Quasigroups and Related Systems **28** (2020), 229 – 236

A note on the construction of right conjugacy closed loops

Gerhard Hiss and *Lucia Ortjohann*

Abstract. We describe a group theoretical construction of non-associative right conjugacy closed loops with abelian inner mapping groups.

1. Introduction

A *loop* is a quasigroup with an identity element. If the multiplication of the loop is associative, it is a group. In the following, every loop, and in particular every group, will be assumed to be finite.

Let $(\mathcal{L}, *)$ be a loop with identity element $1_{\mathcal{L}}$. For every $x \in \mathcal{L}$, we denote by R_x the *right multiplication* by x in $\mathcal{L}$, i.e. $R_x : \mathcal{L} \to \mathcal{L}, y \mapsto y * x$, and we set $R_{\mathcal{L}} := \{R_x \mid x \in \mathcal{L}\}$. Then $\mathrm{RM}(\mathcal{L}) := \langle R_{\mathcal{L}} \rangle \leqslant \mathrm{Sym}(\mathcal{L})$ and its subgroup $\mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}})$ are called the *right multiplication group*, and the *inner mapping group* of $\mathcal{L}$, respectively. The *envelope* of $\mathcal{L}$ consists of the triple $(\mathrm{RM}(\mathcal{L}), \mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}}), R_{\mathcal{L}})$. To simplify notation, let us put $G := \mathrm{RM}(\mathcal{L})$, $H := \mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}})$ and $T := R_{\mathcal{L}}$. Clearly, G acts faithfully and transitively on $\mathcal{L}$, which may hence be identified with the set of right cosets of H in G. Notice that $\mathcal{L}$ is a group if and only if $|G| = |\mathcal{L}|$, or, equivalently, $H = \{1\}$. By definition, T generates G, and one can check that T is a transversal for the set of right cosets of H^g in G for every $g \in G$. Envelopes of loops are generalized to *loop folders*.

The connection between loops and loop folders, summarized below, goes back to Baer [3], and is described in detail by Aschbacher in [2, Section 1]. In the following, G denotes a finite group and H a subgroup of G; we write $H\backslash G$ for the set of right cosets of H in G. The triple (G, H, T) is called a *loop folder* if $T \subseteq G$ is a transversal for $H^g\backslash G$ for every $g \in G$, and if $1 \in T$. We call (G, H, T) *faithful* if G acts faithfully on $H\backslash G$, i.e. if $\mathrm{core}_G(H) = \{1\}$.

By construction, the envelope (G, H, T) of a loop $\mathcal{L}$ is a faithful loop folder with $G = \langle T \rangle$, and there is a natural bijection between T and $\mathcal{L}$. Conversely, given a loop folder (G, H, T), one can construct a loop $(T, *)$ on the set T in such a way that (G, H, T) is isomorphic to the envelope of $(T, *)$, provided (G, H, T) is faithful and $G = \langle T \rangle$. This motivates the following definition. A transversal T for $H\backslash G$ is called a *generating transversal* if $G = \langle T \rangle$.

2010 Mathematics Subject Classification: Primary: 20N05, Secondary: 20D99, 20K01
Keywords: Finite loops, conjugacy closed, invariant transversals

A loop $\mathcal{L}$ is called *right conjugacy closed* or an RCC-*loop* if the set $R_{\mathcal{L}}$ is closed under conjugation, i.e. if $R_x^{-1}R_yR_x \in R_{\mathcal{L}}$ for all $x, y \in \mathcal{L}$. Analogously, a loop folder (G, H, T) is called *right conjugacy closed*, or an RCC-*loop folder* if T is G-invariant under conjugation, i.e. $g^{-1}tg \in T$ for all $g \in G, t \in T$. Clearly, a loop is right conjugacy closed if and only if its envelope is an RCC-loop folder.

In this paper we construct envelopes of RCC-loops with abelian inner mapping groups. The following trivial observations form the starting point of our construction.

