From quotient trigroups to groups

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Abstract. In this paper, we study the notion of normality in the category of trigroups, and construct quotient trigroups. This allows us to establish analogues for trigroups of some useful results on groups, namely, the first, second and third isomorphism theorems as well as some of their related corollaries. Our construction provides a new functorial link between the categories of groups and trigroups.

1. Introduction

The concept of digroups originated from the work of J. L. Loday on dialgebras [9], and were formally axiomatized by M. Kinyon in his contribution to the Coquecigrue problem; an analogue of Lie's third theorem which consists to associate a grouplike object to a given Leibniz algebra by "antidifferentiation". More precisely, Kinyon showed in [4] that conjugating digroups equipped with a manifold structure differentiate to Leibniz algebras [7]. Digroups was also independently introduced by K. Liu [5] and R. Felipe [3], and further studied in [10].

In their study of trialgebras and families of polytopes [8], Loday and Ronco provided an axiomatic definition of associative trioids. This led the authors to introduce the category of trigroups as associative trioid – also called trisemigroups– equipped with bar-units and in which each element has a bar-inverse. Trigroups are generalizations of digroups to algebraic structures with three operations, since forgetting one operation of a trigroup yields a digroup structure. Analogue to the relationship between digroups and Leibniz algebras provided by Kinyon in [4], it is shown in [2] that conjugating linear trigroups yields Lie 3-racks [1], which produce Leibniz 3-algebras [6] when differentiated with respect to the distinguish bar-unit.

At the beginning of the last century, Evarist Galois introduced in the classical theory of groups the notion of normal subgroups which played a fundamental role in defining quotient groups and in the so-called isomorphism theorems which are very important in the general development of Group Theory (see [12]). In 2016, Ongay, Velasquez and Wills-Toro defined normal subdigroups [11] and studied a construction of quotient digroups and the corresponding analogues of Isomorphism Theorems. Our aim in this paper is to conduct a similar study on trigroups using a different approach. Our study produces a different quotient on the underlying digroup associated to a trigroup. More precisely, we use the notion of conjugation

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of trigoups provided in [2] to define the concept of normality on trigroups. This allows us to define a congruence for which the quotient set has a group structure, i.e. a trivial trigroup structure. It is worth mentioning that our construction of quotient trigroup produces a functor from the category of trigroups to the category of groups, other than the functor provided in [2].

2. Trigroups

Recall from [2] that a *trisemigroup* $(A, \vdash, \perp, \dashv)$ is a set A equipped with three binary associative operations $\vdash$, $\perp$ and $\dashv$ respectively called left, middle and right, and satisfying the following conditions:

$$\begin{cases} x \vdash (y \vdash z) = (x \dashv y) \vdash z & (p_1) \\ x \vdash (y \vdash z) = (x \perp y) \vdash z & (p_2) \\ x \vdash (y \dashv z) = (x \vdash y) \dashv z & (p_3) \\ x \vdash (y \perp z) = (x \vdash y) \perp z & (p_4) \\ x \dashv (y \dashv z) = x \dashv (y \vdash z) & (p_5) \\ x \dashv (y \dashv z) = x \dashv (y \perp z) & (p_6) \\ (x \perp y) \dashv z = x \perp (y \dashv z) & (p_7) \\ (x \dashv y) \perp z = x \perp (y \vdash z) & (p_8) \end{cases}$$

for all $x, y, z \in A$.

A trisemigroup A is a *trigroup* if there exists an element $1 \in A$ satisfying

$$1 \vdash x = x = x \dashv 1 \quad \text{for all} \quad x \in A \qquad (I)$$

and for all $x \in A$, there exists $x^{-1} \in A$ (called *inverse* of x) such that

$$x \vdash x^{-1} = 1 = x^{-1} \dashv x \quad \text{and} \quad x \perp x^{-1} = 1 = x^{-1} \perp x.$$

Let $\mathfrak{U}_A := \{e \in A : e \vdash x = x = x \dashv e \text{ for all } x, y \in A\}$ be the set of *bar-units* of A.

Recall also that a morphism between two trigroups is a map that preserves the three binary operations and is compatible with bar-units and inverses.

Remark 2.1. [2, Lemma 4.5]

(a) The set $J_A = \{x^{-1} : x \in A\}$ is a group in which $\vdash = \perp = \dashv$.

(b) The mapping $\phi : A \to J_A$ defined by $x \mapsto (x^{-1})^{-1}$ is an epimorphism of trigroups that fixes J_A, and $\mathsf{Ker}\,\phi = \mathfrak{U}_A$.

(c) $x \vdash 1 = 1 \perp x = x \perp 1 = 1 \dashv x = (x^{-1})^{-1}$ for all $x \in A$.

(d) $(x \perp y)^{-1} = y^{-1} \perp x^{-1}$ for all $x, y \in A$.

(e) $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1} = (x \dashv y)^{-1}$ for all $x, y \in A$. Consequently, $((x^{-1})^{-1})^{-1} = x^{-1}$.

(f) $x^{-1} \vdash x \vdash y = x \vdash x^{-1} \vdash y = y$ for all $x, y \in A$.

The following results are consequences of Remark 2.1 and will be heavily used without reference throughout the paper to simplify proofs.

Remark 2.2.

(a) $x^{-1} \vdash 1 = x^{-1} = 1 \dashv x^{-1}$ for all $x \in A$.

(b) $x \vdash y = (x^{-1})^{-1} \vdash y$ for all $x, y \in A$.

