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Maximal cyclic subgroups of a finite abelian p-group of rank two

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Abstract. Let G be a finite group. A cyclic subgroup of G that is not a proper subgroup of any other proper cyclic subgroup of G is called a maximal cyclic subgroup and the set of all maximal cyclic subgroups of G is denoted by $\mathcal{M}_G$. In this paper, we find the cardinality of the set $\mathcal{M}_G$, where G is a finite abelian p-group of rank two. As an application, we obtain the independence number of the power graph of the group G.

1. Introduction

Counting the number of subgroups of finite groups is one of the old problems in finite group theory and it is still frequently studied. In [2], Bhowmik gave a method to determine the total number of subgroups of a finite abelian p-group. A simple formula, in the case of a finite abelian p-group of rank two was obtained by Călugăreanu [3], Petrillo [10] and Tóth [14] by using Goursat's lemma. In [13], Tóth obtained the number of cyclic subgroups of a finite abelian group.

Let G be a finite group. A cyclic subgroup of G that is not a proper subgroup of any other proper cyclic subgroup of G is called a *maximal cyclic subgroup* and the set of all maximal cyclic subgroups of G is denoted by $\mathcal{M}_G$. Let Γ be a graph. A set of pairwise non-adjacent vertices of Γ is called an *independent set*. The maximum size of an independent set in a graph Γ is called the *independence number* of Γ and denoted by $\beta(\Gamma)$.

Let G be a group. The *undirected power graph* $\mathcal{P}(G)$ has the vertex set G and two distinct vertices x and y are adjacent if $x = y^m$ or $y = x^m$ for some positive integer m. The concepts of a power graph and an undirected power graph were first considered by Kelarev and Quinn [8] and Chakrabarty et al. [6], respectively. Since this paper deals only with undirected graphs, for convenience throughout we use the term "power graph" to refer to an undirected power graph. Recently, a lot of interesting results on the power graphs have been obtained, see for example [4, 5]. A detailed list of open problems and results can be found in [1]. Chakrabarty et al. [6], found that the power graph $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order p^n, where p is a prime number and n is a non-negative integer. Sehgal and Singh [12] obtained the degree of a vertex in the power graph of a finite abelian

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group. Chelevam and Sattanathan [7] determined the finite abelian groups whose power graphs are planar. They have also characterized the finite abelian groups G with $\beta(\mathcal{P}(G)) = 2$. In [9], X. Ma et al. obtained that the independence number of the power graph of a finite p-group G is equal to the cardinality of the set $\mathcal{M}_G$. For generalized extraspecial p-groups G with $p > 2$, $\beta(\mathcal{P}(G))$ had been determined in [?] by calculating the cardinality of the set $\mathcal{M}_G$.

In this paper, we find the cardinality of the set $\mathcal{M}_G$, where G is an abelian p-group of rank two. Equivalently, we find the independence number of $\mathcal{P}(G)$.

Throughout the paper p denotes a prime number. Let $|X|$ denote the cardinality of the set X and $o(x)$ denote the order of the element x in the group G. Let $\langle g\rangle$ denote the cyclic subgroup of the group G generated by $g \in G$ and the identity element of the group G is denoted by e. For a positive integer n, $\phi(n)$ denotes the Euler's totient function. Let $\mathcal{C}(G)$ denote the set of all distinct cyclic subgroups of the group G. Note that $(\mathcal{C}(G), \subseteq)$ is a poset.

2. Preliminaries

We will start with the basic facts that will be needed later.

Lemma 2.1. *Let $G \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \cong \langle x\rangle \times \langle y\rangle$ where $o(x) = p^{\beta_1}$ and $o(y) = p^{\beta_2}$ and $\beta_1 \geqslant \beta_2 \geqslant 1$. Let $g = x^{p^{k_1}\alpha_1} y^{p^{k_2}\alpha_2} \neq e \in G$. If $0 < k_i$ and $p \nmid \alpha_i$ $\forall\, i \in \{1, 2\}$, then there are p cyclic subgroups of order $o(g)p$ containing $\langle g\rangle$. Further, if for some $i = i_o$, $k_{i_o} = 0$ and $\alpha_{i_o} \neq 0$, then $\langle g\rangle$ doesn't contained in any cyclic subgroup of order $o(g)p$.*