Proposition 1. *Let G be a finite group, $Q \trianglelefteq G$ and $H \leqslant G$ with $H \cap Q = \{1\}$. Let $\hat{T} = \{t_1, \ldots, t_n\}$ be a transversal for HQ in G. Then $T := \hat{T}Q$ is a transversal for H in G and $\{t_1Q, \ldots, t_nQ\}$ is a transversal for HQ/Q in G/Q. Furthermore, we have the following two statements.*

- (a) *The transversal $\{t_1Q, \ldots, t_nQ\}$ is G/Q-invariant if and only if T is G-invariant.*
- (b) *The transversal $\{t_1Q, \ldots, t_nQ\}$ generates G/Q if and only if T generates G.*

Thus if $\operatorname{core}_G(H) = \{1\}$ *and the transversal $\{t_1Q, \ldots, t_nQ\}$ is G/Q-invariant and generates G/Q, then (G, H, T) is an envelope of an* RCC-*loop, which is non-associative if $\{1\} \lneq H \lneq G$.* □

Notice that $\operatorname{core}_G(H) \leqslant C_H(Q)$ under our assumption $H \cap Q = \{1\}$, so that $C_H(Q) = \{1\}$ implies $\operatorname{core}_G(H) = \{1\}$. If G/Q is abelian, then H is abelian and T is G-invariant by part (a) of Proposition 1. This holds in particular for Q equal to the commutator subgroup $[G, G]$ of G. We conjecture that the converse of this statement holds.

Conjecture 1. *Let G be a finite group, $H \leqslant G$ an abelian subgroup such that there exists a G-invariant transversal T for $H\backslash G$ with $1 \in T$, i.e. (G, H, T) is an* RCC-*loop folder. Then $[G, G] \cap H = \{1\}$.* □

In Section 3 we prove this conjecture in special cases. The conjecture also makes sense if G is an infinite group, but this more general question is out of the scope of this paper.

2. Generating transversals for abelian groups

In this section we investigate the existence of generating transversals in abelian groups. Let p be a prime. We first show that if G is an abelian p-group, and the index of H in G is larger than the minimal size of a generating set of G, there exists a generating transversal for $H\backslash G$ containing 1. We generalize this result for an arbitrary abelian group G, however with a stronger condition on the index of H in G.

The minimal size of a generating set of G is called the *rank* of G, i.e.

$$\mathrm{rk}(G) := \min\{|S| \mid S \subseteq G, G = \langle S \rangle\}.$$

A cyclic group of order n is denoted by C_n. If T is a generating transversal for $H\backslash G$ containing 1, then necessarily, $|G : H| > \mathrm{rk}(G)$. For the sake of clarity in the proofs to follow, we write the elements of a direct product $A \times B$ of groups as pairs (a, b) with $a \in A$, $b \in B$.

Proposition 2. *Let G be an abelian p-group. Suppose that $H \leqslant G$ is a subgroup of G such that $|G : H| > \mathrm{rk}(G)$. Then there exists a generating transversal T for $H\backslash G$ with $1 \in T$.*

Proof. We proceed by induction on the order of G, where the base case is trivial. For the induction step, we assume that $G \neq \{1\}$, that the statement holds for every abelian p-group of order less than $|G|$, and distinguish two cases.
Case 1: For every decomposition

$$G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}, \tag{1}$$

with $\{1\} \neq C_{m_i} \leqslant G$ for $1 \leqslant i \leqslant r$, we have $C_{m_i} \nleqslant H$ for all $1 \leqslant i \leqslant r$.

Consider an arbitrary decomposition of G as in (1), and let a_i be a generator of C_{m_i} for every $1 \leqslant i \leqslant r$. Then $G = \langle a_1, \ldots, a_r \rangle$ and our assumption implies that $a_i \notin H$ for all $1 \leqslant i \leqslant r$. Suppose that for any $1 \leqslant i \neq j \leqslant r$, the generators a_i and a_j of G lie in distinct cosets of H in G. Then there is a transversal for $H\backslash G$ containing $\{1, a_1, \ldots, a_r\}$ and we are done. Otherwise, $Ha_i = Ha_j$ for some $1 \leqslant i \neq j \leqslant r$. Without loss of generality, we may assume that $|a_j| \geq |a_i|$. Then

$$G = \langle a_1 \rangle \times \cdots \times \langle a_{j-1} \rangle \times \langle a_j a_i^{-1} \rangle \times \langle a_{j+1} \rangle \times \cdots \times \langle a_r \rangle,$$

and we have $\langle a_j a_i^{-1} \rangle \leqslant H$. We have thus reduced the assertion to the following situation.
Case 2: There exist $\{1\} \neq C_{m_i} \leqslant G$ for $1 \leqslant i \leqslant r$ such that

$$G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r},$$

and $C_{m_j} \leqslant H$ for some $1 \leqslant j \leqslant r$.

