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On ordered semigroups satisfying certain regularity conditions

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Abstract. In terms of ideals, this paper investigates ordered semigroups satisfying certain regularity conditions. In particular we study regularity, complete regularity, quasi-regularity, intra-regularity as well as left (right) regularity, left (right) quasi-regularity, and left (right) reproduce.

1. Preliminaries

Regular rings and semigroups have been introduced and studied by Neumann [10]. These lead to study other types of regularity, for example, completely regularity, intra-regularity and quasi-regularity ([9], [12]). Using the so-called *linear words*, Bogdanovíc et al. classified all the types of regularity of semigroups [2]; based on the results obtained the authors then described semigroups satisfying certain regularity conditions [1]. In [11], Phochai and Changphas determined all the types of regularity conditions for ordered semigroups. This paper then examines ordered semigroups satisfying each of the types of regularity conditions. Some types of regularity of ordered semigroups have been studied ([3], [4]).

An *ordered semigroup* $(S, \cdot, \leqslant)$ consists of a semigroup $(S, \cdot)$ together with a relation $\leqslant$ that is *compatible* with the semigroup operation (cf. [7]), meaning that, for any $a, b, c \in S$, $a \leqslant b$ implies $ac \leqslant bc$ and $ca \leqslant cb$. For $\emptyset \neq A, B \subseteq S$, $AB := \{ab \in S \mid a \in A, b \in B\}$ and $(A] := \{x \in S \mid \exists a \in A, x \leqslant a\}$. It is observed that (1) $A \subseteq (A]$; (2) $(A](B] \subseteq (AB]$; (3) $((A](B]] = (AB]$.

A non-empty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *left* (resp. *right*) *ideal* of S if (i) $SA \subseteq A$ (resp. $AS \subseteq A$); (ii) $(A] = A$. We say that A is a (*two-sided*) *ideal* of S if A is both a left and a right ideal of S. S is said to be *simple* if S contains no proper ideals. If $a \in S$ then $L(a) = (a \cup Sa]$ (resp. $R(a) = (a \cup aS]$, $J(a) = (a \cup aS \cup Sa \cup SaS]$) is a left (resp. right, two-sided) ideal containing a. An ordered semigroup S is simple if and only if $S = (SaS]$ for all $a \in S$ [8]. A subsemigroup B of S (that is, $BB \subseteq B$) is called a *bi-ideal* of S if (i)

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$BSB \subseteq B$; (ii) $(B] = B$ ([5], p. 242). If $a \in S$ then $(a \cup a^2 \cup aSa]$ is a bi-ideal containing a .

Let X be a countable alphabet whose elements are called *variables*. Let c be a symbol such that $c \notin X$, called a *constant*. Consider $(X \cup \{c\})^+$, free semigroup generated by $X \cup \{c\}$, let L be the set of all words $u \in (X \cup \{c\})^+$ satisfying the following conditions:

(i) The constant c appears at least once in u.
(ii) There is at least one occurrence of a variable in u.
(iii) Any variable appears at most once in u.

A word $u \in L$ is called *linear*, and we shall write $u(c, x_1, \ldots, x_n)$ instead of u to emphasize that $\{x_1, \ldots, x_n\}$ is the set of all variables appearing in u. For $u(c, x_1, \ldots, x_n) \in L$, an expression of the form $c \leqslant u(c, x_1, \ldots, x_n)$ is called a *regularity condition*. For an ordered semigroup $(S, \cdot, \leqslant)$ and $a \in S$, an expression of the form $a \leqslant u(a, x_1, \ldots, x_n)$ is *solvable* in S if there exist $a_1, \ldots, a_n \in S$ such that $a \leqslant u(a, a_1, \ldots, a_n)$. Two regularity conditions $c \leqslant u(c, x_1, \ldots, x_n)$ and $c \leqslant v(c, y_1, \ldots, y_m)$ are *equivalent* if for every ordered semigroup $(S, \cdot, \leqslant)$, and for every $a \in S$,

$$a \leqslant u(a, x_1, \ldots, x_n) \text{ is solvable in } S \Longleftrightarrow a \leqslant v(a, y_1, \ldots, y_m) \text{ is solvable in } S.$$

