.6)$$

After computation of (7.6), one easily obtains the conditions of (7.3). Hence the proposition is established. □

We denote such a hom-antiassociative algebra $(\mathcal{A} \oplus V, *, \alpha + \beta)$ or $\mathcal{A} \ltimes^{-1}_{l,r,\alpha,\beta} V$.

Example 7.6. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-antiassociative algebra. Let $L.(x)$ and $R.(x)$ denote the left and right multiplication operators, respectively, that is, $L.(x)(y) = x \cdot y, R.(x)(y) = y \cdot x$ for any $x, y \in \mathcal{A}$. Let $L. : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto L.(x)$ and $R. : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto R.(x)$ (for every $x \in \mathcal{A}$) be two linear maps. Then $(L., 0, \alpha), (0, R., \alpha)$ and $(L., R., \alpha)$ are bimodules of $(\mathcal{A}, \cdot, \alpha)$.

Proposition 7.7. *Let (l, r, β, V) be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Then $(l \circ \alpha^n, r \circ \alpha^n, \beta, V)$ is a bimodule of $\mathcal{A}$ for any entiger $n \geqslant 1$.*

Proof. We have:

$$\begin{aligned} l \circ \alpha^n(x \cdot y)\beta(v) &= l(\alpha^n(x) \cdot \alpha^n(y))\beta(v) = -l(\alpha(\alpha^n(x)))l(\alpha^n(y))v \\ &- l(\alpha^{n+1}(x))l(\alpha^n(y))v = -l \circ \alpha^n(\alpha(x))l \circ \alpha^n(y)v. \end{aligned}$$

Similarly, the other relations are established. □

Example 7.8. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-antiassociative algebra. Then $(L. \circ \alpha^n, R. \circ \alpha^n, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{A}$ for any entiger $n \geqslant 1$.

Example 7.9. Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative antiassociative algebra. Also let $\beta : \mathcal{A} \to \mathcal{A}$ be a morphism. Then $\mathcal{A}_\beta = (\mathcal{A}, \cdot_\beta = \beta \circ \cdot, \alpha_\beta = \beta \circ \alpha)$ is also a multiplicative hom-antiassociative algebra. Hence $(L_{\cdot_\beta} \circ \alpha_\beta^n, R_{\cdot_\beta} \circ \alpha_\beta^n, \alpha_\beta, \mathcal{A})$ is a bimodule of $\mathcal{A}$ for any integer $n \geqslant 0$.

Theorem 7.10. *Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two hom-antiassociative algebras. Suppose that there are linear maps $l_\mathcal{A}, r_\mathcal{A} : \mathcal{A} \to gl(\mathcal{B})$ and $l_\mathcal{B}, r_\mathcal{B} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_\mathcal{A}, r_\mathcal{A}, \beta, \mathcal{B})$ is a bimodule of $\mathcal{A}$ and $(l_\mathcal{B}, r_\mathcal{B}, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{B}$, satisfying the following conditions:*

$$l_\mathcal{A}(\alpha(x))(a \circ b) = -l_\mathcal{A}(r_\mathcal{B}(a)x)\beta(b) - (l_\mathcal{A}(x)a) \circ \beta(b), \quad (7.7)$$

$$r_\mathcal{A}(\alpha(x))(a \circ b) = -r_\mathcal{A}(l_\mathcal{B}(b)x)\beta(a) - \beta(a) \circ (r_\mathcal{A}(x)b), \quad (7.8)$$

$$l_{\mathcal{B}}(\beta(a))(x\cdot y) = -l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)\alpha(y) - (l_{\mathcal{B}}(a)x)\cdot\alpha(y), \tag{7.9}$$
$$r_{\mathcal{B}}(\beta(a))(x\cdot y) = -r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)\alpha(x) - \alpha(x)\cdot(r_{\mathcal{B}}(a)y), \tag{7.10}$$
$$l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)\beta(b) + (r_{\mathcal{A}}(x)a)\circ\beta(b) + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)\beta(a) + \beta(a)\circ(l_{\mathcal{A}}(x)b) = 0, \tag{7.11}$$
$$l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)\alpha(y) + (r_{\mathcal{B}}(a)x)\cdot\alpha(y) + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)\alpha(x) + \alpha(x)\cdot(l_{\mathcal{B}}(a)y) = 0, \tag{7.12}$$

for any $x, y \in \mathcal{A}, a, b \in \mathcal{B}$. Then, there is a hom-antiassociative algebra structure on the direct sum $\mathcal{A}\oplus\mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given by

$$(x+a)*(y+b) = (x\cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a\circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a), \tag{7.13}$$
$$(\alpha\oplus\beta)(x+a) = \alpha(x)+\beta(a) \tag{7.14}$$

for all $x, y \in \mathcal{A}, a, b \in \mathcal{B}$.