Note that the generators of these cyclic groups form a minimal generating set of G of size r. Thus, it follows from Burnside's basis theorem [5, III, Satz 3.15] that $r = \mathrm{rk}(G)$.

Set $U := C_{m_j}$,

$$\widetilde{G} := C_{m_1} \times \cdots \times C_{m_{j-1}} \times C_{m_{j+1}} \times \cdots \times C_{m_r},$$

and $\widetilde{H} := \widetilde{G} \cap H$. Clearly, $\widetilde{H}$ is a complement to U in H and thus, without loss of generality, we may assume that

$$G = \widetilde{G} \times U \quad \text{and} \quad H = \widetilde{H} \times U.$$

By construction,

$$\operatorname{rk}(\widetilde{G}) = r - 1 < r = \operatorname{rk}(G).$$

Since $|\widetilde{G}| < |G|$ and

$$|\widetilde{G} : \widetilde{H}| = |G : H| > \operatorname{rk}(G) > \operatorname{rk}(\widetilde{G}), \tag{2}$$

we can apply the induction hypothesis to $\widetilde{G}$ and hence there exists a generating transversal $\widetilde{T}$ for $\widetilde{H}\backslash\widetilde{G}$ with $1 \in \widetilde{T}$. Moreover, from Equation (2) we obtain

$$|\widetilde{T} - \{1\}| = |\widetilde{G} : \widetilde{H}| - 1 > \operatorname{rk}(\widetilde{G}). \tag{3}$$

Suppose that $\widetilde{T} - \{1\}$ is a minimal generating set for $\widetilde{G}$. Then Burnside's basis theorem [5, III, Satz 3.15] implies that $|\widetilde{T} - \{1\}| = \operatorname{rk}(\widetilde{G})$, contradicting Equation (3). Thus there exists $1 \neq t \in \widetilde{T}$ such that $t = t_1 \cdots t_k$ for certain $t_1, \ldots, t_k \in \langle \widetilde{T} \backslash \{1, t\} \rangle$.

Now $\widetilde{T} \times \{1\}$ is a transversal for $H\backslash G$ and we set

$$T := (\widetilde{T} \times \{1\} \setminus \{(t, 1)\}) \cup \{(t, u)\},$$

where u is a generator of U. Clearly, $(1, 1) \in T$, and T is a transversal for $H\backslash G$ since $(t, 1)$ and (t, u) lie in the same coset of H in G. It remains to show that T generates G. Recall that $t = t_1 \cdots t_k$ with $t_1, \ldots, t_k \in \langle \widetilde{T} \setminus \{1, t\} \rangle$. As $(t_1, 1), \ldots, (t_k, 1) \in \langle T \rangle$, we also have $(t^{-1}, 1) \in \langle T \rangle$. Hence $(1, u) = (t^{-1}, 1)(t, u) \in \langle T \rangle$ and then $(t, 1) = (t, u)(1, u^{-1}) \in \langle T \rangle$. We conclude that $\langle T \rangle \geqslant \langle \widetilde{T} \rangle \times \langle u \rangle = \widetilde{G} \times U = G$ and we are done. □

Let G be an abelian group and let $p_1, \ldots, p_n$ be the distinct prime divisors of G. Assume that $G = G_1 \times \cdots \times G_n$ with $G_i := O_{p_i}(G)$ for all $1 \leqslant i \leqslant n$. Then an easy induction on n shows that

$$\operatorname{rk}(G) = \max\{\operatorname{rk}(G_i) \mid 1 \leqslant i \leqslant n\}. \tag{4}$$

We now transfer the result of Proposition 2 to an arbitrary abelian group.

