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On regularities in *po*-ternary semigroups

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Abstract. In this paper, we show the way to get into some results of partially ordered (in short, *po*-) ternary semigroup based on quasi-ideals, bi-ideals and semiprime ideals. We extend some results of *po*-semigroup into *po*-ternary semigroup under certain methodology. In particular, we characterize some properties of regular *po*-ternary semigroup, left (resp. right) regular *po*-ternary semigroup, completely regular *po*-ternary semigroup and intra-regular *po*-ternary semigroup by using quasi-ideal, bi-ideal and semiprime ideal of *po*-ternary semigroup.

1. Introduction

The ideal theory of ternary semigroup was introduced and studied by Sioson in [12]. Dixit and Dewan [2] studied the notion of quasi-ideals and bi-ideals in ternary semigroup. Later on Santiago, Sri Bala [11] developed the theory of ternary semigroup and semiheaps. Further Kar and Maity developed the ideal theory on ternary semigroup in [6]. Some properties of regular ternary semigroup were discussed by Dutta, Kar and Maity in [4]. Ternary semigroups were studied by many authors, semiheaps (and similar) by V. Vagner [13], W.A. Dudek [3], A. Knorbel [9] and many others.

Kehayapulu ([7], [8]) introduced and studied the notion of completely regular ordered semigroup. In 2012, Daddi and Power [1] studied the concept of ordered quasi-ideals and ordered bi-ideals in ordered ternary semigroup and also discussed about their properties. The result on the minimality and maximality theory of ordered quasi-ideal in odered ternary semigroup was developed by Jailoka and Iampan in [5].

In this paper, we study the notion of regular ordered ternary semigroups. We also introduce the notion of completely regular and intra-regular ordered ternary semigroups. Finally we characterize these classes of ordered ternary semigroups in terms of ideals, quasi-ideals, bi-ideals, semiprime ideals of ternary semigroup.

2. Preliminaries and Prerequisites

Here we provide some definitions and relevant results of *po*-ternary semigroup which will be required to develop our paper.

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A ternary semigroup S is called a *partially ordered ternary semigroup* (*po-ternary semigroup*) if there is a partial order " $\leq$ " on S such that for $x, y \in S$; $x \leqslant y \Longrightarrow xx_1x_2 \leqslant yx_1x_2,\ x_1xx_2 \leqslant x_1yx_2,\ x_1x_2x \leqslant x_1x_2y$ for all $x_1, x_2 \in S$.

For a *po*-ternary semigroup $(S, \cdot, \leqslant)$ and a subset H of S, we define

$$(H] := \{t \in S \mid t \leqslant h \text{ for some } h \in H\}.$$

A nonempty subset A of S is called a *left ideal* of S if (i) $SSA \subseteq A$ and (ii) $(A] = A$, a *right ideal* of S if (i) $ASS \subseteq A$ and (ii) $(A] = A$ and a *lateral ideal* of S if (i) $SAS \subseteq A$ and (ii) $(A] = A$. A nonempty subset A of S is called an *ideal* of S if it is a left ideal, right ideal and lateral ideal of S.

For a *po*-ternary semigroup S and $a \in S$, we denote by $R(a)$ (resp. $L(a)$, $M(a)$) the right (resp. left, lateral) ideal of S generated by a and $I(a)$ the ideal generated by a.

It can be easily proved that for an element a of S the right (resp. left, lateral) ideal and the ideal $I(a)$ of S generated by a have the form

$R(a) = (a \cup aSS], \quad L(a) = (a \cup SSa], \quad M(a) = (a \cup SaS \cup SSaSS],$

$I(a) = (a \cup SSa \cup SaS \cup SSaSS \cup aSS] = (a \cup S^2a \cup SaS \cup S^2aS^2 \cup aS^2].$

If A, B, C are subsets of a *po*-ternary semigroup $(S, \cdot, \leqslant)$, then (cf. [5])

(1) $A \subseteq (A]$.

(2) If $A \subseteq B$ then $(A] \subseteq (B]$.

(3) $((A]] = (A]$.

(4) $(A](B](C] \subseteq (ABC]$.

(5) $((A](B](C]] = ((A](B]C] = (AB(C]] = (ABC]$.

(6) $(A \cup B] = (A] \cup (B]$.