Proof. Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Set

$$[(x_1+v_1)*(x_2+v_2)]*(\alpha(x_3)+\beta(v_3)) = -(\alpha(x_1)+\beta(v_1))*[(x_2+v_2)*(x_3+v_3)]. \tag{7.15}$$

After computation of (7.15), we obtain (7.7) – (7.12). Hence the theorem is proved. □

This hom-antiassociative algebra will be denoted by $(\mathcal{A}\bowtie_{-1}\mathcal{B}, *, \alpha+\beta)$ or by $\mathcal{A}\bowtie^{-1,l_{\mathcal{A}},r_{\mathcal{A}},\beta}_{l_{\mathcal{B}},r_{\mathcal{B}},\alpha}\mathcal{B}$.

Definition 7.11. Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two hom-antiassociative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta)$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is a bimodule of $\mathcal{B}$. If the equations (7.7) - (7.12) are satisfied, then $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is called a *matched pair of hom-antiassociative algebras.*

8. Quadratique Hom-antiassociative Algebras

In this section, we consider the multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$ such that α involutive, i.e., $\alpha^2 = \mathrm{id}_{\mathcal{A}}$.

Definition 8.1. Let V_1, V_2 be two vector spaces. For a linear map $\phi : V_1 \to V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \to V_1^*$ given by

$$\langle v, \phi^*(u^*)\rangle = \langle \phi(v), u^*\rangle \text{ for all } v \in V_1, u^* \in V_2^*.$$

Lemma 8.2. *Let (l, r, β, V) be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$.*

(i) Let $l^*, r^* : \mathcal{A} \to gl(V^*)$ be the linear maps given by

$$\langle l^*(x)u^*, v\rangle = \langle l(x)v, u^*\rangle, \langle r^*(x)u^*, v\rangle = \langle r(x)v, u^*\rangle \tag{8.1}$$

for all $x \in \mathcal{A}$, $u^* \in V^*, v \in V$. Then, (r^*, l^*, β^*, V^*) is a bimodule of $(\mathcal{A}, \cdot, \alpha)$.

(ii) $(r^*, 0, \beta^*, V^*)$ and $(0, l^*, \beta^*, V^*)$ are also bimodules of $\mathcal{A}$.

Proof. (i): Let (l, r, β, V) be a bimodule of a multiplicative hom-antiassociative algebra $(\mathcal{A}, \cdot, \alpha)$. Show that (r^*, l^*, β^*, V^*) is a bimodule of $\mathcal{A}$.

Let $x, y \in \mathcal{A}, u^* \in V^*, v \in V$, we have

$$\begin{aligned}\langle r^*(x\cdot y)\beta^*(u^*), v\rangle &= \langle \beta(r(x\cdot y)v), u^*\rangle = \langle r(\alpha(x\cdot y))\beta(v), u^*\rangle\\ &= \langle r(\alpha(x)\cdot\alpha(y))\beta(v), u^*\rangle = \langle -r(\alpha^2(y))r(\alpha(x))v, u^*\rangle\\ &= \langle -(r(y)r(\alpha(x)))^*u^*, v\rangle = \langle -r^*(\alpha(x))r^*(y)u^*, v\rangle\end{aligned}$$

leading to $r^*(x\cdot y)\beta^*(u^*) = -r^*(\alpha(x))r^*(y)u^*$.

$$\begin{aligned}\langle l^*(x\cdot y)\beta^*(u^*), v\rangle &= \langle \beta(l(x\cdot y)(v)), u^*\rangle = \langle l(\alpha(x\cdot y))\beta(v), u^*\rangle\\ &= \langle l(\alpha(x)\cdot\alpha(y))\beta(v), u^*\rangle = \langle -l(\alpha^2(x))l(\alpha(y))\beta(v), u^*\rangle\\ &= \langle -(l(x)l(\alpha(y)))^*u^*, v\rangle = \langle -l^*(\alpha(y))l^*(x)u^*, v\rangle\end{aligned}$$

giving $l^*(x\cdot y)\beta^*(u^*) = -l^*(\alpha(y))l^*(x)u^*$.