(c) $x \dashv y = x \dashv (y^{-1})^{-1}$ for all $x, y \in A$.

Proof. The assertion (a) follows by Remark 2.1(c). For (b) and (c), we have again by Remark 2.1(c), $(x^{-1})^{-1} \vdash y = (x \vdash 1) \vdash y = x \vdash (1 \vdash y) = x \vdash y$ and $x \dashv (y^{-1})^{-1} = x \dashv (1 \dashv y) = (x \dashv 1) \dashv y = x \dashv y$. □

3. Subtrigroups

In this section we define sub-objects in the category of trigroups, and study the concept of normality on these sub-objects.

Definition 3.1. We say that a trigroup A is *trivial* if $A = J_A$.

Proposition 3.2. *A trigroup $(A, \vdash, \perp, \dashv)$ is trivial if and only if $(x^{-1})^{-1} = x$ for all $x \in A$.*

Proof. The proof is straightforward by Definition 3.1. □

For the rest of the paper, all trigroups are assumed to be non-trivial unless otherwise stated.

Definition 3.3. Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subset S of A is said to be a *subtrigroup* of A if $(S, \vdash, \perp, \dashv)$ is a trigroup with distinguish bar-unit 1.

Proposition 3.4. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1, and H a nonempty subset of A. H is a subtrigroup of A if and only H is closed under the operations $\vdash, \perp, \dashv$, and $x^{-1} \in H$ for all $x \in H$.*

Proof. The proof of the forward direction is obvious. For the converse, it is enough to verify that $1 \in H$. Indeed, since H is nonempty there is some $x_0 \in H$, which yields $x_0^{-1} \in H$, and thus $1 = x_0 \vdash x_0^{-1} \in H$. □

Proposition 3.5. *Let A be a trigroup. Then $(J_A, \vdash = \dashv = \perp)$ and $(\mathfrak{U}_A, \vdash, \dashv, \perp)$ are subtrigroups of A.*

Proof. J_A is a subtrigroup of A since by Remark 2.1(a), $J_A \subseteq A$ and J_A is a group in which $\vdash = \perp = \dashv$. To show that $\mathfrak{U}_A$ is a subtrigroup of A, notice that for all $e, e' \in \mathfrak{U}_A$, $e \vdash e' = e', \quad e \dashv e' = e, \quad (e \perp e') \vdash x \overset{p2}{=} e \vdash (e' \vdash x) = e \vdash x = x$ and $x \dashv (e \perp e') \overset{p6}{=} x \dashv (e \dashv e') = (x \dashv e) \dashv e' = x \dashv e = x$ for all $x \in A$. So $\mathfrak{U}_A$ is closed under the operations $\vdash, \perp, \dashv$. In addition, $e^{-1} \in \mathfrak{U}_A$ by [2, Lemma 4.6]. The result follows by Proposition 3.4. $\square$

Proposition 3.6. *Let $\phi : A \to A'$ be a morphism of trigroups. Then:*

(a) $\mathsf{Ker}\,\phi$ *is a subtrigroup of A.*

(b) *If S is a subtrigroup of A, then $\phi(S)$ is a subtrigroup of A'.*

(c) *If S' is a subtrigroup of A', then $\phi^{-1}(S')$ is a subtrigroup of A.*

Proof. To prove (a), first notice that $\phi(1_A) = 1_{A'}$, so $\mathsf{Ker}\,\phi \neq \emptyset$. Now Let $x, y \in \mathsf{Ker}\,\phi$. Then $\phi(x \vdash y) = \phi(x) \vdash \phi(y) = 1_{A'} \vdash 1_{A'} = 1_{A'}$, $\phi(x \dashv y) = \phi(x) \dashv \phi(y) = 1_{A'} \dashv 1_{A'} = 1_{A'}$, $\phi(x \perp y) = \phi(x) \perp \phi(y) = 1_{A'} \perp 1_{A'} = 1$ and $\phi(x^{-1}) = (\phi(x))^{-1} = 1_{A'}$. Thus by proposition 3.4, $\mathsf{Ker}\,\phi$ is a subtrigroup of A. The proofs of (b) and (c) are similar. $\square$

Consider the following sets: $x \star S = \{x \star s,\ s \in S\}$ and $S \star x = \{s \star x,\ s \in S\}$, where $\star \in \{\vdash, \perp, \dashv\}$. In [2], the operation $[-,-,-] : A \times A \times A \to A$ given by $[x, y, z] = (x \perp y) \vdash z \dashv (y^{-1} \perp x^{-1})$, was defined as a generalization of the conjugation on digroups [4, Equation (13)] to trigroups. Using this operation, we define normality of subtrigroups as follows:

Definition 3.7. Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subtrigroup S of A is said to be *normal* if $(x \perp y) \vdash S \dashv (y^{-1} \perp x^{-1}) \subseteq S$ for all $x, y \in A$.

This definition extends the following definition of normality in digroups to trigroups.

Definition 3.8. [11, Definition 4] A subdigroup S of a digroup $(A, \vdash, \dashv)$ is said to be *normal* if $x \vdash S \dashv x^{-1} \subseteq S$ for all $x \in A$.

It turns out that normality in a trigroup is completely determined by its underlying digroup structure, as proven in the following Lemma.