We denote by $\mathbb{N}$ the set of all positive integers. It was proved in [11] that an arbitrary regularity condition $c \leqslant u(c, x_1, \ldots, x_n)$ is equivalent to one of the regularity conditions (C1)–(C16):

Number	Condition	Name
C1	$c \leqslant xcy$	
C2	$c \leqslant xc$	Left Reproduce
C3	$c \leqslant cx$	Right Reproduce
C4	$c \leqslant xcycz$	Intra-quasi-regular
C5	$c \leqslant xcyc$	Left Quasi-regular
C6	$c \leqslant cxcy$	Right Quasi-regular
C7	$c \leqslant cxc$	Regular
C8	$c \leqslant xc^ky$, for some $k \in \mathbb{N}$	Intra-regular ($k = 2$)
C9	$c \leqslant xc^kyc$, for some $k \in \mathbb{N}$	Left Regular
C10	$c \leqslant cxc^ky$, for some $k \in \mathbb{N}$	Right Regular
C11	$c \leqslant xc^2$	
C12	$c \leqslant xc^2$	
C13	$c \leqslant c^2x$	
C14	$c \leqslant c^2xc^2$	
C15	$c \leqslant cxc^2$	
C16	$c \leqslant c^2xc$	

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. S is said to satisfy a regularity condition $c \leqslant u(c, x_1, ..., x_n)$, if for every $a \in S$, the expression $a \leqslant u(a, x_1, ..., x_n)$ is solvable in S. It is observed that an ordered semigroup $(S, \cdot, \leqslant)$ is intra-quasi-regular (resp. left quasi-regular) if $a \in (SaSaS]$ (resp. $a \in (SaSa]$) for all $a \in S$. Any other types of regularity can be observed similarly. Finally, we call an element a of S a *left* (resp. *right*) *reproduce element* if $a \leqslant xa$ (resp. $a \leqslant ax$) is solvable. Any other types of elements can be defined similarly.

The main results can be described shortly as the following: Theorem 1 shows that every ordered semigroup in a semilattice satisfies the same regularity condition of such semilattice. In Theorem 2, we consider relationships of ordered semigroups containing intra-quasi-regular elements, intra-regular elements, left [resp. right] quasi-regular elements, left [resp. right] regular elements. In Theorem 3, we characterize several regularities of elements by its principal ideal, principal left ideal and principal right ideal. The rest of this paper shows characterizations of regularities of semigroups and regularity conditions of semigroups as well.

2. Main Results

Let Y be a semilattice. An ordered semigroup $(S, \cdot, \leqslant)$ is a *semilattice* Y of ordered semigroups $(S_\alpha, \cdot_\alpha, \leq_\alpha)$, $\alpha \in Y$ if (i) $S_\alpha \cap S_\beta = \emptyset$ for all different $\alpha, \beta \in Y$; (ii) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$; (iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Theorem 1. *Assume an ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice Y of ordered semigroups $(S_\alpha, \cdot_\alpha, \leq_\alpha)$, $\alpha \in Y$. If $(S, \cdot, \leqslant)$ satisfies one of the regularity conditions $(C4)$–$(C14)$ then $(S_\alpha, \cdot_\alpha, \leq_\alpha)$ satisfies the same regularity condition for all $\alpha \in Y$.*