Proof. Let $g \in G$ such that $g = x^{p^{k_1}\alpha_1} y^{p^{k_2}\alpha_2}$, where $p \nmid \alpha_i$ for $i \in \{1, 2\}$. First, we count the number of elements $h \in G$ such that $h^p = g$. Consider $h = x^{r_1} y^{r_2}$. Now, $h^p = g$ implies $x^{pr_1} y^{pr_2} = x^{p^{k_1}\alpha_1} y^{p^{k_2}\alpha_2}$. So $p^{k_i}\alpha_i = pr_i \mod p^{\beta_i}$ $\forall\, i \in \{1, 2\}$. For fixed i, latter equation has integer solution r_i if and only if $p \mid p^{k_i}\alpha_i$. Thus, if for some $i = i_o$, $k_{i_o} = 0$ and $\alpha_{i_o} \neq 0$, then there doesn't exist any $h \in G$ such that $h^p = g$.

Now, assume $k_i > 0$, $\forall\, i$. So, if $p^{k_i}\alpha_i \equiv pr_i \mod p^{\beta_i}$, then $p^{k_i-1}\alpha_i \equiv r_i \mod p^{\beta_i-1}$. Thus, the latter equation has p distinct solutions for each fixed i and that are $r_i = p^{k_i-1}\alpha_i + kp^{\alpha_1-1}$, where $0 \leqslant k \leqslant p-1$. Thus, for given $g = x^{p^{k_1}\alpha_1} y^{p^{k_2}\alpha_2}$, where $p \nmid \alpha_i$ and $k_i > 0$, there are p^2 elements $h \in G$ such that $h^p = g$ and $o(h) = o(g)p$.

Now, let $\langle h\rangle$ be a cyclic subgroup of order $o(g)p$ such that $\langle g\rangle \subset \langle h\rangle$ and $h^p = g$. Suppose $w \in \langle h\rangle$ such that $w^p = g$, then $w = h^r$ and $h^{rp} = h^p = g$. This implies that $rp \equiv p \mod o(h)$. Thus, $r = 1 + k\frac{o(h)}{p}$, where $1 \leqslant k \leqslant p$. Thus, each cyclic subgroup $\langle h\rangle$ of order $o(g)p$ contains p distinct elements $w \in \langle h\rangle$ such that $w^p = g$. Hence that, there are $\frac{p^2}{p} = p$ cyclic subgroups of order $o(g)p$ containing g for $k_i > 0$ $\forall\, i$. This completes the proof. $\square$

Corollary 2.2. *Suppose $G \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}}$, $\beta_1 > \beta_2$. Then a cyclic subgroup $H = \langle x^{p^{\beta_1 - t}} y^b \rangle$ (where $\beta_2 \leqslant t < \beta_1, 1 \leqslant b \leqslant p^{\beta_2}$) of order p^t is contained in a cyclic subgroup of order p^{t+1} if and only if $p \mid b$.*

Proof. This follows from Lemma 2.1. □

Recall that the set of all maximal cyclic subgroups of the finite group G is denoted by $\mathcal{M}_G$ and the independence number of the graph Γ is denoted by $\beta(\Gamma)$.

Theorem 2.3. [9, Corollary 2.14] *Let G be a p-group. Then $\beta(\mathcal{P}(G)) = |\mathcal{M}_G|$.*

3. Maximal cyclic subgroups

In this section, we find the number of maximal cyclic subgroups of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$, $r \geqslant s \geqslant 1$. For the rest of the paper, we fixed that $G \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s} \cong \langle x \rangle \times \langle y \rangle$, where $o(x) = p^r$ and $o(y) = p^s$ and $r \geqslant s \geqslant 1$.

The number of cyclic subgroups of order p in G is $p+1$ and these cyclic groups are given as $\{\langle y^{p^{s-1}} \rangle\} \cup \{\langle x^{p^{r-1}} y^{ip^{s-1}} \rangle \mid 1 \leqslant i \leqslant p\}$. From [11], we know that a cyclic subgroup of order p^t $(t > 1)$ contains exactly one cyclic subgroup of order p. Let X_i be the set of all cyclic subgroups of G containing cyclic subgroup $\langle x^{p^{r-1}} y^{ip^{s-1}} \rangle$ for $1 \leqslant i \leqslant p$ and X_0 be the set of all cyclic subgroups of G containing $\langle y^{p^{s-1}} \rangle$.