Theorem 1. *Let G be an abelian group, let $p_1, \ldots, p_n$ be the distinct prime divisors of G and let $H \leqslant G$. Then*

$$G = G_1 \times \cdots \times G_n \quad \text{and} \quad H = H_1 \times \cdots \times H_n,$$

with $G_i := O_{p_i}(G)$ and $H_i := O_{p_i}(H)$. If

$$\max\{|G_i : H_i| \mid 1 \leqslant i \leqslant n\} > \operatorname{rk}(G),$$

then there exists a generating transversal for $H\backslash G$ containing 1.

Proof. Without loss of generality, we assume that

$$|G_1 : H_1| = \max\{|G_i : H_i| \mid 1 \leqslant i \leqslant n\}$$

and we set $\widetilde{G} := G_2 \times \cdots \times G_n$ and $\widetilde{H} := H_2 \times \cdots \times H_n$. Then $G = G_1 \times \widetilde{G}$ and $H = H_1 \times \widetilde{H}$. Equation (4) yields

$$\begin{aligned} m := |G_1 : H_1| &= \max\{|G_i : H_i| \mid 1 \leqslant i \leqslant n\} > \mathrm{rk}(G) \\ &= \max\{\ \mathrm{rk}(G_i)\ \ \mid 1 \leqslant i \leqslant n\} \geqslant \mathrm{rk}(G_1). \end{aligned}$$

Since G_1 is an abelian p_1-group with $|G_1 : H_1| > \mathrm{rk}(G_1)$, it follows from Proposition 2 that there exists a transversal $T_1 = \{t_1, \dots, t_m\}$ for $H_1 \backslash G_1$ with $t_1 = 1$ and $G_1 = \langle T_1 \rangle$. We are done if $n = 1$. Assume from now on that $n > 1$.

Put $K := H_1 \times \widetilde{G}$. Then $H \leqslant K \leqslant G$ and $|G : K| = |G_1 : H_1| = m$. We next construct a generating transversal for $K \backslash G$ containing 1. Our hypothesis and Equation (4) imply that

$$\begin{aligned} k := \mathrm{rk}(\widetilde{G}) &= \max\{\mathrm{rk}(G_i) \mid 2 \leqslant i \leqslant n\} \leqslant \mathrm{rk}(G) \\ &< \max\{|G_i : H_i| \mid 1 \leqslant i \leqslant n\} = |G_1 : H_1| = m. \end{aligned}$$

Let S be a generating set of $\widetilde{G}$ with $|S| = k$. Then S is a minimal generating set and thus $1 \notin S$. Write $S \cup \{1\} := \{s_1, \dots, s_{k+1}\}$ with $s_1 = 1$. Now $|S \cup \{1\}| = k+1 \leqslant m$, and we set

$$R := \bigcup_{i=1}^{k+1} (t_i, s_i) \cup \bigcup_{j=k+2}^{m} (t_j, s_1).$$

As $t_1 = 1$ and $s_1 = 1$, we have $(1,1) \in R$ and $|R| = m = |G : K|$. We proceed to show that R is a generating transversal for $K \backslash G$. Suppose that $(t_i, s_j), (t_k, s_l) \in R$ such that $(t_i, s_j)(t_k, t_l)^{-1} \in H_1 \times \widetilde{G}$. Then $t_i t_k^{-1} \in H_1$ and as T_1 is a transversal for $H_1 \backslash G_1$, it follows that $i = k$. This implies that $j = l$. We conclude that R is a transversal for $K \backslash G$. The fact $\gcd(|G_1|, |\widetilde{G}|) = 1$ yields that for every $(t, s) \in R$ there exist $a, b \in \mathbb{Z}$ such that $(t, s)^a = (1, s)$ and $(t, s)^b = (t, 1)$. Hence

$$\langle R \rangle \geqslant \langle T_1 \rangle \times \langle S \rangle = G_1 \times \widetilde{G} = G$$

and thus, R is a generating transversal for $K \backslash G$ with $1 \in R$.

Let V be a transversal for $H \backslash K$ with $1 \in V$. Then $T := VR$ is a transversal for $H \backslash G$. Since $1 \in V$, we have $R \subseteq T$ and it follows that $\langle T \rangle \geqslant \langle R \rangle = G$. This implies that T is a generating transversal for $H \backslash G$ with $1 \in T$. $\square$

With this result and Proposition 1 we can construct envelopes of RCC-loops.