$$\begin{aligned}\langle r^*(\alpha(x))l^*(y)u^*, v\rangle &= \langle l(y)r(\alpha(x))v, u^*\rangle\\ &= \langle l(\alpha^2(y))r(\alpha(x))v, u^*\rangle = \langle (l\circ\alpha)(\alpha(y))(r\circ\alpha)(x))v, u^*\rangle\\ &= \langle -r(\alpha^2(x))l(\alpha(y))v, u^*\rangle\\ &= \langle -r(x)l(\alpha(y))v, u^*\rangle = \langle -l^*(\alpha(y))r^*(x)u^*, v\rangle\end{aligned}$$

providing that $r^*(\alpha(x))l^*(y)u^* = -l^*(\alpha(y))r^*(x)u^*$.

$$\begin{aligned}\langle \beta^*(r^*(x))u^*, v\rangle &= \langle r(x)(\beta(v)), u^*\rangle = \langle r(\alpha^2(x))(\beta(v)), u^*\rangle\\ &= \langle (r\circ\alpha)(\alpha(x))(\beta(v)), u^*\rangle\\ &= \langle \beta(r(\alpha(x)))v, u^*\rangle = \langle r^*(\alpha(x))\beta^*(u^*), v\rangle.\end{aligned}$$

Then $\beta^*(r^*(x))u^* = r^*(\alpha(x))\beta^*(u^*)$.

Similarly, we show that $\beta^*(l^*(x))u^* = l^*(\alpha(x))\beta^*(u^*)$. Hence, (r^*, l^*, β^*, V^*) is a bimodule of $\mathcal{A}$.

(ii): Similarly, we can show also that $(r^*, 0, \beta^*, V^*)$ and $(0, l^*, \beta^*, V^*)$ are well bimodules of $\mathcal{A}$. □

Definition 8.3. Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-antiassociative algebra and $B : \mathcal{A}\times\mathcal{A} \to K$ be a non degenerate symmetric bilinear form on $\mathcal{A}$. B is said *α-invariant* if

$$B(\alpha(x)\cdot\alpha(y), \alpha(z)) = B(\alpha(x), \alpha(y)\cdot\alpha(z)).$$

Definition 8.4. We call $(\mathcal{A}, \alpha, B)$ a *double construction of an involutive quadratic hom-antiassociative algebra* associated to $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ if

(1) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;

(2) $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ are hom-antiassociative subalgebras of $(\mathcal{A}, \alpha)$ with $\alpha = \alpha_1 \oplus \alpha_1^*$;

(3) B is the natural non-degenerate $(\alpha_1 \oplus \alpha_1^*)$-invariant symmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given by

$$B(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \tag{8.2}$$
$$B((\alpha + \alpha^*)(x + a^*), y + b^*) = B(x + a^*, (\alpha + \alpha^*)(y + b^*)) \tag{8.3}$$

for all $x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*$ where $\langle\, , \rangle$ is the natural pair between the vector space $\mathcal{A}_1$ and its dual space $\mathcal{A}_1^*$.

Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive hom-antiassociative algebra. Suppose that there is an involutive hom-antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. We construct an involutive hom-antiassociative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ such that $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ are hom-subalgebras and equipped with the non-degenerate $(\alpha_1 \oplus \alpha_1^*)$-invariant symmetric bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$ given by the equation (8.2). That is, $(\mathcal{A} \oplus \mathcal{A}^*, \alpha \oplus \alpha^*, B)$ is an involutive quadratic hom- antiassociative algebra. Such a construction is called a double construction of an involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$.

