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Computational approach for intransitive action of $\Delta(2,4,k)$ on $PL(F_q)$

Tahir Imran, Muhammad Ashiq and Muhammad Asad Zaighum

Abstract In this paper, we have investigated actions of triangle group $\Delta(2,4,k)$ defined by $< r, s : r^2 = s^4 = (rs)^k = 1 >$, on projective line over the finite field $PL(F_q)$ by using the concept of coset diagrams. We will parameterize this action and prove that actions of $\Delta(2,4,4)$ is intransitive on $PL(F_q)$, where q is such a prime that $q+2$ gives a perfect square. We have also developed a useful computational technique to parameterize this action and also to draw coset diagrams. Throughout -1 represents ∞ ,in diagrams as these are computer generated.

1. Introduction

The *linear-fractional group* $\Delta(2,4,k)$ is defined by the transformations $r : z \to \frac{-1}{z}$ and $s : z \to \frac{-1}{2(z+1)}$ that satisfies the relations $r^2 = s^4 = 1$. This group can be extended by adjoining an involution $t : z \to \frac{1}{2z}$ such that $(rt)^2 = (st)^2 = 1$. This extended group is denoted by $\Delta^*(2,4,k)$ [1, 2, 6].

Let $\alpha : PGL\,(2,Z) \longrightarrow PGL\,(2,q)$ be a non-degenerate homomorphism. We know that every non-degenerate homomorphism gives rise to an action. So, this non-degenerate homomorphism gives rise to an action of $PGL\,(2,Z)$ on $PL\,(F_q)$. The action which arises from this non-degenerate homomorphism not only corresponds to the non-degenerate homomorphism but to a conjugacy class of the homomorphisms [3, 5].

Since, there is one-to-one correspondence between the conjugacy classes of elements of order greater than 2 in $PGL\,(2,q)$ and the non-zero elements of F_q, such that the class corresponding to an element θ in F_q consists of all the elements represented by matrices A [6]. It follows that we can actually parameterize the non-degenerate homomorphisms of $PGL\,(2,Z)$ into $PGL\,(2,q)$, except for a few uninteresting ones, by the elements of F_q. If α is any such non-degenerate homomorphism, and R, S and T are in $GL\,(2,q)$, which yield the elements $\overline{r}$, $\overline{s}$, $\overline{t}$ then letting $\theta = m_2^2/\Delta$ (where $m_2 = trace(RS)$, $\triangle = det(RS)$), we associate the parameter θ with the homomorphism α. This non-zero element θ of F_q provides a permutation representation of the action corresponding to the homomorphism α. We draw a coset diagram corresponding to this action which is a diagram corresponding to not only one action but to a class of actions whose parameter is θ.

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We are looking for a condition on θ and q which ensures action of $PGL\,(2, Z)$ on $PL\,(F_q)$ evolving the required coset diagrams [4, 6, 7].

2. Conjugacy classes and coset diagrams

In this section, construction of coset diagrams for the generalized triangle group $< r, s, t : r^2 = s^4 = t^2 = (rt)^2 = (st)^2 = (rs)^k = 1 >$ are considered along-with certain observations about this case. The coset diagrams for action of $\Delta^*(2, 4, k)$ on finite space are defined as follows.

The four cycles of s are represented by squares whose vertices are permuted anti-clock wise by S. Any two vertices which are interchanged by involution r is represented by an edge. The action of t is represented by reflection about a vertical axis of symmetry. For example, action of $\Delta^*(2, 4, k)$ on $PL(F_{31})$ gives us the following permutation representations:

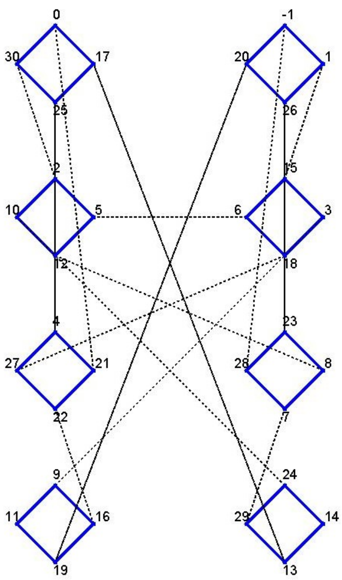

Figure 1: Action of $\Delta^*(2, 4, k)$ on $PL(F_{31})$

Theorem 2.1. *Corresponding to each $\theta = m_4 \in F_q$ there exists a conjugacy class of non-degenerate homomorphism $\alpha : PGL(2, Z) \to PGL(2, q)$ which yields the homomorphic image of $< r, s : r^2 = s^4 = (rs)^4 = 1 >$ under α.*