Lemma 3.1. *The number of cyclic subgroups of order p^t in X_i, $0 \leqslant i \leqslant p$ is p^{t-1} where $1 \leqslant t \leqslant s$.*

Proof. By Lemma 2.1, each cyclic subgroup of order p^t is contained in p cyclic subgroups of order p^{t+1}, $1 \leqslant t < s$. Thus, it is immediate that each X_i contains p^{t-1} cyclic subgroups of order p^t, $1 \leqslant t \leqslant s$. □

Let $\mathcal{M}(X_i, \subseteq)$ denote the set of all maximal elements of the poset $(X_i, \subseteq)$.

Lemma 3.2. $|\mathcal{M}_G| = \sum_{i=0}^{p} |\mathcal{M}(X_i, \subseteq)|$.

Proof. Recall that $\mathcal{C}(G)$ is the set of all distinct cyclic subgroups of the group G. Let $\mathcal{C}^*(G)$ be the set $\mathcal{C}(G) \backslash \langle e \rangle$. Define a relation R on $\mathcal{C}^*(G)$ such that $\langle x \rangle, \langle y \rangle \in \mathcal{C}^*(G)$ are said to be related if $\langle x \rangle$ and $\langle y \rangle$ contain a unique cyclic subgroup of order p. It is immediate the R is an equivalence relation. Since, G has $p+1$ cyclic subgroups of order p, $\mathcal{C}^*(G)$ has $p+1$ equivalence classes. Clearly, X_i, $0 \leqslant i \leqslant p$ are these equivalence classes. It is easy to observe that if $\langle x \rangle \in X_i$ and $\langle y \rangle \in X_j$ for $i \neq j$, $0 \leqslant i, j \leqslant p$, then $\langle x \rangle \nsubseteq \langle y \rangle$ and $\langle y \rangle \nsubseteq \langle x \rangle$. Thus, a maximal element of the poset $(X_i, \subseteq)$ is a maximal cyclic subgroup of G. This completes the proof. □

Theorem 3.3. *Let $G \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$, $r > s$. Then*

$$\beta(\mathcal{P}(G)) = |\mathcal{M}_G| = \begin{cases} 2p^s + \phi(p^s)(r - s - 1), & r \geqslant s \\ p^s + p^{s-1}, & r = s. \end{cases}$$

Proof. Suppose $r > s$. Now assume that $s < t \leqslant r$. Take $g = x^a y^b \in G$. If the order of g is p^t, then $g = x^{p^{r-t}k} y^b$, where $\gcd(k,p) = 1$, $1 \leqslant k \leqslant p^t$ and $1 \leqslant b \leqslant p^s$. Thus, the number of elements of order p^t is $\phi(p^t)p^s$. Since, each cyclic subgroup of order p^t contains $\phi(p^t)$ elements of order p^t, so the number of cyclic subgroups of order p^t is $\frac{\phi(p^t)p^s}{\phi(p^t)} = p^s$ and they are $\langle x^{p^{r-t}} y^b \rangle$, $1 \leqslant b \leqslant s$. Further, $(x^{p^{r-t}} y^b)^{p^{t-1}} = x^{p^{r-1}}$. Thus, all cyclic subgroups of order p^t, $t > s$ belong to X_p. By Corollary 2.2, cyclic subgroup $H = \langle x^{p^{r-t}} y^b \rangle$ of order p^t is contained in cyclic subgroup of order p^{t+1} if and only if $p \mid b$ and if $p \mid b$, then H is contained in p cyclic subgroups of order p^{t+1} $(t < r)$. Hence, out of p^s only p^{s-1} cyclic subgroups of order p^t are contained in cyclic subgroups of order p^{t+1}.

Again, the number of cyclic subgroups of order p^s in the set X_p is p^{s-1} (Lemma 3.1) and the number of cyclic subgroups of order p^{s+1} is p^s and each cyclic subgroup of order p^s is contained in at most p cyclic subgroups of order p^{s+1} (Lemma 2.1). Thus, each cyclic subgroup of order p^s is contained in p cyclic subgroups of order p^{s+1} in the set X_p. By Lemmas 2.1 and 3.1, it is clear that X_p has p^{t-1} cyclic subgroups of order p^t and each cyclic subgroup of order p^t is contained in p cyclic subgroups of order p^{t+1} in X_p for $1 \leqslant t < s$.