(7) $(A \cap B] \subseteq (A] \cap (B]$.

In particular, if A and B are some types of ideals in S, then $(A \cap B] = (A] \cap (B]$.

(8) $(SSA]$, $(ASS]$, $(SAS \cup SSASS]$ are left, right and lateral ideal in S.

A nonempty subset Q of S is called a *quasi-ideal* of S, if (i) $(SSQ] \cap (SQS] \cap (QSS] \subseteq Q$, (ii) $(SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$ and (iii) $(Q] = Q$.

Every left, right and lateral ideal of a *po*-ternary semigroup S is a quasi-ideal.

A subsemigroup B of S is called a *bi-ideal* of S, if (i) $BSBSB \subseteq B$ and (ii) $(B] = B$.

Every quasi-ideal is a bi-ideal. Since every left, right and lateral ideal is a quasi-ideal, it follows that every left, right and lateral ideal is a bi-ideal.

A proper ideal T of a *po*-ternary semigroup S is called *semiprime* if for any ideal A of S with $A^3 \subseteq T$, we have $A \subseteq T$.

3. Regular *po*-ternary semigroups

A *po*-ternary semigroup S is said to be *regular* (*left, right regular*) if $A \subseteq (ASA]$ (respectively, $A \subseteq (SA^2]$, $A \subseteq (A^2S]$) for every $A \subseteq S$.

Lemma 3.1. *A lateral ideal of a regular po-ternary semigroup is regular.*

Proof. Let I be a lateral ideal of a regular *po*-ternary semigroup S. Let $A \subseteq I$. Since S is regular, $A \subseteq (ASA]$. Now $A \subseteq (ASA] \subseteq (AS(ASA]] = (ASASA] = (A(SAS)A] \subseteq (A(SIS)A] \subseteq (AIA]$. Consequently, I is regular. □

Corollary 3.2. *In a regular po-ternary semigroup S, every ideal is regular.*

Theorem 3.3. (cf. [10]) *In a po-ternary semigroup S, the following are equivalent:*
- (i) *S is regular,*
- (ii) *$(RML] = R \cap M \cap L$ where R, M, L are right ideal, lateral ideal and left ideal of S respectively,*
- (iii) *for every bi-ideal B of S, $(BSBSB] = B$,*
- (iv) *for every quasi-ideal Q of S, $(QSQSQ] = Q$.*

Theorem 3.4. *A po-ternary subsemigroup B of a regular po-ternary semigroup S is a bi-ideal of S if and only if $B = (BSB]$.*

Proof. Let S be a regular *po*-ternary semigroup and $B \subseteq S$. Let $B = (BSB]$. Then $B = (BSB] = (BS(BSB]] = (BS(BSB)] = (BSBSB]$. Thus $BSBSB \subseteq (BSBSB] = B$. It remains to show that $(B] = B$. Let $x \in (B]$. Then $x \in ((BSB]] = (BSB] = B$. Thus $(B] \subseteq B$. Hence B is a bi-ideal of S.

Conversely, let B be any bi-ideal of a regular *po*-ternary semigroup S. Since S is regular and $B \subseteq S$ we have $B \subseteq (BSB]$. Again $(BSB] \subseteq (BS(BSB]] = (BS(BSB)] = (BSBSB] \subseteq (B] = B$. Thus $B = (BSB]$. □

Theorem 3.5. *In a regular po-ternary semigroup S, every bi-ideal of S is a quasi-ideal of S.*