Theorem 8.5. *Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive hom-antiassociative algebra. Suppose that there is an involutive hom-antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, there is a double construction of an involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ if and only if $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, \alpha^*, R_\circ^*, L_\circ^*, \alpha)$ is a matched pair of involutive hom-antiassociative algebras.*

Proof. Let us consider the four maps

$$\begin{aligned}
&L_\cdot^* : \mathcal{A} \to gl(\mathcal{A}^*), \langle L_\cdot^*(x)u^*, v \rangle = \langle L_\cdot(x)v, u^* \rangle = \langle x \cdot v, u^* \rangle, \\
&R_\cdot^* : \mathcal{A} \to gl(\mathcal{A}^*), \langle R_\cdot^*(x)u^*, v \rangle = \langle R_\cdot(x)v, u^* \rangle = \langle v \cdot x, u^* \rangle, \\
&R_\circ^* : \mathcal{A}^* \to gl(\mathcal{A}), \langle R_\circ^*(x^*)u, v^* \rangle = \langle R_\circ(x^*)v^*, u \rangle = \langle v^* \circ x^*, u \rangle, \\
&L_\circ^* : \mathcal{A}^* \to gl(\mathcal{A}), \langle L_\circ^*(x^*)u, v^* \rangle = \langle L_\circ(x^*)v^*, u \rangle = \langle x^* \circ v^*, u \rangle,
\end{aligned}$$

for all $x, v, u \in \mathcal{A}$, $x^*, v^*, u^* \in \mathcal{A}^*$.

If $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, \alpha^*, R_\circ^*, L_\circ^*, \alpha)$ is a matched pair of multiplicative hom-anti-associative algebras, then $(\mathcal{A} \bowtie_{-1} \mathcal{A}^*, *, \alpha + \alpha^*)$ is a multiplicative hom-antiassociative algebra with its product $*$ given by the equation (7.13) and the bilinear form $\mathcal{B}(\cdot, \cdot)$ defined by the equation (8.2) is $(\alpha \oplus \alpha^*)$-invariant, that is

$$\begin{aligned}
&\mathcal{B}[(\alpha(x) + \alpha^*(a^*)) * (\alpha(y) + \alpha^*(b^*)), (\alpha(z) + \alpha^*(c^*))] \\
&\qquad = \mathcal{B}[\alpha(x) + \alpha^*(a^*), (\alpha(y) + \alpha^*(b^*)) * (\alpha(z) + \alpha^*(c^*))]
\end{aligned}$$

for all $x, y \in \mathcal{A}^*, a^*, b^* \in \mathcal{A}^*$ and

$$(x + a^*) * (y + b^*) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)$$

with $l_{\mathcal{A}} = R_\cdot^*, r_{\mathcal{A}} = L_\cdot^*, l_{\mathcal{B}} = R_\circ^*, r_{\mathcal{B}} = L_\circ^*$.

Indeed,

$$\begin{aligned}
&\mathcal{B}[(\alpha(x)+\alpha^*(a^*))*(\alpha(y)+\alpha^*(b^*)),(\alpha(z)+\alpha^*(c^*))]\\
&=\mathcal{B}[(\alpha(x)\cdot\alpha(y)+l_{A^*}(\alpha^*(a^*))\alpha(y)+r_{A^*}(\alpha^*(b^*))\alpha(x))+(\alpha^*(a^*)\circ\alpha^*(b^*)\\
&\qquad\qquad+l_{\mathcal{A}}(\alpha(x))\alpha^*(b^*)+r_{\mathcal{A}}(\alpha(y))\alpha^*(a^*)),\alpha(z)+\alpha^*(c^*)]\\
&=\langle\alpha(x)\cdot\alpha(y),\alpha^*(c^*)\rangle+\langle\alpha^*(c^*)\circ\alpha^*(a^*),\alpha(y)\rangle+\langle\alpha^*(b^*)\circ\alpha^*(c^*),\alpha(x)\rangle\\
&\qquad+\langle\alpha^*(a^*)\circ\alpha^*(b^*),\alpha(z)\rangle+\langle\alpha(z)\cdot\alpha(x),\alpha^*(b^*)\rangle+\langle\alpha(y)\cdot\alpha(z),\alpha^*(a^*)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}[\alpha(x)+\alpha^*(a^*),(\alpha(y)+\alpha^*(b^*))*(\alpha(z)+\alpha^*(c^*))]\\
&=\mathcal{B}[\alpha(x)+\alpha^*(a^*),(\alpha(y)\cdot\alpha(z)+l_{\mathcal{A}^*}(\alpha^*(b^*))\alpha(z)+r_{\mathcal{A}^*}(\alpha^*(c^*))\alpha(y))\\
&\qquad\qquad+(\alpha^*(b^*)\circ\alpha^*(c^*)+l_A(\alpha(y))\alpha^*(c^*)+r_{\mathcal{A}}(\alpha(z))\alpha^*(b^*))]\\
&=\langle\alpha(x),\alpha^*(b^*)\circ\alpha^*(c^*)\rangle+\langle\alpha^*(c^*),\alpha(x)\cdot\alpha(y)\rangle+\langle\alpha^*(b^*),\alpha(z)\cdot\alpha(x)\rangle\\
&\qquad+\langle\alpha(y)\cdot\alpha(z),\alpha^*(a^*)\rangle+\langle\alpha^*(a^*)\circ\alpha^*(b^*),\alpha(z)\rangle+\langle\alpha(c^*)\circ\alpha^*(a^*),\alpha(y)\rangle.
\end{aligned}$$