Lemma 3.9. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1 and S a subtrigroup of A. Then S is a normal subtrigroup of A iff S is a normal subdigroup of the underlying digroup $(A, \vdash, \dashv)$.*

Proof. Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then for all $x \in A$,

$$\begin{aligned} x \vdash S \dashv x^{-1} &= x \vdash (1 \vdash S) \dashv x^{-1} \overset{p_2}{=} (x \perp 1) \vdash S \dashv x^{-1} \\ &= (x \perp 1) \vdash (S \dashv 1) \dashv x^{-1} = (x \perp 1) \vdash S \dashv (1 \dashv x^{-1}) \\ &\overset{p_6}{=} (x \perp 1) \vdash S \dashv (1 \perp x^{-1}) \subseteq S. \end{aligned}$$

The converse is obvious since for all $x, y \in A$ we have by setting $z = x \perp y$, $(x \perp y) \vdash S \dashv (x \perp y)^{-1} = z \vdash S \dashv z^{-1} \subseteq S$ □

Lemma 3.10. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup with distinguish bar-unit 1. A subtrigroup S of A is said to be normal if and only if $(x \perp y) \vdash S = S \dashv (x \perp y)$ for all $x, y \in A$.*

Proof. Assume that S is a normal subtrigroup of A. Let $x, y \in A$ and set $z = x \perp y$. For all $s \in S$, we have: $z \vdash s \dashv z^{-1} = s'$ for some $s' \in S$, i.e., $z \vdash s = z \vdash (s \dashv 1)$ $= z \vdash (s \dashv (z^{-1} \dashv z)) = z \vdash ((s \dashv z^{-1}) \dashv z) \overset{p_3}{=} (z \vdash s \dashv z^{-1}) \dashv z = s' \dashv z$. So $(x \perp y) \vdash S \subseteq S \dashv (x \perp y)$.

For the reverse inclusion,

$$\begin{aligned} S \dashv z &= ((z \vdash z^{-1}) \vdash S \dashv 1) \dashv z = (z \vdash (z^{-1} \vdash S) \dashv 1) \dashv z \\ &= z \vdash ((z^{-1} \vdash S) \dashv 1) \dashv z)) = z \vdash (z^{-1} \vdash S \dashv (1 \dashv z)) \\ &= z \vdash (z^{-1} \vdash S \dashv (z^{-1})^{-1}) \subseteq z \vdash S \quad \text{since} \quad S \quad \text{is normal.} \end{aligned}$$

Conversely, assume that $(x \perp y) \vdash S = S \dashv (x \perp y)$ for all $x, y \in A$. Then,

$$\begin{aligned} (x \perp y) \vdash S \dashv (y^{-1} \perp x^{-1}) &= ((x \perp y) \vdash S) \dashv (x \perp y)^{-1} \\ &= (S \dashv (x \perp y)) \dashv (x \perp y)^{-1} \\ &= S \dashv ((x \perp y) \dashv (x \perp y)^{-1}) \\ &= S \quad \text{since} \quad (x \perp y) \dashv (x \perp y)^{-1} \in \mathfrak{U}_A. \end{aligned}$$

Therefore S is a normal subtrigroup of A. □

The following Lemma is the normality transfer condition for trigroups.

Lemma 3.11. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup. If S is a subtrigroup of A and R is a normal subtrigroup of A, then $S \cap R$ is a normal subtrigroup of S.*

Proof. The proof is obvious since for all $s \in S$, we have $s \vdash S \cap R \dashv s^{-1} \subseteq S$ due to closure under the operations $\vdash, \dashv$, and $s \vdash S \cap R \dashv s^{-1} \subseteq R$ since R is normal in A. The result follows by Lemma 3.9. □

Remark 3.12. Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then $S \perp x^{-1} = x^{-1} \vdash S$ for all $x \in A$.

Proof. Let $x \in A$. Since $x^{-1} \vdash x \in \mathfrak{U}_A$, we have

$$\begin{aligned} S \perp x^{-1} &= (x^{-1} \vdash x) \vdash (S \perp x^{-1}) = x^{-1} \vdash (x \vdash (S \perp x^{-1})) \\ &\overset{p_7}{=} x^{-1} \vdash ((x \vdash S) \perp x^{-1}) = x^{-1} \vdash ((S \dashv x) \perp x^{-1}) \\ &= x^{-1} \vdash (S \perp (x \vdash x^{-1})) = x^{-1} \vdash (S \perp 1) = x^{-1} \vdash (1 \dashv S) \\ &\overset{p_3}{=} (x^{-1} \vdash 1) \dashv S = x^{-1} \dashv S. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.13. *Let $\phi : A \to A'$ be a morphism of trigroups. Then $\mathsf{Ker}\,\phi$ is a normal subtrigroup of A. Consequently, the set $\mathfrak{U}_A$ of bar-units of A is a normal subtrigroup of A.*

Proof. By Proposition 3.5, Proposition 3.6 and Lemma 3.9, it remains to show that for all $x \in A$, $x \vdash \mathsf{Ker}\,\phi \dashv x^{-1} \subseteq \mathsf{Ker}\,\phi$. Indeed, let $z \in \mathsf{Ker}\,\phi$,

$$\begin{aligned}\phi(x \vdash z \dashv x^{-1}) &= \phi(x) \vdash \phi(z) \dashv \phi(x^{-1}) = \phi(x) \vdash 1 \dashv (\phi(x))^{-1}\\ &= (\phi(x) \vdash 1) \dashv (\phi(x))^{-1} = ((\phi(x))^{-1})^{-1} \dashv (\phi(x))^{-1} = 1.\end{aligned}$$