Proof. Assume that S satisfies (C4); that is S is left intra-quasi-regular. Let $\alpha \in Y$. To show that $(S_\alpha, \cdot_\alpha, \leq_\alpha)$ satisfies (C4), let $a \in S_\alpha$. By assumption, $a \leqslant xayaz$ for some $x, y, z \in S$. Since $S = \bigcup\limits_{\nu \in Y} S_\nu$, there exist $\beta, \gamma, \delta \in Y$ such that $x \in S_\beta, y \in S_\gamma, z \in S_\delta$. We have $\alpha\beta = \alpha\gamma = \alpha\delta = \alpha$ because $a \in S_\alpha$ and $a \leqslant axayaz$, so

$$\begin{aligned} a &\leqslant xayaz \leqslant x(xayaz)y(xayaz)z = (xxay)a(zyx)a(yazz) \\ &\leqslant (xxay)(xayaz)(zyx)a(yazz) = (xxayx)a(yazzyx)a(yazz) \in S_\alpha a S_\alpha a S_\alpha. \end{aligned}$$

Then $a \in (S_\alpha a S_\alpha a S_\alpha]$. So S_α satisfies (C4). The rest of the assertions can be proved similarly. □

Theorem 2. *The following statements hold for an ordered semigroup $(S, \cdot, \leqslant)$:*

(1) *S has an intra-quasi-regular element if and only if S has an intra-regular element.*
(2) *S has a left quasi-regular element if and only if S has a left regular element.*

(3) *S has a right quasi-regular element if and only if S has a right regular element.*

Proof. (1) Assume that S has an intra-quasi-regular element a; then $a \leqslant xayaz$ for some $x, y, z \in S$. We have

$$\begin{aligned} yaz &\leqslant y(xayaz)z = (yx)a(yazz) \leqslant (yx)(xayaz)(yazz) \\ &= (yxxa)(yaz)(yaz)z \in S(yaz)^2S. \end{aligned}$$

Then $yaz \in (S(yaz)^2S]$, and S has an intra-regular element yaz.

Conversely, assume that S has an intra-regular element a. That is, $a \leqslant xa^2y$ for some $x, y \in S$. We have

$$a \leqslant xa^2y \leqslant xa(xa^2y)y = xaxa(ayy) \in SaSaS.$$

Then $a \in (SaSaS]$, and S has an intra-quasi-regular element a.

(2) Assume that S has a left quasi-regular element a; then $a \leqslant xaya$ for some $x, y \in S$. Thus,

$$\begin{aligned} ya &\leqslant y(xaya) = (yx)a(ya) \leqslant (yx)(xaya)(ya) \\ &= (yxxa)(ya)(ya) \in S(ya)^2. \end{aligned}$$

So $ya \in (S(ya)^2]$, and S has a left regular element ya.

Conversely, assume that S has a left regular element a. Then $a \leqslant xa^2$ for some $x \in S$. Since

$$a \leqslant xa^2 \leqslant xaxa^2 = xa(xa)a \in SaSa$$

then $a \in (SaSa]$, and a is a left quasi-regular element. That (3) holds can be proved analogously. □

Theorem 3. *For an element a of an ordered semigroup $(S, \cdot, \leqslant)$, the following statements hold:*

(1) *a is intra-quasi-regular if and only if the principal two-sided ideal $J(a)$ of S has an intra-regular generator.*
(2) *a is left quasi-regular if and only if the principal left ideal $L(a)$ of S has a left regular generator.*
(3) *a is right quasi-regular if and only if the principal right ideal $R(a)$ of S has a right regular generator.*

Proof. (1) Assume that $a \in S$ is intra-quasi-regular; then $a \leqslant xayaz$ for some $x, y, z \in S$. Since $a \in J(a)$, $J(yaz) \subseteq J(a)$. By $a \leqslant xayaz$, $J(a) \subseteq J(yaz)$. Then $J(a) = J(yaz)$. As the proof of Theorem 2 we have yaz is intra-regular, and then $J(a)$ has an intra-regular generator.

Conversely, assume that the principal ideal $J(a)$ has an intra-regular generator b. Then $J(a) = J(b)$ and $b \leqslant pb^2q$ for some $p, q \in S$. We have

$$a \in J(b) = J(pb^2q) = (pb^2q \cup S(pb^2q) \cup (pb^2q)S \cup S(pb^2q)S] \subseteq (Sb^2S].$$

Since $b \in J(a)$, then $b^2 \in J(a)J(a) \subseteq (SaSaS]$. Hence, a is intra-quasi-regular. That (2) and (3) hold can be proved similarly. $\square$

Next, we deal with ordered semigroups satisfying $c \leqslant xc^2yc$, and satisfying both of $c \leqslant xc^2yc$ and $c \leqslant cxc^2y$.