Proof. Define a homomorphism $\alpha : PGL(2, Z) \longrightarrow PGL(2, q)$ such that $\overline{r} = r\alpha$, $\overline{s} = s\alpha$ and $\overline{t} = t\alpha$ satisfying the relations:

$$\overline{r}^2 = \overline{s}^4 = \overline{t}^2 = \left(\overline{r}\overline{t}\right)^2 = \left(\overline{s}\overline{t}\right)^2 = 1. \tag{1}$$

So, there is requirement to see for elements $\overline{r}$, $\overline{s}$, $\overline{t} \in PGL(2,q)$ satisfying the relations 1 with $\overline{r}\,\overline{s}$ in given conjugacy class. Let $\overline{r}$, $\overline{s}$ and $\overline{t}$ be represented by matrices,
$R = \begin{bmatrix} r_1 & kr_3 \\ r_3 & -r_1 \end{bmatrix}$, $S = \begin{bmatrix} s_1 & ks_3 \\ s_3 & -s_1 - \sqrt{2} \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$ respectively, as defined in [4], where r_1, r_3, s_1, s_3, $k \in F_q$. Let $det(R) = \Delta$ and $det(S) = 1$, then

$$det(R) = \Delta = -r_1^2 - kr_3^2 = r_1^2 + kr_3^2 \neq 0 \tag{2}$$

and,

$$\begin{aligned} det(S) = 1 = -s_1^2 - \sqrt{2}s_1 - ks_3^2 \\ s_1^2 + \sqrt{2}s_1 + ks_3^2 + 1 = 0. \end{aligned} \tag{3}$$

This surely, yields such elements that satisfy the relations (1). Now the product of matrices R and S is given by,

$$RS = \begin{bmatrix} r_1 & kr_3 \\ r_e & -r_1 \end{bmatrix} \begin{bmatrix} s_1 & ks_3 \\ s_3 & -s_1 - 1 \end{bmatrix} = \begin{bmatrix} r_1s_1 + kr_3s_3 & kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 \\ r_3s_1 - r_1s_3 & kr_3s_3 + r_1s_1 + \sqrt{2}r_1 \end{bmatrix}$$

As already supposed that $tr(RS) = m_2$, therefore

$$m_2 \quad = \quad 2r_1s_1 + 2kr_3s_3 + \sqrt{2}r_1. \tag{4}$$

The matrix RST is given by

$$\begin{aligned} RST &= \begin{bmatrix} r_1s_1 + kr_3s_3 & kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 \\ r_3s_1 - r_1s_3 & kr_3s_3 + r_1s_1 + \sqrt{2}r_1 \end{bmatrix} \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 & -k(r_1s_1 + kr_3s_3) \\ kr_3s_3 + r_1s_1 + \sqrt{2}r_1 & -k(r_3s_1 - r_1s_3) \end{bmatrix} \end{aligned}$$

and so the trace of RST is given by

$$tr(RST) = kr_1s_3 - kr_3s_1 - \sqrt{2}kr_3 - k(r_3s_1 - r_1s_3) = 2kr_1s_3 - kr_3\left(2s_1 + \sqrt{2}\right)$$

and as already considered, $m_3k = trace(RST)$ so

$$\begin{aligned} m_3k &= 2kr_1s_3 - kr_3\left(2s_1 + \sqrt{2}\right) \\ m_3 &= \ 2r_1s_3 - r_3\left(2s_1 + \sqrt{2}\right). \end{aligned} \tag{5}$$