Corollary 1. *Let G be a group and let H be a subgroup of G. Let Q be a normal subgroup of G such that G/Q is abelian, $H \cap Q = \{1\}$, $C_H(Q) = \{1\}$ and*

$$\max\{|O_p(G/Q) : O_p(HQ/Q)| \mid p \text{ prime divisor of } G/Q\} > \mathrm{rk}(G/Q).$$

Then there exists a G-invariant generating transversal T for $H \backslash G$ with $1 \in T$, and G acts faithfully on $H\backslash G$; thus (G, H, T) is an envelope of a non-associative RCC*-loop.* □

Recall that T in Corollary 1 arises from multiplying a generating transversal for $HQ\backslash G$ with Q; see Proposition 1. If G is a Frobenius group with kernel Q (in which case $Q = [G, G]$, the commutator subgroup of G), every G-invariant transversal for $H\backslash G$ has this form (see [7, Theorem 3.6]). In general there may be G-invariant transversals, which are not obtained in this way. Since $[G, G]$ is the smallest normal subgroup of G with abelian quotient, we can replace Q by $[G, G]$.

Let us investigate the range of Corollary 1 by comparing with the examples presented in [1, Appendix B]. Let $n \in \{6, 8, 9, 10, 12, 14, 15, 21\}$ and let G be one of the transitive groups of degree n listed in this appendix. In each case, let H denote the stabilizer of 1 in G. Assume that H is abelian, and put $Q := [G, G]$. Using GAP, once checks that $H \cap Q = \{1\}$ in each case. Trivially, $\mathrm{core}_G(H) = \{1\}$, although, in general, $C_H(Q)$ is non-trivial. (The latter just means that the sufficent condition for $\mathrm{core}_G(H)$ to be trivial mentioned after Proposition 1 is not necessary.) Now, unless the GAP-identity number of G is one of $(8, 17)$, $(12, 15)$, $(12, 28)$ or $(12, 42)$, the displayed condition in Corollary 1 is satisfied, so that one of the RCC-loops arising from G and H is of the form constructed in Proposition 1.

Finally, notice that the construction of RCC-loops arising from Proposition 1 is, of course, not restricted to the case G/Q abelian. For example, consider the transitive group L of degree 6 with GAP-identity number $(6, 5)$, and let H denote the stabilizer of 1 in L. Then H is cyclic of order 3, and $L/Z(L) \cong \mathrm{SL}_2(2)$; thus L acts on the Klein four group Q in such a way that $C_H(Q) = \{1\}$. Letting G denote the semidirect product $G = L \ltimes Q$ and identifying H with a subgroup of G, Proposition 1, applied to the invariant transversals for $H\backslash L$, yields invariant transversals for $H\backslash G$, and thus RCC-loops of order 24.

3. A conjecture for RCC-loop folders

In this final section we discuss Conjecture 1. Using GAP [4], this conjecture has been verified for all non-abelian groups of order smaller than 40 by the second author in her master thesis [7], and for the multiplication groups of RCC-loops of order up to 30, by Artic in her dissertation [1].

It follows from a result of Zappa, that Conjecture 1 holds in case H is a Hall subgroup of G. Indeed, Zappa shows that if H is a nilpotent Hall subgroup of G such that there exists a transversal for $H\backslash G$ which is invariant under conjugation by H, then H has a normal complement; see [8, Proposizione XIV 12.1]. Now if H

is abelian, the commutator subgroup of G is contained in this normal complement. In [6], Kochendörffer generalizes Zappa's result. We present the essence of Zappa's and Kochendörffer's argument in the following theorem.

Theorem 2. *Let G be a finite group and let H be an abelian Hall subgroup of G. Suppose that there exists transversal T for $H\backslash G$ which is invariant under conjugation by H. Then $[G,G] \cap H = \{1\}$.*

Proof. This is very much inspired by the proof of [6, Theorem]. The transfer map

$$\tau : G \to H, x \mapsto \prod_{t \in T} \lambda_x^T(t),$$

where $\lambda_x^T(t)$ is the unique element in H such that $tx = \lambda_x^T(t)t'$ for some $t' \in T$, is a group homomorphism; see [5, IV, Hauptsatz 1.4].