Theorem 4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *$a \in (SbSa]$ for all $a, b \in S$;*
(2) *S is simple and left quasi-regular;*
(3) *S is simple and left reproduce;*
(4) *every left ideal of S is simple;*
(5) *S is simple and every left ideal of S is intra-regular.*

Proof. (1) $\Longleftrightarrow$ (2): Assume that $a \in (SbSa]$ for all $a, b \in S$. Since $a \in (SaSa]$ for any $a \in S$, S is left quasi-regular. To show that S is simple, let $a \in S$. Clearly, $(SaS] \subseteq S$. By assumption, $a \in (SaSa] \subseteq (SaS]$, and so $S \subseteq (SaS]$. Then $S = (SaS]$.

Conversely, assume that S is simple and left quasi-regular. Let $a, b \in S$. Since S is simple, $S = (SaS]$ and $S = (SbS]$. Since S is left quasi-regular, $a \in (SaSa]$, and so $S = (SaSa]$. Thus

$$a \in S = (SbS] \subseteq (Sb(SaSa]] \subseteq ((S](b](SaSa]] = (SbSaSa] \subseteq (SbSa],$$

as required.

(1) $\Longleftrightarrow$ (3): If $a \in (SbSa]$ for all $a, b \in S$, then $a \in (SaSa]$ for all $a \in S$; hence $a \in (Sa]$ for all $a \in S$. We have S is left reproduce. As in the proof of (1) $\Longleftrightarrow$ (2), S is simple.

Conversely, assume that S is simple and left reproduce. Let $a, b \in S$. Since S is simple, $S = (SaS]$ and $S = (SbS]$. Since S is left reproduce, $a \in (Sa]$, so $S = (Sa]$. Thus

$$a \in S = (SbS] \subseteq (Sb(Sa]] \subseteq ((S](b](Sa]] = (SbSa],$$

as required.

(1) $\Longleftrightarrow$ (4): Assume that $a \in (SbSa]$ for all $a, b \in S$. Let L be a left ideal of S, and let $a \in L$. Clearly, $(LaL] \subseteq (LLL] \subseteq (L] = L$. If $b \in L$, then $b \in (SaaSb]$ by assumption. Since

$$(SaaSb] \subseteq (SLaSL] \subseteq (LaL]$$

it follows that $b \in (LaL]$, and $(LaL] = L$. Hence, L is simple.

Conversely, assume that every left ideal of S is simple. Let $a, b \in S$. Since $(a \cup Sa]$ is a left ideal of S, it follows by assumption that

$$\begin{aligned}(a \cup Sa] &= ((a \cup Sa](ba)(a \cup Sa]] \subseteq ((a \cup Sa](ba](a \cup Sa]] \\ &= ((a \cup Sa)(ba)(a \cup Sa)] = (abaa \cup abaSa \cup Sabaa \cup SabaaSa] \\ &\subseteq (SbSa].\end{aligned}$$

Hence, (1) holds.

(1) $\Longleftrightarrow$ (5): Assume that $a \in (SbSa]$ for all $a, b \in S$. As in the proof of (1) $\Longleftrightarrow$ (2) we have S is simple. Let L be a left ideal of S, and let $a \in L$. By assumption,

$$a \in (SaaaSa] \subseteq (SLaaSL] \subseteq (La^2L].$$

Then L is intra-regular.