Thus, $\mathcal{B}$ is well $(\alpha\oplus\alpha^*)$-invariant. Conversely, let

$$x*a^*=l_{\mathcal{A}}(x)a^*+r_{\mathcal{A}^*}(a^*)x, a^**x=l_{\mathcal{A}^*}(a^*)x+r_{\mathcal{A}}(x)a^\star,$$

for $x\in\mathcal{A},a^*\in\mathcal{A}^*$. Then, $(\mathcal{A},\mathcal{A}^*,R^*_\cdot,L^*_\cdot,\alpha^*,R^*_\circ,L^*_\circ,\alpha)$ is a matched pair of multiplicative hom-antiassociative algebras, since the double construction of the involutive quadratic hom-antiassociative algebra associated to $(\mathcal{A},\cdot,\alpha)$ and $(\mathcal{A}^*,\circ,\alpha^*)$ produces the equations (7.7) – (7.12). □

Theorem 8.6. *Let $(\mathcal{A},\cdot,\alpha)$ be an involutive hom-associative algebra. Suppose that there is an involutive hom-associative algebra structure "$\circ$" on its dual space $(\mathcal{A}^*,\alpha^*)$. Then, $(\mathcal{A},\mathcal{A}^*,R^*_\cdot,L^*_\cdot,\alpha^*,R^*_\circ,L^*_\circ,\alpha)$ is a matched pair of involutive hom-associative algebras if and only if for any $x\in\mathcal{A}$ and $a^*,b^*\in\mathcal{A}^*$,*

$$R^*_\cdot(\alpha(x))(a^*\circ b^*)=-R^*_\cdot(L^*_\circ(a^*)x)\alpha^*(b^*)-(R^*_\cdot(x)a^*)\circ\alpha^*(b^*),\tag{8.4}$$

$$R^*_\cdot(R^*_\circ(a^*)x)\alpha^*(b^*)+L^*_\cdot(x)a^*\circ\alpha^*(b^*)=-L^*_\cdot(L^*_\circ(b^*)x)\alpha^*(a^*)-\alpha^*(a^*)\circ(R^*_\cdot(x)b^*).\tag{8.5}$$

Proof. Obviously, (8.4) gives (7.7) and (8.5) reduces to (7.11) when $l_{\mathcal{A}}=R^*_\cdot$, $r_{\mathcal{A}}=L^*_\cdot$, $l_{\mathcal{B}}=l_{\mathcal{A}^*}=R^*_\circ$, $r_{\mathcal{B}}=r_{\mathcal{A}^*}=L^*_\circ$. Now, show that

$$(7.7)\Leftrightarrow(7.8)\Leftrightarrow(7.9)\Leftrightarrow(7.10)\quad\text{and}\quad(7.11)\Leftrightarrow(7.12).$$

Suppose (8.4) and (8.5) are satisfied and show that one has:

$$\begin{aligned}
L^*_\cdot(\alpha(x))(a^*\circ b^*)&=-L^*_\cdot(R^*_\circ(b^*)x)\alpha^*(a^*)-\alpha^*(a^*)\circ(L^*_\cdot(x)b^*),\\
R^*_\circ(\alpha^*(a^*))(x\cdot y)&=-R^*_\circ(L^*_\cdot(x)a^*)\alpha(y)-(R^*_\circ(a)x)\cdot\alpha(y),\\
L^*_\circ(\alpha^*(a^*))(x\cdot y)&=L^*_\circ(R^*_\cdot(y)a^*)\alpha(x)+\alpha(x)\cdot(L^*_\circ(a^*)y),\\
R^*_\circ(R^*_\cdot(x)a^*)\alpha(y)+(L^*_\circ(a^*)x)\cdot\alpha(y)&+L^*_\circ(L_\cdot(y)a^*)\alpha(x)+\alpha(x)\cdot(R^*_\circ(a)y)=0.
\end{aligned}$$