Proof. Let B be a bi-ideal of a regular *po*-ternary semigroup S. Then $BSBSB \subseteq B$ and $(B] = B$. Now $S^2(S^2B] \subseteq (S](S](SSB] \subseteq (S^4B] \subseteq (SSB]$ and $((SSB]] = (SSB]$. Hence $(SSB]$ is a left ideal of S. Also $(BS^2]S^2 \subseteq (BS^2](S](S] \subseteq (BS^4] \subseteq (BS^2]$ and $((BS^2]] = (BS^2]$. Thus $(BSS]$ is a right ideal of S. Again $S(SBS \cup S^2BS^2]S \subseteq (S](SBS \cup S^2BS^2](S] \subseteq (S^2BS^2 \cup S^3BS^3] \subseteq (S^2BS^2 \cup SBS]$ and $((SBS \cup S^2BS^2]] = (SBS \cup S^2BS^2]$. So $(SBS \cup S^2BS^2]$ is a lateral ideal of S. From Theorem 3.3, we have $(BS^2] \cap (SBS \cup S^2BS^2] \cap (S^2B] = ((BS^2](SBS \cup S^2BS^2](S^2B]] = ((BS^2)(SBS \cup S^2BS^2)(S^2B)] = (BS^3BS^3B \cup BS^4BS^4B] \subseteq (BSBSB \cup BS^2BS^2B] \subseteq (BSBSB \cup BSB] = (BSBSB] \cup (BSB] = B \cup B = B$, by using Theorem 3.3 and Theorem 3.4. Consequently, B is a quasi-ideal of S. □

Theorem 3.6. *Let S be a po-ternary semigroup. Then S is left* (*resp. right*) *regular if and only if every left* (*resp. right*) *ideal of S is semiprime.*

Proof. Let S be a left regular *po*-ternary semigroup and L be a left ideal of S. Let $A^3 \subseteq L$ for some left ideal A of S. Since S is left regular, we have $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSL] \subseteq (L] = L$. Thus L is semiprime.

Conversely, suppose that every left ideal of S is semiprime. Let $A \subseteq S$. Then $SS(SAA] \subseteq (S](S](SAA] \subseteq (S^3AA] \subseteq (SAA]$ and $((SAA]] = (SAA]$. Therefore,

$(SAA]$ is a left ideal of S. Now $A^3 \subseteq SA^2 \subseteq (SA^2]$. Since every left ideal of S is semiprime, we have $A \subseteq (SA^2]$. Thus S is a left regular *po*-ternary semigroup.

Similarly, we can also prove the same for right ideal of S. □

Theorem 3.7. *Let S be a commutative po-ternary semigroup. Then S is regular if and only if every ideal of S is semiprime.*

Proof. Let S be a commutative regular *po*-ternary semigroup and I be any ideal of S. Let $A^3 \subseteq I$ for $A \subset S$. Since S is regular and $A \subseteq S$ we have $A \subseteq (ASA] = (AAS] \subseteq (A(ASA]S] = (A(ASA)S] = (A(A^2S)S] = (A^3SS] \subseteq (ISS] \subseteq (I] = I$. Thus I is a semiprime ideal of S.

Conversely, we assume that every ideal of commutative *po*-ternary semigroup S is semiprime. Let $A \subseteq S$. Then $(ASA]$ is an ideal of S. If $(ASA] = (S] = S$, we get our conclusion. If $(ASA] \neq S$, then by hypothesis, $(ASA]$ is a semiprime ideal of S. Now $A^3 \subseteq ASA \subseteq (ASA]$ implies that $A \subseteq (ASA]$. Consequently, S is regular. □

Definition 3.8. Let S be a *po*-ternary semigroup. A nonempty subset B_w of S is called a *weak bi-ideal* of S, if (i) $bSbSb \subseteq B_w$ for all $b \in B_w$ and (ii) $(B_w] = B_w$.

Clearly, we have the following results:

Lemma 3.9. *Every bi-ideal of a po-ternary semigroup S is a weak bi-ideal of S.*

Lemma 3.10. *The intersection of arbitrary set of weak bi-ideals of a po-ternary semigroup S is either empty or a weak bi-ideal of S.*

Theorem 3.11. *Let S be a po-ternary semigroup. Then S is regular if and only if $B_w = (\bigcup_{b\in B_w} bSbSb]$ for any weak bi-ideal B_w of S.*

Proof. Let S be a regular *po*-ternary semigroup and B_w be any weak bi-ideal of S. Then $bSbSb \subseteq B_w$ for all $b \in B_w$. So $\bigcup_{b\in B_w} bSbSb \subseteq B_w$. This implies that $(\bigcup_{b\in B_w} bSbSb] \subseteq (B_w] = B_w$. Let $b \in B_w$. Since S is regular, there exists $x \in S$ such that $b \leqslant bxb$. So $b \leqslant bxb \leqslant bxbxb \in bSbSb \subseteq \bigcup_{b\in B_w} bSbSb$. Therefore, $b \in (\bigcup_{b\in B_w} bSbSb]$. Thus $B_w \subseteq (\bigcup_{b\in B_w} bSbSb]$. Hence $B_w = (\bigcup_{b\in B_w} bSbSb]$.