Conversely, assume that S is simple and every left ideal of S is intra-regular. Let $a, b \in S$. Since S is simple, $(SaS] = (SbS]$. Since $(a \cup Sa]$ is a left ideal of S, we have $(a \cup Sa]$ is intra-regular, and so $a \in ((a \cup Sa]aa(a \cup Sa]]$. Consider:

$$\begin{aligned} a &\in ((a \cup Sa]aa(a \cup Sa]] \subseteq ((a \cup Sa](aa](a \cup Sa]] = ((a \cup Sa)(aa)(a \cup Sa)] \\ &= (aaaa \cup aaaSa \cup Saaaa \cup SaaaSa] \subseteq (SaSa] \subseteq ((SaS]a] = ((SbS]a] \\ &\subseteq ((SbS](a]] = (SbSa]. \end{aligned}$$

Hence, (1) holds. $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is called *completely quasi-regular* (resp. *completely reproduce*) if S is left and right quasi-regular (resp. left and right reproduce). Using Theorem 4 and its dual, we have:

Corollary 1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *$a \in (SbSa] \cap (bSaS]$ for all $a, b \in S$;*
(2) *S is simple and completely quasi-regular;*
(3) *S is simple and completely reproduce;*
(4) *every one-sided ideal of S is simple;*
(5) *S is simple and every one-sided ideal of S is intra-regular.*

Theorem 5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *S satisfies the regularity condition $c \leqslant xc^2yc$;*
(2) *S is intra-regular and left quasi-regular;*
(3) *every left ideal of S is intra-regular;*
(4) *every left ideal of S is intra-quasi-regular.*

Proof. (1) $\Longleftrightarrow$ (2): That (1) $\Longrightarrow$ (2) is clear. To prove (2) $\Longrightarrow$ (1), assume that S is intra-regular and left quasi-regular. Let $a \in S$. Since S is intra-regular, $a \leqslant xa^2y$ for some $x, y \in S$. Since S is left quasi-regular, $a \leqslant uava$ for some $u, v \in S$. Then

$$a \leqslant uava \leqslant uxa^2yva.$$

Hence S satisfies the regularity condition $c \leqslant xc^2yc$.

(1) $\Longrightarrow$ (3): Assume that S satisfies the regularity condition $c \leqslant xc^2yc$. Let L be a left ideal of S. If $a \in L$, then by assumption we have $a \leqslant xa^2ya$ for some $x, y \in S$, and $a^2 \leqslant ua^4va^2$ for some $u, v \in S$. Thus,

$$a \leqslant xa^2ya \leqslant x(ua^4va^2)ya = (xua^2)a^2(va^2ya) \in La^2L$$

it follows that $a \in (La^2L]$, and L is intra-regular.

(3) $\Longrightarrow$ (4): This is easy to see.

(4) $\Longrightarrow$ (1): Assume that every left ideal of S is intra-quasi-regular. Let $a \in S$. Since $(a \cup Sa]$ is a left ideal of S, $(a \cup Sa]$ is intra-quasi-regular. Then

$$\begin{aligned} a &\in ((a \cup Sa]a(a \cup Sa]a(a \cup Sa]] \subseteq ((a \cup Sa](a](a \cup Sa](a](a \cup Sa]] \\ &= ((a \cup Sa)a(a \cup Sa)a(a \cup Sa)] \subseteq (Sa^2Sa]. \end{aligned}$$

Hence S satisfies the condition $c \leqslant xc^2yc$. □

Using Theorem 5 and its dual, we have the following.

Corollary 2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *S satisfies the regularity conditions $c \leqslant xc^2yc$ and $c \leqslant cxc^2y$;*
(2) *S is intra-regular and completely quasi-regular;*
(3) *every one-sided ideal of S is intra-regular;*
(4) *every one-sided ideal of S is intra-quasi-regular.*

We next consider ordered semigroups satisfying the regularity $c \leqslant cxc^2yc$.

Theorem 6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *$a \in (aSbSa]$ for all $a, b \in S$;*
(2) *S is simple and regular;*
(3) *every bi-ideal of S is simple.*

Proof. (1) $\Longrightarrow$ (2): Assume that $a \in (aSbSa]$ for all $a, b \in S$. Then, for any $a \in S$, we have $a \in (aSaSa] \subseteq (aSa]$. Hence, S is regular. Since $(aSaS] \subseteq (SaS]$, $S \subseteq (SaS]$, and so S is simple.