We have:

$$\langle R^*_\cdot(x)a^*, y\rangle = \langle L^*_\cdot(y)a^*, x\rangle = \langle y\cdot x, a^*\rangle,$$
$$\langle R^*_\circ(b^*)x, a^*\rangle = \langle L^*_\circ(a^*)x, b^*\rangle = \langle a^*\circ b^*, x\rangle,$$
$$\alpha^*(R^*_\cdot(x)a^*) = R^*_\cdot(\alpha(x))\alpha^*(a^*),\ \ \alpha^*(L^*_\cdot(x)a^*) = L^*_\cdot(\alpha(x))\alpha^*(a^*), \tag{8.6}$$
$$\alpha(R^*_\circ(a^*)x) = R^*_\circ(\alpha^*(a^*))\alpha(x),\ \ \alpha(L^*_\circ(a^*)x) = L^*_\circ(\alpha^*(a^*))\alpha(x), \tag{8.7}$$

for all $x, y \in \mathcal{A}, a^*, b^* \in \mathcal{A}^*$. Set $\alpha(x) = z, \alpha(y) = t$, $\alpha^*(a^*) = c^*$ and $\alpha^*(b^*) = d^*$. Then

$$\begin{aligned}\langle -R^*_\cdot(\alpha(x))(a^*\circ b^*), y\rangle &= \langle -y\cdot\alpha(x), a^*\circ b^*\rangle = \langle -(L_\cdot(y)\circ\alpha)x, a^*\circ b^*\rangle\\ &= \langle -x, \alpha^*(L^*_\cdot(y)(a^*\circ b^*))\rangle = \langle -L^*_\cdot(\alpha(y))\alpha^*(a^*\circ b^*), x\rangle\\ &= \langle -L^*_\cdot(\alpha(y))(\alpha^*(a^*)\circ\alpha^*(b^*)), x\rangle = \langle -L^*_\cdot(\alpha(y))(c^*\circ d^*), x\rangle,\end{aligned}$$

$$\begin{aligned}\langle -R^*_\cdot(L^*_\circ(a^*)x)\alpha(b^*), y\rangle &= \langle -y\cdot L^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle -L^*_\cdot(y)(\alpha^*(b^*)), L^*_\circ(a^*)x\rangle\\ &= \langle -L^*_\circ(a^*)x, L^*_\cdot(y)(\alpha^*(b^*))\rangle = \langle -a^*\circ(L^*_\cdot(y)(\alpha^*(b^*))), x\rangle\\ &= \langle -\alpha^*(c^*)\circ(L^*_\cdot(y)(d^*)), x\rangle,\end{aligned}$$

$$\begin{aligned}\langle (R^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &= \langle -R^*_\circ(\alpha^*(b^*))y, R^*_\cdot(x)a^*\rangle = \langle -a^*, (R^*_\circ(\alpha^*(b^*))y)\cdot x\rangle\\ &= \langle -L^*_\cdot[R^*_\circ(\alpha^*(b^*))y]a^*, x\rangle = \langle -L^*_\cdot(R^*_\circ(d^*)y)\alpha^*(c^*), x\rangle\end{aligned}$$

leading to (7.7) $\Leftrightarrow$ (7.8).

$$\begin{aligned}\langle L^*(\alpha(x))(a^*\circ b^*), y\rangle &= \langle -a^*\circ b^*, \alpha(x)\cdot y\rangle = \langle -R^*_\circ(b^*)(\alpha(x)\cdot y), a^*\rangle\\ &= \langle -R^*_\circ(\alpha^*(d^*))(z\cdot y), a^*\rangle,\end{aligned}$$

$$\begin{aligned}\langle \alpha^*(a^*)\circ(L^*_\cdot(x)b^*), y\rangle &= \langle \alpha^*(a^*), R^*_\circ(L^*_\cdot(x)b^*)y\rangle = \langle a^*, \alpha[R^*_\circ(L^*_\cdot(x)b^*)y]\rangle\\ &= \langle a^*, R^*_\circ[\alpha^*(L^*_\cdot(x)b^*)]\alpha(y)\rangle = \langle a^*, R^*_\circ[L^*_\cdot(\alpha(x))\alpha^*(b^*)]\alpha(y)\rangle\\ &= \langle a^*, R^*_\circ(L^*_\cdot(z)d^*)\alpha(y)\rangle,\end{aligned}$$