Conversely, let $B_w = (\bigcup_{b\in B_w} bSbSb]$, where B_w is a weak bi-ideal of S. Let R be a right ideal, M be a lateral ideal and L be a left ideal of S. Since every left, right and lateral ideal of a *po*-ternary semigroup S is a bi-ideal of S, it follows that every left, right and lateral ideal of a *po*-ternary semigroup S is a weak bi-ideal of S. So R, M, L are weak bi-ideals of S. Thus by Lemma 3.10, $R \cap M \cap L$ is a weak bi-ideal of S. Clearly, $(RML] \subseteq R \cap M \cap L$. Now let $a \in R \cap M \cap L$. Since $R \cap M \cap L$ is weak bi-ideal of S, by hypothesis we have $R \cap M \cap L = (\bigcup_{x \in R \cap M \cap L} xSxSx]$. Then $a \leqslant xs_1xs_2x$ for some $x \in R \cap M \cap L$ and $s_1, s_2 \in S$. So $a \leqslant xs_1xs_2ys_3ys_4y$ for some $x, y \in R \cap M \cap L$ and $s_1, s_2, s_3, s_4 \in S$. This implies that $a \in (RML]$. Thus $R \cap M \cap L \subseteq (RML]$ and hence $(RML] = R \cap M \cap L$. Consequently, S is a regular *po*-ternary semigroup by Theorem 3.3. □

4. Completely regular *po*-ternary semigroups

In this section, we characterize completely regular *po*-ternary semigroup by using quasi-ideals, bi-ideals and semiprime ideals.

Definition 4.1. A *po*-ternary semigroup S is said to be *completely regular* if it is regular, left regular and right regular i.e., $A \subseteq (ASA]$, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$ for every $A \subseteq S$.

Example 4.2. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with the ternary operation defined on S as $abc = a * (b * c)$, where the binary operation * is defined by

*	a	b	c	d	e
a	a	a	c	d	a
b	a	b	c	d	a
c	a	a	c	d	a
d	a	a	c	d	a
e	a	a	c	d	e

and the order defined as

$\leqslant = \{(a,a),(a,c),(a,d),(b,b),(b,d),(b,a),(b,c),(c,c),(c,d),(d,d),(e,a),(e,c),(e,d),(e,e)\}$.

Then S is a completely regular *po*-ternary semigroup.

Theorem 4.3. *In a po-ternary semigroup S, the following conditions are equivalent:*

(i) *S is completely regular;*

(ii) *$A \subseteq (A^2SA^2]$ for every $A \subseteq S$.*

Proof. $(i) \Rightarrow (ii)$. Then for any $A \subseteq S$, we have $A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2S^3A^2] \subseteq (A^2SA^2]$.

$(ii) \Rightarrow (i)$. Let $A \subseteq S$. Then $A \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (ASA]$, $A \subseteq (A^2SA^2] = ((A^2S)A^2] \subseteq (SA^2]$ and $A \subseteq (A^2SA^2] = (A^2(SA^2)] \subseteq (A^2S]$. This implies that S is regular, left regular and right regular. Consequently, S is completely regular. □

In the following result we provide another characterization of completely regular *po*-ternary semigroup in terms of quasi-ideal.

Theorem 4.4. *Let S be a po-ternary semigroup. Then S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S.*