Let $h \in H$ and let $h' := \lambda_h^T(t)$ for some $t \in T$. Then $th = h't'$ for some $t' \in T$. It follows that $hh'^{-1} = t^{-1}h't'h'^{-1} \in H$ and since T is H-invariant, we have $h't'h'^{-1} \in T$. Thus, $t = h't'h'^{-1}$. This yields that $h = h' = \lambda_h^T(t)$ and hence

$$\tau(h) = \prod_{t \in T} \lambda_h^T(t) = \prod_{t \in T} h = h^{|G:H|}.$$

As H is an abelian Hall subgroup of G, the map $f : H \to H, h \mapsto h^{|G:H|}$ is an isomorphism. Thus $\ker \tau \cap H = \{1\}$. Furthermore, $G/\ker \tau$ is abelian, because the image of τ is abelian as subgroup of H. Hence, $[G,G] \leqslant \ker \tau$. We conclude that $[G,G] \cap H \leqslant \ker \tau \cap H = \{1\}$. $\square$

In the next example we show that for the conclusion of Conjecture 1 to be true, it is not enough to require the existence of an H-invariant transversal for $H\backslash G$.

Example 1. Let $G := Q_8$ and let $H := Z(G)$. Then H is abelian and every transversal of $H\backslash G$ is H-invariant. However, $H = Z(G) = [G,G]$. Notice that there does not exist any G-invariant transversal for $H\backslash G$.

However, if p is a prime, G is a group of order p^3 and there exists a G-invariant transversal for $H\backslash G$, then Conjecture 1 holds.

Lemma 1. *Let G be a p-group with $[G,G] = Z(G)$ and $|Z(G)| = p$. Suppose that $H \leqslant G$ is abelian and that there exists a G-invariant transversal T of $H\backslash G$ containing 1. Then $[G,G] \cap H = \{1\}$.*

Proof. Since T is G-invariant, T is a union of conjugacy classes of G. As $1 \in T$, we conclude that T contains at least p conjugacy classes with exactly one element, i.e. T contains at least p elements of $Z(G)$. Hence $[G,G] = Z(G) \subseteq T$ and thus $[G,G] \cap H \subseteq T \cap H = \{1\}$. $\square$

This lemma shows that Conjecture 1 holds for groups of order p^3.

Acknowledgements

We are grateful to Gabriele Nebe for a crucial hint in the proof of Theorem 1. We thank Klaus Lux for carefully reading a first version of this article and for his comments which considerably improved the exposition. Finally, we thank the referee for interesting suggestions.

References

[1] **K. Artic**, *On right conjugacy closed loops and right conjugacy closed loop folders*, Dissertation, RWTH Aachen University, 2017.

[2] **M. Aschbacher**, *On Bol loops of exponent 2*, J. Algebra, **288** (2005), 99 − 136.

[3] **R. Baer**, *Nets and groups*, Trans. Amer. Math. Soc., **46** (1939), 110 − 141.

[4] **The GAP Group**, *GAP – Groups, Algorithms, and Programming, Version 4.10.2*; 2019, `https://www.gap-system.org`.

[5] **B. Huppert**, *Endliche Gruppen I*, Springer-Verlag, 1967.

[6] **R. Kochendörffer**, *On supplements in finite groups*, J. Austral. Math. Soc., **3** (1963), 63 − 67.

[7] **L. Ortjohann**, *Invariant transversals in finite groups*, Master Thesis, RWTH Aachen University, 2019, `https://arxiv.org/abs/2005.01380`.

[8] **G. Zappa**, *Fondamenti di teoria dei gruppi, Vol. II*, 18 Edizioni Cremonese, Rome, 1970.

Received May 05, 2020

Lehrstuhl für Algebra und Zahlentheorie
RWTH Aachen University
Pontdriesch 14/16
52062 Aachen
Germany
E-mails: gerhard.hiss@math.rwth-aachen.de, lucia.ortjohann@rwth-aachen.de