(2) $\Longrightarrow$ (1): Assume that S is simple and regular. Let $a, b \in S$. Since S is simple, $S = (SbS]$. Since S is regular, $a \in (aSa]$. Then

$$a \in (a(SbS]a] \subseteq ((a](SbS](a]] = (aSbSa]$$

as required.

(1) $\Longrightarrow$ (3): Assume that $a \in (aSbSa]$ for all $a, b \in S$. Let B be a bi-ideal of S. Then $(BSB] \subseteq B$. If $b \in B$, then $b \in (bSbSb] \subseteq (BSBSB] \subseteq (BSB]$. Thus $B \subseteq (BSB]$, and B is simple.

(3) $\Longrightarrow$ (1): Assume that every bi-ideal of S is simple. Let $a, b \in S$. Since $(a \cup a^2 \cup aSa]$ is a bi-ideal of S, it follows by assumption that

$$\begin{aligned} a \in (a \cup a^2 \cup aSa] &= ((a \cup a^2 \cup aSa]aba(a \cup a^2 \cup aSa]] \\ &\subseteq ((a \cup a^2 \cup aSa](aba](a \cup a^2 \cup aSa]] \\ &= ((a \cup a^2 \cup aSa)aba(a \cup a^2 \cup aSa)] \subseteq (aSbSa]. \end{aligned}$$

Thus (1) holds. □

Theorem 7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *S satisfies the regularity condition $c \leqslant cxc^2yc$;*
(2) *S is intra-regular and regular;*
(3) *every left ideal of S is right quasi-regular;*
(4) *every right ideal of S is left quasi-regular;*
(5) *every bi-ideal of S is intra-regular;*
(6) *every bi-ideal of S is intra-quasi-regular.*

Proof. (1) $\Longleftrightarrow$ (2): That (1) $\Longrightarrow$ (2) is clear. To show that (2) $\Longrightarrow$ (1), assume that S is intra-regular and regular. Let $a \in S$. Since S is regular, $a \leqslant axa$ for some $x \in S$. Since S is intra-regular, $a \leqslant ya^2z$ for some $y, z \in S$. We have

$$a \leqslant (ax)a \leqslant (ax)a(xa) \leqslant (ax)ya^2z(xa).$$

Then (1) holds.

(1) $\Longrightarrow$ (3): Assume that S satisfies the regularity condition $c \leqslant cxc^2yc$. Let L be a left ideal of S, and let $a \in L$. By assumption, $a \leqslant axa^2ya$ for some $x, y \in S$. By $xa, ya \in L$, it follows that

$$a \leqslant axa^2ya = a(xa)a(ya) \in aLaL.$$

Thus $a \in (aLaL]$. Hence, L is right quasi-regular.

(3) $\Longrightarrow$ (1): Assume that every left ideal of S is right quasi-regular. Let $a \in S$. Since $(a \cup Sa]$ is a left ideal of S, $(a \cup Sa]$ is right quasi-regular. Then

$$a \in (a(a \cup Sa]a(a \cup Sa]] \subseteq ((a](a \cup Sa](a](a \cup Sa]] = (a(a \cup Sa)a(a \cup Sa)].$$

This implies that $a \in (aSa^2Sa]$, and S satisfies the regularity condition $c \leqslant cxc^2yc$.

(1) $\Longrightarrow$ (4): Assume that S satisfies the regularity condition $c \leqslant cxc^2yc$. Let R be a right ideal of S, and let $a \in R$. By assumption, $a \leqslant axa^2ya$ for some x, y in S. By $ax, ay \in R$, it follows that

$$a \leqslant axa^2ya = (ax)a(ay)a \in RaRa.$$

Thus $a \in (RaRa]$, whence R is left quasi-regular.