Proof. Let S be a completely regular *po*-ternary semigroup and Q be a quasi-ideal in S. Since $\phi \neq Q \subseteq S$ and $Q^3 \subseteq QSS \cap SQS \cap SSQ \subseteq (QSS] \cap (SQS] \cap (SSQ] \subseteq Q$, Q is a subsemigroup of S. Let $A \subseteq Q \subseteq S$. We have to show that Q is completely regular. Since S is completely regular and $A \subseteq S$, we have $A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2SSSA^2] \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (A(ASA]SAA] = (A(ASA)SAA] = (A(ASASA)A]$. Now $ASASA \subseteq SSASS \subseteq SSQSS$, $ASASA \subseteq SSA \subseteq SSQ$ and $ASASA \subseteq ASS \subseteq QSS$. Therefore, $ASASA \subseteq SSQ \cap SSQSS \cap QSS \subseteq (SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$. Hence $A \subseteq (AQA]$. Again $A \subseteq (ASA] \subseteq (AS(SA^2]] = (AS(SA^2)] \subseteq (ASS(SA^2]A] = (AS^2(SA^2)A] = ((AS^3A)A^2] \subseteq ((ASA)A^2] \subseteq (AS(ASA]A^2] = (AS(ASA)A^2] = ((ASASA)A^2] \subseteq (QA^2]$ and $A \subseteq (ASA] \subseteq ((A^2S]SA] = ((A^2S)SA] \subseteq (A(A^2S]SSA] = (A(A^2S)SSA] = (A^2(ASSSA)] \subseteq (A^2(ASA)] \subseteq (A^2(ASA]SA] = (A^2(ASA)SA] = (A^2(ASASA)] \subseteq (A^2Q]$. Thus Q is regular, left regular and right regular. Consequently, Q is completely regular subsemigroup.

Conversely, suppose that every quasi-ideal of S is a completely regular subsemigroup of S. Since S itself a quasi-ideal in S, S is completely regular. □

Theorem 4.5. *Let S be a po-ternary semigroup. Then S is left regular and right regular if and only if every quasi-ideal of S is semiprime.*

Proof. Let S be a left regular and right regular *po*-ternary semigroup and Q be a quasi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq Q$. Since S is left regular and right regular, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$. Now $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSQ]$, $A \subseteq (A^2S] \subseteq (A(A^2S]S] = (A(A^2S)S] = (A^3SS] \subseteq (QSS]$ and $A \subseteq (SA^2] \subseteq (SA(A^2S]] = (SA^3S] \subseteq (SQS]$. Therefore, $A \subseteq (SSQ] \cap (SQS] \cap (QSS] \subseteq Q$. Hence Q is semiprime.

Conversely, suppose that every quasi-ideal of S is semiprime. Since every right ideal and left ideal of S is a quasi-ideal of S, every right ideal and left ideal are semiprime. Now by using Theorem 3.6, we find that S is left regular and right regular. □

Corollary 4.6. *If S is a completely regular po-ternary semigroup then quasi-ideals of S are semiprime.*

The converse of the above result does not hold.

Example 4.7. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with ternary operation product defined on S by $abc = a * (b * c)$, where binary operation * is defined as

*	a	b	c	d	e
a	a	e	e	a	e
b	d	b	b	d	b
c	d	b	b	d	b
d	d	b	b	d	b
e	a	e	e	a	e

and the order defined by

$\leqslant\ := \{(a,a),(b,a),(b,b),(b,d),(b,e),(c,a),(c,c),(c,d),(c,e),(d,d),(d,a),(e,a),(e,e)\}$.

Then S is a left regular and right regular *po*-ternary semigroup. So every quasi-ideal of S is semiprime by Theorem 4.5 but S is not completely regular. In fact it is not regular since $c \in S$ is not regular.

Theorem 4.8. *A po-ternary semigroup S is completely regular if and only if every bi-ideal of S is semiprime.*

Proof. Let S be a completely regular *po*-ternary semigroup and B be any bi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq B$. Since S is completely regular *po*-ternary semigroup and $A \subseteq S$ we have $A \subseteq (A^2SA^2] \subseteq (A(A^2SA^2]S(A^2SA^2]A] = (A(A^2SA^2)S(A^2SA^2)A] = ((A^3SA^2S)(A^2S)A^3] \subseteq ((A^3SA^2S)(A^2SA^2](A^2SA^2]SA^3] = ((A^3SA^2S)(A^2SA^2)(A^2SA^2)SA^3] = (A^3(SA^2SA^2S)A^3(ASA^2S)A^3] \subseteq (BSBSB] \subseteq (B] = B$. Therefore B is semiprime.