$$\begin{aligned}\langle L^*_\cdot(R^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &= \langle (R^*_\circ(b^*)x)\circ y, \alpha^*(a^*)\rangle = \langle \alpha[(R^*_\circ(b^*)x)\circ y], a^*\rangle\\ &= \langle (R^*_\circ(\alpha^*(b^*))\alpha(x))\circ\alpha(y), a^*\rangle = \langle R^*_\circ(d^*)z\cdot\alpha(y), a^*\rangle\end{aligned}$$

giving (7.8) $\Longleftrightarrow$ (7.9).

$$\begin{aligned}\langle R^*(\alpha(x))(a^*\circ b^*), y\rangle &= \langle a^*\circ b^*, y\cdot\alpha(x)\rangle = \langle L_\cdot(a^*)b^*, y\cdot z\rangle\\ &= \langle L^*_\circ(a^*)(y\cdot z)\rangle = \langle L^*_\circ(\alpha^*(c^*))(y\cdot z)\rangle,\end{aligned}$$

$$\begin{aligned}\langle (R^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &= \langle \alpha^*(b^*), L^*_\cdot(R^*_\cdot(x)a^*)y\rangle = \langle b^*, \alpha^*[L^*_\cdot(R^*_\cdot(x)a^*)y]\rangle\\ &= \langle b^*, L^*_\cdot(R^*_\cdot(\alpha(x))\alpha^*(a^*))\alpha(y)\rangle\\ &= \langle b^*, L^*_\cdot(R^*_\cdot(z)c^*)\alpha(y)\rangle,\end{aligned}$$

$$\begin{aligned}\langle R^*_\cdot(L^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &= \langle y\cdot L^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle \alpha(y)\cdot\alpha(L^*_\circ(a^*)x), b^*\rangle\\ &= \langle \alpha(y)\cdot L^*_\circ(\alpha^*(a^*))\alpha(x), b^*\rangle = \langle \alpha(y)\cdot L^*_\circ(c^*)z, b^*\rangle\end{aligned}$$

providing that (7.7) $\Longleftrightarrow$ (7.10).

$$\begin{aligned}\langle L^*_\cdot(L^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &= \langle (L^*_\circ(b^*)x)\cdot y, \alpha^*(a^*)\rangle = \langle a^*, \alpha(L^*_\circ(b^*)x)\cdot\alpha(y)\rangle\\ &= \langle a^*, L^*_\circ(\alpha^*(b^*))\alpha(x)\cdot\alpha(y)\rangle = \langle a^*, L^*_\circ(d^*)z\cdot\alpha(y)\rangle,\\ \langle \alpha^*(a^*)\circ(R^*_\cdot(x)b^*), y\rangle &= \langle R^*_\circ(R^*_\circ(x)b^*)y, \alpha^*(a^*)\rangle = \langle \alpha^*(a^*)\circ(R^*_\cdot(x)b^*), y\rangle\\ &= \langle \alpha[R^*_\circ(R^*_\circ(x)b^*)y], a^*\rangle = \langle R^*_\circ[R^*_\circ(\alpha(x))\alpha^*(b^*)]\alpha(y), a^*\rangle\\ &= \langle R^*_\circ(R^*_\cdot(z)d^*)\alpha(y), a^*\rangle,\\ \langle (L^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &= \langle R^*_\circ(\alpha^*(b^*))y, L^*_\cdot(x)a^*\rangle = \langle x\cdot(R^*_\circ(d^*)y), a^*\rangle\\ &= \langle \alpha(z)\cdot(R^*_\circ(d^*)y), a^*\rangle,\\ \langle R^*_\cdot(R^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &= \langle y\cdot R^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle \alpha^*(b^*), L_\cdot(y)(R^*_\circ(a^*)x)\rangle\\ &= \langle (L^*_\cdot(y)(d^*), R^*_\circ(a^*)x\rangle = \langle L^*_\cdot(y)d^*\circ a^*, x\rangle\\ &= \langle L^*_\circ(L^*_\cdot(y)d^*)x, a^*\rangle = \langle L^*_\circ(L^*_\cdot(y)d^*)\alpha(z), a^*\rangle\end{aligned}$$

implying that (7.11) $\Longleftrightarrow$ (7.12). □

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