The number of cyclic subgroups of order p^t in X_i for $0 \leqslant i \leqslant p-1$ is p^{t-1}, for $1 \leqslant t \leqslant s$ (Lemma 3.1) and none of cyclic subgroups of order p^t for $t > s$ belong to $X_i (0 \leqslant i \leqslant p-1)$. Further, each cyclic subgroup of order p^t is contained in p cyclic subgroups of order p^{t+1} for $1 \leqslant t < s$ in X_i.

Collecting all arguments, the Hasse diagram of the poset $(X_p, \subseteq)$ is given in Figure 1 and the Hasse diagram of the poset $(X_i, \subseteq)$ $(0 \leqslant i \leqslant p-1)$ is given in Figure 2.

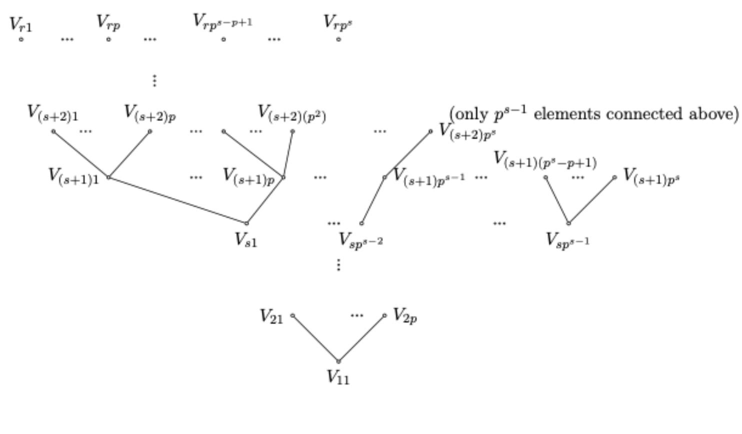

Figure 1: The Hasse diagram of the poset $(X_p, \subseteq)$

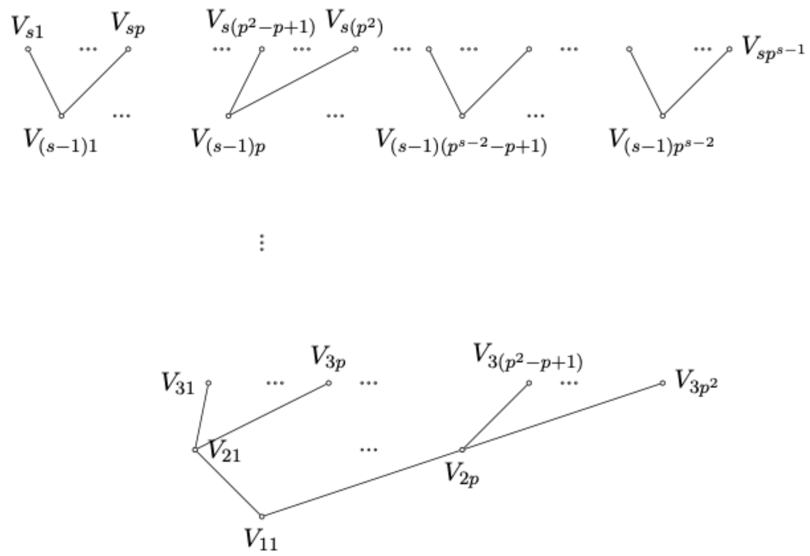

Figure 2: The Hasse diagram of the poset $(X_i, \subseteq)$, for $0 \leqslant i \leqslant p-1$

In Figures 1 and 2, V_{tj} denotes the element of X_i, $0 \leqslant i \leqslant p$ of cardinality p^t. $(X_p, \subseteq)$ has $p^s + \phi(p^s)(r-s-1)$ maximal elements (see Figure 1) and $(X_i, \subseteq)$ for $0 \leqslant i \leqslant p-1$ has p^{s-1} maximal elements. Thus, by Lemma 3.2, $|\mathcal{M}_G| = 2p^s + \phi(p^s)(r-s-1)$ for $r > s$. Now, for $r = s$. Only the cyclic subgroups of order p^s are maximal elements in X_i, $0 \leqslant i \leqslant p$ and each X_i has p^{s-1} cyclic subgroups of order p^s. Thus, by Lemma 3.2, $|\mathcal{M}_G| = p^s + p^{s+1}$, for $r = s$. Hence by Theorem 2.3, we complete the proof. □

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