(4) $\Longrightarrow$ (1): Assume that every right ideal of S is left quasi-regular. Let $a \in S$. Since $(a \cup aS]$ is a right ideal of S, $(a \cup aS]$ is left quasi-regular. Then

$$a \in ((a \cup aS]a(a \cup aS]a] \subseteq ((a \cup aS](a](a \cup aS](a]] = ((a \cup aS)a(a \cup aS)a].$$

This implies that $a \in (aSa^2Sa]$, and S satisfies the regularity condition $c \leqslant cxc^2yc$.

(1) $\Longrightarrow$ (5): Assume that S satisfies the regularity condition $c \leqslant cxc^2yc$. Let B be a bi-ideal of S, and let $a \in B$. Then $a \leqslant axa^2ya$ for some $x, y \in S$ and $a^2 \leqslant a^2ua^4va^2$ for some $u, v \in S$. We have

$$a \leqslant axa^2ya \leqslant ax(a^2ua^4va^2)ya = (axa^2ua)a^2(ava^2ya) \in (BSB)a^2(BSB) \subseteq Ba^2B$$

Then $a \in (Ba^2B]$, and B is intra-regular.

(5) $\Longrightarrow$ (6): This is easy to see.

(6) $\Longrightarrow$ (1): Assume that every bi-ideal of S is intra-quasi-regular. Let $a \in S$. Since $(a \cup a^2 \cup aSa]$ is a bi-ideal of S, $(a \cup a^2 \cup aSa]$ is intra-quasi-regular. Consider:

$$\begin{array}{rcl} a & \in & ((a \cup a^2 \cup aSa]a(a \cup a^2 \cup aSa]a(a \cup a^2 \cup aSa]] \\ & \subseteq & ((a \cup a^2 \cup aSa](a](a \cup a^2 \cup aSa](a](a \cup a^2 \cup aSa]] \\ & = & ((a \cup a^2 \cup aSa)a(a \cup a^2 \cup aSa)a(a \cup a^2 \cup aSa)] \\ & \subseteq & (aSa^2Sa] \end{array}$$

Thus, S satisfies the regularity condition $c \leqslant cxc^2yc$. □

Finally, ordered semigroups satisfying left regularity and complete regularity conditions will be characterized.

Theorem 8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *S is left regular;*
(2) *every left ideal of S is left quasi-regular;*
(3) *every left ideal of S is left reproduce.*

Proof. (1) $\Longrightarrow$ (2): Assume that S is left regular. Let L be a left ideal of S, and let $a \in L$. By assumption, $a \leqslant xa^2$ for some $x \in S$. We have

$$a \leqslant xaa \leqslant x(xaa)a \leqslant x(x(xaa)a)a = (xxxa)a(aa) \in LaLa.$$

That is: $a \in (LaLa]$. Hence L is left quasi-regular.

(2) $\Longrightarrow$ (3): This is easy to see.

(3) $\Longrightarrow$ (1): Assume that every left ideal of S is left reproduce. Let $a \in S$. By assumption, $(a \cup Sa]$ is left reproduce. Then

$$a \in ((a \cup Sa]a] \subseteq ((a \cup Sa](a]] = ((a \cup Sa)a] = (a^2 \cup Sa^2].$$

This implies $a \in (Sa^2]$, and S is left regular. □

An ordered semigroup $(S, \cdot, \leqslant)$ is called *completely regular* if S satisfies both of the regularity conditions $c \leqslant c^2x$ and $c \leqslant xc^2$, equivalently, if S satisfies the regularity condition $c \leqslant c^2xc^2$. The proof of the following assertion will be omitted.

Theorem 9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following are equivalent:*

(1) *S is completely regular;*
(2) *S is left regular, right regular, left quasi-regular, and right quasi-regular;*
(3) *every left ideal of S is left regular, and every right ideal is right regular;*
(4) *every left and right ideal of S is completely quasi-regular;*
(5) *every bi-ideal of S is left and right quasi-regular.*

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