So $x \vdash z \dashv x^{-1} \in \mathsf{Ker}\,\phi$. Consequently, $\mathfrak{U}_A$ is a normal subtrigroup by Remark 2.1. $\square$

Lemma 3.14. *Let A be a trigroup. Then the group J_A of inverses of elements in A is a normal subtrigroup of A.*

Proof. By Proposition 3.5 and Lemma 3.9, it is enough to show that if $x \in A$, then $x \vdash J_A \dashv x^{-1} \subseteq J_A$. Notice that for all $y \in A$,

$$x \vdash y = x \vdash (1 \vdash y) = (x \vdash 1) \vdash y = (x^{-1})^{-1} \vdash y.$$

So $x \vdash J_A \dashv x^{-1} = (x^{-1})^{-1} \vdash J_A \dashv x^{-1} \subseteq J_A$ since $x^{-1}, (x^{-1})^{-1} \in J_A$. $\square$

Lemma 3.15. *Let $\phi : A \to A'$ be a morphism of trigroups. Then,*

(a) *If S is a normal subtrigroup of A and ϕ is surjective, then $\phi(S)$ is a normal subtrigroup of A'.*

(b) *If S' is a normal subtrigroup of A', then $\phi^{-1}(S')$ is a normal subtrigroup of A.*

Proof. To prove (a), assume that S is a normal subtrigroup of A and ϕ is surjective. By Proposition 3.6 and Lemma 3.9, it remains to show that $y \vdash \phi(S) \dashv y^{-1} \subseteq \phi(S)$ for all $y \in A'$. let $y \in A'$ and $s \in S$. Then, $y = \phi(x)$ for some $x, \in A$. We have

$$\begin{aligned}y \vdash \phi(s) \dashv y^{-1} &= \phi(x) \vdash \phi(s) \dashv (\phi(x))^{-1} = \phi(x) \vdash \phi(s) \dashv \phi(x^{-1})\\ &= \phi(x \vdash s \dashv x^{-1}) \in \phi(S) \text{ since } S \text{ is normal in } A.\end{aligned}$$

The proof of (b) is similar. $\square$

4. Quotient trigroups

4.1. From quotient trigroups to groups

In an effort to study the notion of quotient of a given trigroup by a normal subtrigroup, we define an equivalence relation for which the equivalence classes are the cosets of the normal subtrigroup, and the equivalence class of the identity element is the normal subtrigroup.

Lemma 4.1. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup, and S a subtrigroup of A. Then the following assertions are true:*

(*a*) $g \vdash S = S \iff g^{-1} \in S \iff S \dashv g = S$ *for all* $g \in A$.

(*b*) $g \vdash S = h \vdash S \iff g^{-1} \dashv h \in S$.

(*c*) $S \dashv g = S \dashv h, \iff g \vdash h^{-1} \in S$

Proof. For (a), it is clear that for all $g \in A$, $(g^{-1})^{-1} = g \vdash 1 \in g \vdash S$. So if $g \vdash S = S$, then $(g^{-1})^{-1} \in S$ which implies $g^{-1} \in S$. Conversely, let $g \in A$ such that $g^{-1} \in S$. So $g \vdash 1 = (g^{-1})^{-1} \in S$. Then $g \vdash S = g \vdash (1 \vdash S) = (g \vdash 1) \vdash S \subseteq S$ since S is closed under the operation $\vdash$. For the reverse inclusion, we have for all $s \in S$, that $s = 1 \vdash s = (g \vdash g^{-1}) \vdash s = g \vdash (g^{-1} \vdash s) \in g \vdash S$. This proves that $g \vdash S = S \iff g^{-1} \in S$. The proof of the other equivalence is similar.

To prove (b), let $g, h \in A$ such that $g \vdash S = h \vdash S$, then there exists $s \in S$ such that $h \vdash 1 = g \vdash s$. So

$$\begin{aligned} g^{-1} \dashv h &= g^{-1} \dashv (h \dashv 1) \overset{p_5)}{=} g^{-1} \dashv (h \vdash 1) = g^{-1} \dashv (g \vdash s) \\ &\overset{p_5}{=} g^{-1} \dashv (g \dashv s) = (g^{-1} \dashv g) \dashv s = 1 \dashv s \in S. \end{aligned}$$

Conversely, let $g, h \in A$ such that $g^{-1} \dashv h \in S$. Then

$$\begin{aligned} h \vdash S &= ((g \vdash g^{-1}) \vdash h) \vdash S = (g \vdash (g^{-1} \vdash h)) \vdash S \\ &= g \vdash ((g^{-1} \vdash h) \vdash S) = g \vdash (g^{-1} \vdash (h \vdash S)) \\ &\overset{p_1}{=} g \vdash ((g^{-1} \dashv h) \vdash S) \subseteq g \vdash S. \end{aligned}$$

The reverse inclusion holds also since $h^{-1} \dashv g = h^{-1} \dashv (g^{-1})^{-1} = (g^{-1} \dashv h)^{-1} \in S$.

The proof of (c) is similar to the proof of (b). $\square$

Proposition 4.2. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a subtrigroup of A. Define the relation: For $x, y \in A$,*

$$x \sim y \iff x^{-1} \dashv y \quad \in \ S.$$

Then $\sim$ is an equivalence relation and the equivalence classes are the left cosets $x \vdash S$, $x \in A$ (orbits of the action of S on A.)