Conversely, suppose that every bi-ideal of S is semiprime. Let $\phi \neq A \subseteq S$. Then we have $A^2SA^2 \subseteq S$ i.e. $(A^2SA^2] \subseteq S$. Now $(A^2SA^2]S(A^2SA^2]S(A^2SA^2] \subseteq (A^2SA^2](S](A^2SA^2](S](A^2SA^2] \subseteq (A^2SA^2SA^2SA^2SA^2SA^2] \subseteq (A^2SA^2]$ and also $((A^2SA^2]] = (A^2SA^2]$. Thus $(A^2SA^2]$ is a bi-ideal in S. Now $A^9 = A^2(A^5)A^2 \subseteq A^2SA^2 \subseteq (A^2SA^2]$. By hypothesis, since every bi-ideal is semiprime, $A^9 = (A^3)^3 \subseteq (A^2SA^2] \Longrightarrow A^3 \subseteq (A^2SA^2] \Longrightarrow A \subseteq (A^2SA^2]$. Since A is arbitrary, $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Hence S is completely regular. □

5. Intra-regular *po*-ternary semigroups

In this section, we characterize intra-regular *po*-ternary semigroup by using properties of ideals.

Definition 5.1. A *po*-ternary semigroup S is called intra-regular if for every $a \in S$, there exists $x, y \in S$ such that $a \leqslant xa^3y$ or equivalently, $a \in (Sa^3S]$ for all $a \in S$.

In other words, a *po*-ternary semigroup S is intra-regular if $A \subseteq (SA^3S]$ for every $A \subseteq S$.

Lemma 5.2. *If S is a left (resp. right) regular po-ternary semigroup, then S is intra-regular.*

Proof. Let S be left regular *po*-ternary semigroup and $A \subseteq S$. Then $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] \subseteq (SS(SA^2]AA] = (SS(SA^2)AA] = (SSSA^3A] \subseteq (SSSA^3S] \subseteq (SA^3S]$. Thus S is intra-regular.

Similarly, we can prove the result for right regular *po*-ternary semigroup. □

But the converse of the above result is not true.

Example 5.3. Let $S = \{a, b, c, d, e\}$ be a *po*-ternary semigroup with ternary operation defined on S by $abc = a * (b * c)$, where the binary operation * is defined as

*	a	b	c	d	e
a	a	b	a	d	a
b	a	b	a	d	a
c	a	b	a	d	a
d	a	b	a	d	a
e	a	b	a	d	a

and the order defined by

$$\leqslant \ := \{(a,a),(a,b),(a,c),(a,e),(b,b),(c,c),(c,b),(c,e),(d,d),(e,b),(e,e)\}.$$

Then $(S, \cdot, \leqslant)$ is an intra-regular *po*-ternary semigroup but not left regular, since c and e are not left regular elements of S.

Now we can easily prove the following fact:

Theorem 5.4. *In an intra-regular po-ternary semigroup S, $L \cap M \cap R \subseteq (LMR]$, where L, M, R are left ideal, lateral ideal and right ideal of S respectively.*

Clearly, every ideal of a *po*-ternary semigroup S is also a lateral ideal of S. Certainly a lateral ideal of S is not necessarily an ideal of S. But in particular, for intra-regular *po*-ternary semigroup S we have the following result:

Theorem 5.5. *Let S be an intra-regular po-ternary semigroup. Then a non-empty subset I of S is an ideal of S if and only if I is a lateral ideal of S.*

Proof. Clearly, if I is an ideal of S, then I is a lateral ideal of S.

Conversely, assume that I is a lateral ideal of an intra-regular *po*-ternary semigroup S. Then $SIS \subseteq I$ and $(I] = I$. Since S is intra-regular and $I \subseteq S$ we have $I \subseteq (SI^3S]$. Now $SSI \subseteq (SSI] \subseteq (SS(SI^3S]] = (SS(SI^3S)] = (S^3I^3S] \subseteq (S^3(SI^3S]I^2S] = (S^3(SI^3S)I^2S] = ((S^4I)I(ISIIS)] \subseteq (SIS] \subseteq (I] = I$ and $ISS \subseteq (ISS] \subseteq ((SI^3S]SS] = ((SI^3S)SS] = (SI^3S^3] \subseteq (SI^2(SI^3S]S^3] = (SI^2(SI^3S)S^3] = ((SIISI)I(IS^4] \subseteq (SIS] \subseteq (I] = I$. Thus I is a left ideal as well as a right ideal of S. Consequently, I is an ideal of S. □