Proof. For all $x, y, z \in A$, we have

i) $x^{-1} \dashv x = 1 \in S$,

ii) if $x^{-1} \dashv y \in S$ then $y^{-1} \dashv x = y^{-1} \dashv (x^{-1})^{-1} = (x^{-1} \dashv y)^{-1} \in S$,

iii) if $x^{-1} \dashv y \in S$ and $y^{-1} \dashv z \in S$, then

$$\begin{aligned} x^{-1} \dashv z &= (x^{-1} \vdash 1) \dashv z = (x^{-1} \vdash (y \vdash y^{-1})) \dashv z \overset{p_1}{=} ((x^{-1} \dashv y) \vdash y^{-1}) \dashv z) \\ &\overset{p_3}{=} (x^{-1} \dashv y) \vdash (y^{-1} \dashv z) \in S. \end{aligned}$$

These prove that $\sim$ is respectively reflexive, symmetric and transitive, and by Lemma 4.1(b), the equivalence classes are left cosets $x \vdash S$ □

By the fundamental theorem of equivalence relations, the relation $\sim$ partitions A into the left cosets $x \vdash S$, $x \in A$. Let A/S be the set of left cosets. Define the following binary operations $\rhd, \vartriangle, \lhd$: $A/S \times A/S \to A/S$ by:

$$(g \vdash S) \rhd (h \vdash S) = (h \vdash g) \vdash S$$

$$(g \vdash S) \lhd (h \vdash S) = (h \dashv g) \vdash S$$

$$(g \vdash S) \vartriangle (h \vdash S) = (h \perp g) \vdash S.$$

We have the following result.

Lemma 4.3. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then for all $x, y \in A$, $x \sim y \iff x^{-1} \vdash S \dashv y \subseteq S$.*

Proof. Let $x, y \in A$ such that $x \sim y$ i.e. $x^{-1} \dashv y \in S$. Since $y \dashv y^{-1} \in \mathfrak{U}_A$, it follows that for all $s \in S$,

$$\begin{aligned} (x^{-1} \vdash s) \dashv y &= (x^{-1} \vdash ((y \dashv y^{-1}) \vdash s)) \dashv y \overset{p_1}{=} (x^{-1} \vdash (y \vdash (y^{-1} \vdash s))) \dashv y \\ &\overset{p_1}{=} ((x^{-1} \dashv y) \vdash (y^{-1} \vdash s)) \dashv y \overset{p_3}{=} (x^{-1} \dashv y) \vdash (y^{-1} \vdash (s \dashv y)) \in S \end{aligned}$$

since S is normal and S is closed under $\vdash$. For the converse, if $x, y \in A$ such that $x^{-1} \vdash S \dashv y \subseteq S$, then $x^{-1} \dashv y = (x^{-1} \vdash 1) \dashv y \in (x^{-1} \vdash S) \dashv y \subseteq S$. □

Proposition 4.4. *Let $(A, \vdash, \perp, \dashv)$ be a trigroup and S a normal subtrigroup of A. Then the binary operations $\rhd, \vartriangle, \lhd$ are well-defined and equip A/S with a structure of a group with unit S and the inverse of the class $g \vdash S$ is the class $g^{-1} \vdash S$.*

Proof. First we verify that the operations $\rhd$, $\vartriangle$, and $\lhd$ are equal, then we verify their well-definition and their compatibility with the equivalence relation $\sim$. Indeed, let $x, y \in A$. Then, since $y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1} = y^{-1} \perp x^{-1}$ as $\vdash = \dashv = \perp$ in J_A, It follows that

$$(x \vdash y)^{-1} \dashv (x \dashv y) = (x \perp y)^{-1} \dashv (x \dashv y) = (x \dashv y)^{-1} \dashv (x \dashv y) = 1 \in S.$$

So $(x \vdash y) \sim (x \perp y) \sim (x \dashv y)$. Therefore,

$$(x \vdash S) \rhd (y \vdash S) = (x \vdash S) \vartriangle (y \vdash S) = (x \vdash S) \lhd (y \vdash S).$$

To show the well-definition, let $x, y, a, b \in A$ such that $x \sim y, a \sim b$.

So $z := a^{-1} \dashv b \in S$ and thus $x^{-1} \vdash z \dashv y \in S$ by Lemma 4.3. Then

$$\begin{aligned}(a \vdash x)^{-1} \dashv (b \vdash y) &= (x^{-1} \vdash a^{-1}) \dashv (b \vdash y) \overset{p_3}{=} x^{-1} \vdash (a^{-1} \dashv (b \vdash y)) \\ &\overset{p_5}{=} x^{-1} \vdash (a^{-1} \dashv (b \dashv y)) = x^{-1} \vdash ((a^{-1} \dashv b) \dashv y) \\ &= x^{-1} \vdash (z \dashv y) \in S.\end{aligned}$$

So $(a \vdash x) \sim (b \vdash y)$.

To show that S is the unique bar-unit , we prove that $\mathfrak{U}_{A/S} = \{S\}$. Indeed, notice that for all $a, x \in A$,

$$(x \dashv a)^{-1} \dashv x = (a^{-1} \dashv x^{-1}) \dashv x = a^{-1} \dashv (x^{-1} \dashv x) = a^{-1} \dashv 1 = a^{-1}$$

and

$$(a \vdash x)^{-1} \dashv x = (x^{-1} \vdash a^{-1}) \dashv x = (a^{-1} \vdash x^{-1}) \dashv x \overset{p_3}{=} a^{-1} \vdash (x^{-1} \dashv x) = a^{-1}.$$

So $x \dashv a \ \sim x \iff a^{-1} \in S \iff a \vdash x \sim x$. Therefore,

$$\mathfrak{U}_{A/S} = \{a \vdash S : a^{-1} \in S\} = \{S\}$$

by the first property of Lemma 4.1. That the inverse of the class $g \vdash S$ is the class $g^{-1} \vdash S$ is straighforward. We can now conclude that if $(A, \vdash, \perp, \dashv)$ is a trigroup, then $(A/S, \rhd{=}\vartriangle{=}\lhd)$ is a group. $\square$

Remark 4.5. Proposition 4.4 provides another functor from the category of trigroups to the category of groups.

Remark 4.6. Note that every normal subtrigroup is the kernel of some trigroup homomorphism. More precisely, if S is a normal subtrigroup of a trigroup A, then the natural projection $A \to A/S$ is a homomorphism with kernel equal to S.

4.2. A First Isomorphism Theorem for trigroups

Lemma 4.7. *Let $\phi : A \to A'$ be a morphism of trigroups and S a normal subtrigroup of A containing $\mathsf{Ker}\,\phi$. If $t \in A$ such that $\phi(t) \in \phi(S)$, then $t^{-1} \in S$.*

Proof. Under the hypothesis, we have $\phi(t) = \phi(s)$ for some $s \in S$. So $\phi(t \vdash s^{-1}) = \phi(t) \vdash \phi(s^{-1}) = 1$. Thus $t \vdash s^{-1} \in \mathsf{Ker}\,\phi \subseteq S$. Therefore $t^{-1} = ((t^{-1})^{-1})^{-1} \in S$ since $(t^{-1})^{-1} = t \vdash 1 = t \vdash (s^{-1} \dashv s)) = (t \vdash s^{-1}) \dashv s \ \ \in S$. $\square$

Proposition 4.8. *Let A and A' be two trigroups and S a normal subtrigroup of A. Let $\phi : A \to A'$ be a morphism of trigroups such that* $\mathsf{Ker}(\phi) \subseteq S$. *Then there is an isomorphism of groups $\hat{\phi} : A/S \to Im\phi/\phi(S)$. In particular, if $S = ker(\phi)$ then this isomorphism becomes $\hat{\phi} : A/ker(\phi) \to Im\phi/\{1\}$.*

Proof. Since S is a normal subtrigroup of A and $\phi : A \to A$ a morphism of trigroups, then $\phi(S)$ is normal subtrigroup of $Im\phi$ by Lemma 3.15. Moreover

$$\begin{aligned} x \sim y &\Longleftrightarrow x^{-1} \dashv y \in S \Longleftrightarrow \phi(x^{-1} \dashv y) \in \phi(S) \Longleftrightarrow \phi(x^{-1}) \dashv \phi(y) \in \phi(S) \\ &\Longleftrightarrow (\phi(x))^{-1} \dashv \phi(y) \in \phi(S) \Longleftrightarrow \phi(x) \sim \phi(y). \end{aligned}$$

Note that the implication $x^{-1} \dashv y \in S \Longleftarrow \phi(x^{-1} \dashv y) \in \phi(S)$ above is due to Lemma 4.7 since $y^{-1} \dashv x = y^{-1} \dashv (x^{-1})^{-1} = (x^{-1} \dashv y)^{-1} \in S$ and the relation $\sim$ is symmetric. Therefore ϕ induces the isomorphism: $\hat{\phi} : A/S \longrightarrow Im\phi/\phi(S)$ such that $x \vdash S \longmapsto \hat{\phi}(x \vdash S) = \phi(x) \vdash \phi(S)$. □

Corollary 4.9. *Let A be a trigroup. Then there is a group isomorphism*

$$A/\mathfrak{U}_A \cong J_A.$$

Proof. By the assertion (b) of Remark 2.1, the mapping $A \to J_A$ defined by $x \mapsto (x^{-1})^{-1}$ is an epimorphism of trigroups with kernel $\mathfrak{U}_A$. Moreover $J_A/\{1\} = J_A$ since J_A is a group. We conclude the proof using Proposition 4.8. □

Corollary 4.10. *Let A be a trigroup. Then there is a group isomorphism*

$$A/\{1\} \cong A/\mathfrak{U}_A.$$

Proof. Clearly, the map $A \xrightarrow{\pi} A/\mathfrak{U}_A$, $a \longmapsto a \vdash \mathfrak{U}_A$ is a trigroup epimorphism whose kernel is $ker(\pi) = \{1\}$ since by the first property of Lemma 4.1, we have $a \vdash \mathfrak{U}_A = \mathfrak{U}_A \iff a^{-1} \in \mathfrak{U}_A \cap J_A = \{1\} \iff a = 1$. By proposition 4.8, there is a group isomorphism $A/\{1\} \cong A/\mathfrak{U}_A$. □

Corollary 4.11. *Let A and B be two trigroups. Then A can be identified with a normal subtrigroup $A \times \mathfrak{U}_B$ of $A \times B$ and there is a group isomorphism $\frac{A \times B}{A \times \mathfrak{U}_B} \cong B/\{1\}$.*

Proof. Assume that $(A, \vdash, \dashv, \perp)$ and $(B, \vdash', \dashv', \perp')$ are two trigroups. Then clearly $(A \times B, \rhd, \lhd, \trianglerighteq)$ is a trigroup with operations given by

$(a_1, b_1) \rhd (a_2, b_2) = (a_1 \vdash a_2, b_1 \vdash' b_2)$,
$(a_1, b_1) \lhd (a_2, b_2) = (a_1 \dashv a_2, b_1 \dashv' b_2)$,
$(a_1, b_1) \trianglerighteq (a_2, b_2) = (a_1 \perp a_2, b_1 \perp' b_2)$.