Lemma 5.6. *Let S be an intra-regular po-ternary semigroup and I be a lateral ideal of S. Then I is an intra-regular po-ternary semigroup.*

Proof. Let S be an intra-regular *po*-ternary semigroup and I be a lateral ideal of S. Let $A \subseteq I \subseteq S$. Since S is intra-regular, it follows that $A \subseteq (SA^3S]$. Now we have $A \subseteq (SA^3S] \subseteq (S(SA^3S](SA^3S](SA^3S]S] = (S(SA^3S)(SA^3S)(SA^3S)S] = ((SSA^3S^2)A^3(S^2A^3S^2)] \subseteq ((S^3AS^3)A^3(S^3AS^3)] \subseteq ((SAS)A^3(SAS)] \subseteq ((SIS)A^3(SIS)] \subseteq (IA^3I]$. Consequently, I is intra-regular. □

Corollary 5.7. *Let S be an intra-regular po-ternary semigroup and I be an ideal of S. Then I is an intra-regular po-ternary semigroup.*

Theorem 5.8. *Let S be an intra-regular po-ternary semigroup. Let I be an ideal of S and J be an ideal of I. Then J is an ideal of the entire po-ternary semigroup S.*

Proof. It is sufficient to show that J is a lateral ideal of S. Now $J \subseteq I \subseteq S$ and $SJS \subseteq SIS \subseteq I$. We have to show that $SJS \subseteq J$. From Corollary 5.7, it follows that I is an intra-regular *po*-ternary semigroup. Also $SJS \subseteq I$. So we have $(SJS) \subseteq (I(SJS)^3I] = (I(SJS)(SJS)(SJS)I] = ((ISJSS)J(SSJSI)] \subseteq ((ISISS)J(SSISI)] \subseteq ((IIS)J(SII)] \subseteq ((ISS)J(SSI)] \subseteq (IJI] \subseteq (J] = J$. Consequently, J is a lateral ideal of S. □

Theorem 5.9. *Let S be a po-ternary semigroup. Then S is intra-regular if and only if every ideal of S is semiprime.*

Proof. Let S be an intra-regular *po*-ternary semigroup and I be an ideal of S. Let $A^3 \subseteq I$ for $A \subseteq S$. Since S is intra-regular *po*-ternary semigroup, we have $A \subseteq (SA^3S] \subseteq (SIS] \subseteq (I] = I$. Hence I is a semiprime ideal of S.

Conversely, suppose that every ideal of S is semiprime. Let $A \subseteq S$. Since $A^3 \subseteq I(A^3)$, where $I(A^3)$ is the ideal generated by A^3 and by hypothesis $I(A^3)$ is a semiprime ideal of S, so $A \subseteq I(A^3)$.

Now $I(A^3) = (A^3 \cup SSA^3 \cup SA^3S \cup SSA^3SS \cup A^3SS] = (A^3] \cup (SSA^3] \cup (SA^3S] \cup (SSA^3SS] \cup (A^3SS]$.

1) If $A \subseteq (A^3]$. Then $A \subseteq (A(A^3]A] = (A(A^3)A] \subseteq (SA^3S]$.

2) If $A \subseteq (S^2A^3]$ then $A^3 \subseteq (S^2A^3]A^2$. Hence $A \subseteq (S^2(S^2A^3]A^2] = (S^2(S^2A^3)A^2] = (S^4A^5] \subseteq (S^5A^3S] \subseteq (SA^3S]$.

3) If $A \subseteq (SA^3S]$ we get our conclusion.

4) If $A \subseteq (SSA^3SS]$, then $A^3 \subseteq A(S^2A^3S^2]A$. Hence $A \subseteq (S^2A(S^2A^3S^2]AS^2] = (S^2A(S^2A^3S^2)AS^2] = (S^2AS^2A^3S^2AS^2] \subseteq (S^5A^3S^5] \subseteq (SA^3S]$.

5) If $A \subseteq (A^3SS]$, then $A^3 \subseteq A^2(A^3SS]$. Hence $A \subseteq (A^2(A^3SS]SS] = (A^2(A^3SS)SS] = (A^5S^4] \subseteq (SA^3S^5] = (SA^3S^5] \subseteq (SA^3S]$.

In each case, S is intra-regular. Consequently, S is an intra-regular. □

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