It is easy to verify that the map $A \times B \xrightarrow{\theta} B/\mathfrak{U}_B$, $(a, b) \longmapsto b \vdash \mathfrak{U}_B$ is a trigroup epimorphism whose kernel is $ker(\theta) = A \times \mathfrak{U}_B$ by the first property of Lemma 4.1 and since $e \in \mathfrak{U}_B \iff e^{-1} \in \mathfrak{U}_B$. By proposition 4.8, there is a group isomorphism $\frac{A \times B}{A \times \mathfrak{U}_B} \cong B/\mathfrak{U}_B$. Now since $B/\mathfrak{U}_B \cong B/\{1\}$ thanks to Corollary 4.10, the proof is complete. □

4.3. A Second Isomorphism Theorem for trigroups

In this section, we use our construction of quotients on trigroups to prove an analogue of the second isomorphism theorem for trigroups. Consider the following set:

$S \star S' = \{x \star x',\ x \in S \text{ and } x' \in S'\}$ where $\star \in \{\vdash, \dashv\}$.

Lemma 4.12. *Let A be a trigroup, and S, R two subtrigroups of A such that $s \vdash R = R \dashv s$ for all $s \in S$. Then the following hold:*

(*a*) *The set $\widehat{R} =: \{x \in A \ :\ x^{-1} \in R\}$ is a subtrigroup of A containing R.*

(*b*) *$S \vdash R$ is a subtrigroup of A.*

(*c*) *R is a normal subtrigroup of $S \vdash R$.*

(*d*) *$S \cap \widehat{R}$ is a normal subtrigroup of S.*

Proof. The proof of (a) is straightforward since R is a subtrigroup of A.

To show (b), we verify the properties of Proposition 3.4. Indeed, Let $s, s_1 \in S$ and $r, r_1 \in R$. Since $R \dashv s_1 = s_1 \vdash R$, it follows that $r \dashv s_1 = s_1 \vdash r_2$ for some $r_2 \in R$.

1) $$\begin{aligned}(s \vdash r) \vdash (s_1 \vdash r_1) &\overset{p_1}{=} ((s \vdash r) \dashv s_1) \vdash r_1 \overset{p_3}{=} (s \vdash (r \dashv s_1)) \vdash r_1 \\ &= (s \vdash (s_1 \vdash r_2)) \vdash r_1 = (s \vdash s_1) \vdash (r_2 \vdash r_1) \in S \vdash R.\end{aligned}$$

2) $$\begin{aligned}(s \vdash r) \dashv (s_1 \vdash r_1) &\overset{p_3}{=} s \vdash (r \dashv (s_1 \vdash r_1)) \overset{p_5}{=} s \vdash (r \dashv (s_1 \dashv r_1)) \\ &= s \vdash ((r \dashv s_1) \dashv r_1) = s \vdash ((s_1 \vdash r_2) \dashv r_1) \\ &\overset{p_3}{=} (s \vdash (s_1 \vdash r_2)) \dashv r_1 = ((s \vdash s_1) \vdash r_2) \dashv r_1 \\ &\overset{p_3}{=} (s \vdash s_1) \vdash (r_2 \dashv r_1) \in S \vdash R.\end{aligned}$$

3) $$\begin{aligned}(s \vdash r) \perp (s_1 \vdash r_1) &\overset{p_8}{=} ((s \vdash r) \dashv s_1) \perp r_1 \overset{p_3}{=} (s \vdash (r \dashv s_1)) \perp r_1 \\ &= (s \vdash (s_1 \vdash r_2)) \perp r_1 = ((s \vdash s_1) \vdash r_2) \perp r_1 \\ &\overset{p_4}{=} (s \vdash s_1) \vdash (r_2 \perp r_1) \in S \vdash R.\end{aligned}$$

4) Since $R \dashv s^{-1} = s^{-1} \vdash R$, then $r^{-1} \dashv s^{-1} = s^{-1} \vdash r_0$ for some $r_0 \in R$. So $(s \vdash r)^{-1} = r^{-1} \vdash s^{-1} = r^{-1} \dashv s^{-1} = s^{-1} \vdash r_0 \in \ S \vdash R$.

To show (c), we first notice that $R \subseteq S \vdash R$ since $r = 1 \vdash r$ for all $r \in R$. Now let $s \in S$ and $r, r_0 \in R$. Then

$$\begin{aligned}(s \vdash r) \vdash r_0 \dashv (s \vdash r)^{-1} &= (s \vdash r) \vdash r_0 \dashv (r^{-1} \vdash s^{-1}) \\ &\overset{p_5}{=} (s \vdash r) \vdash r_0 \dashv (r^{-1} \dashv s^{-1}) \\ &= s \vdash (r \vdash r_0 \dashv r^{-1}) \dashv s^{-1} \in s \vdash R \dashv s^{-1} \subseteq R.\end{aligned}$$

To show (d), we first notice that $S \cap \widehat{R} \neq \emptyset$ as $1 \in S \cap \widehat{R}$. Also it is clear that $S \cap \widehat{R} \subseteq S$. Now for all $s \in S$ and $t \in S \cap \widehat{R}$, we have $s \vdash t \dashv s^{-1} \in S$ since

S is a subtrigroup of A. Also, since $s \vdash R = R \dashv s$, $s \vdash t^{-1} = t' \dashv s$ for some $t' \in R$. So $s \vdash t^{-1} \dashv s^{-1} = (t' \dashv s) \dashv s^{-1} = t' \dashv (s \dashv s^{-1}) = t' \in R$. We now have $(s \vdash t \dashv s^{-1})^{-1} = (s^{-1})^{-1} \vdash t^{-1} \dashv s^{-1} = s \vdash t^{-1} \dashv s^{-1} \in R$, and thus $s \vdash t \dashv s^{-1} \in \widehat{R}$. Therefore, $s \vdash t \dashv s^{-1} \in S \cap \widehat{R}$. $\square$

Corollary 4.13. *Let A be a trigroup, and S and R two subtrigroups of A such that $s \vdash R = R \dashv s$ for all $s \in S$. Then there is a group isomorphism*

$$(S \vdash R)/R \cong S/(S \cap \widehat{R}).$$

Proof. By Lemma 4.12, $S \vdash R$ is a subtrigroup of A having R as a normal subtrigroup, and that $S \cap R$ is a normal subtrigroup of S. The map

$$S \longrightarrow (S \vdash R)/R, \quad s \mapsto s \vdash R$$

is clearly a surjective homomorphism. Its kernel is $S \cap \widehat{R}$ by the first property of Lemma 4.1. The result now follows using Proposition 4.8. $\square$

Corollary 4.14. *Let A be a trigroup, R a normal subtrigroup of A and S a subtrigroup of A such that $A = S \vdash R$. Then*

$$A/R \cong S/(S \cap \widehat{R}).$$

Proof. The proof is straightforward as a direct consequence of Corollary 4.13. $\square$

Corollary 4.15. *Let A be a trigroup. Then there are group isomorphisms*

$$(J_A \vdash \mathfrak{U}_A)/\mathfrak{U}_A \cong J_A \quad \textit{and} \quad (\mathfrak{U}_A \vdash J_A)/J_A \cong \{\mathfrak{U}_A\}$$

Proof. By Lemma 3.13 and Lemma 3.14, J_A and $\mathfrak{U}_A$ are normal subtrigroups of A. This implies that $e \vdash J_A = J_A \dashv e$ and $j \vdash \mathfrak{U}_A = \mathfrak{U}_A \dashv j$ for all $e \in \mathfrak{U}_A$ and $j \in J_A$. So, $\mathfrak{U}_A$ and J_A are respectively normal subgroups of $J_A \vdash \mathfrak{U}_A$ and $\mathfrak{U}_A \vdash J_A$ by Lemma 4.12. Note that $\widehat{J_A} = A$, thus $\mathfrak{U}_A \cap \widehat{J_A} = \mathfrak{U}_A$. Also, since J_A is a group, $J_A \cap \widehat{\mathfrak{U}}_A = \{1\}$. We now have $(J_A \vdash \mathfrak{U}_A)/\mathfrak{U}_A \cong J_A/\{1\} \cong J_A$ and $(\mathfrak{U}_A \vdash J_A)/J_A \cong \mathfrak{U}_A/\mathfrak{U}_A \cong \{\mathfrak{U}_A\}$ by Corollary 4.13. $\square$

4.4. A Third Isomorphism Theorem for trigroups

Lemma 4.16. *Let A be a trigroup, and S, R two normal subtrigroups of A such that S is a subtrigroup of R. Then $\widehat{R}/S$ is a normal subgroup of A/S.*

Proof. By Lemma 3.11, S is a normal subtrigroup of $\widehat{R}$, and $\widehat{R}/S$ is a subtrigroup of A/S. Now, let $a \in A$. Then for all $r \in R$, $r^{-1} \in R$. So, $(a \vdash r \dashv a^{-1})^{-1} = (a^{-1})^{-1} \vdash r^{-1} \dashv a^{-1} \in R$ since R is a normal subtrigroup of A. Hence $a \vdash r \dashv a^{-1} \in \widehat{R}$. We now have

$$\begin{aligned}(a \vdash S) \rhd (r \vdash S) \lhd (a^{-1} \vdash S) &= ((a \vdash r) \vdash S) \lhd (a^{-1} \vdash S)\\ &= (a \vdash r \dashv a^{-1}) \vdash S \in \widehat{R}/S.\end{aligned}$$

Hence $\widehat{R}/S$ is a normal subtrigroup of A/S. $\square$

Proposition 4.17. *Let A be a trigroup, and S and R two normal subtrigroups of A such that S is a normal subgroup of R. Then, there is a group ismorphism*

$$(A/S)/(\hat{R}/S) \cong A/R.$$

Proof. Under the hypothesis of the proposition, S is also a normal subtrigroup of $\hat{R}$. Now consider the map: $A/S \xrightarrow{\tau} A/R, (a \vdash S) \longmapsto (a \vdash R)$. Then τ is obviously a surjective morphism of groups whose kernel is $ker(\tau) = \hat{R}/S$, by the first property of Lemma 4.1. We now conclude by proposition 4.8 that $(A/S)/(\hat{R}/S) \cong A/R$. □

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