Now squaring equations (4) and (5) we get,

$$\begin{aligned} m_2^2 = [2r_1s_1 + 2kr_3s_3 + \sqrt{2}r_1]^2 = 4r_1^2s_1^2 + 4k^2r_3^2s_3^2 + 2r_1^2 + 8kr_1s_1r_3s_3 \\ +4\sqrt{2}r_1r_3s_3 + 4\sqrt{2}r_1^2s_1 \end{aligned}$$

and

$$\begin{aligned} m_3^2 = [2r_1s_3 - r_3(2s_1+\sqrt{2})]^2 &= 4r_1^2s_3^2 + r_3^2(4s_1^2+2+4\sqrt{2}s_1) - 4r_1r_3s_3(2s_1+\sqrt{2})) \\ &= 4r_1^2s_3^2 + 4r_3^2s_1^2 + 2r_3^2 + 4\sqrt{2}r_3^2s_1 - 8r_1r_3s_1s_3 - 4\sqrt{2}r_1r_3s_3. \end{aligned}$$

Multiplying m_3^2 by k and then adding in m_2^2, we get

$$\begin{aligned} m_2^2+km_3^2 =& 4r_1^2s_1^2 + 4k^2r_3^2s_3^2 + 2r_1^2 + 8kr_1s_1r_3s_3 + 4\sqrt{2}r_1r_3s_3 + 4\sqrt{2}r_1^2s_1 \\ &+ 4kr_1^2s_3^2 + 4kr_3^2s_1^2 + 2kr_3^2 + 4\sqrt{2}kr_3^2s_1 - 8kr_1r_3s_1s_3 - 4\sqrt{2}kr_1r_3s_3 \\ =& 4r_1^2s_1^2+4k^2r_3^2s_3^2 + 2r_1^2+4\sqrt{2}r_1^2s_1 + 4kr_1^2s_3^2+4kr_3^2s_1^2 + 2kr_3^2+4\sqrt{2}kr_3^2s_1 \\ =& 2(r_1^2+kr_3^2) + 4s_1^2(r_1^2+kr_3^2) + 4\sqrt{2}s_1(r_1^2+kr_3^2) + 4ks_3^2(r_1^2+kr_3^2) \\ =& (r_1^2+kr_3^2)(2+4s_1^2+4\sqrt{2}s_1+4ks_3^2) \\ =& [r_1^2+kr_3^2][2+4(s_1^2+\sqrt{2}s_1+ks_3^2)]. \end{aligned}$$

By using equations (3), we obtain

$$m_2^2+km_3^2 = [r_1^2+kr_3^2][2+4(-1)] = (-\Delta)(-2) = 2\Delta.$$

That is,

$$2\Delta = m_2^2 + km_3^2. \tag{6}$$

We have

$$R^{-1}S^{-1} = \frac{1}{\Delta}\begin{bmatrix} r_1s_1+\sqrt{2}r_1+kr_3s_3 & kr_1s_3-kr_3s_1 \\ r_3s_1+\sqrt{2}r_3-r_1s_3 & kr_3s_1+r_1s_1 \end{bmatrix}.$$

The product $RSR^{-1}S^{-1}$ is

$$\frac{1}{\Delta}\begin{bmatrix} r_1s_1+kr_3s_3 & kr_1s_3-kr_3s_1-\sqrt{2}kr_3 \\ r_3s_1-r_1s_3 & kr_3s_3+r_1s_1+\sqrt{2}r_1 \end{bmatrix}\begin{bmatrix} r_1s_1+\sqrt{2}r_1+kr_3s_3 & kr_1s_3-kr_3s_1 \\ r_3s_1+\sqrt{2}r_3-r_1s_3 & kr_3s_1+r_1s_1 \end{bmatrix}.$$

Now further as considered in previous section $trace(RSR^{-1}S^{-1}) = m_4$, then $m_4 = \frac{1}{\Delta}[\Delta - km_2^2 - r_1^2 - kr_3^2]$ and consequently, $m_4\Delta = \Delta - km_3^2 - r_1^2 - kr_3^2 = \Delta - km_3^2 - (r_1^2+kr_3^2) = \Delta - km_3^2 - (-\Delta) = 2\Delta - km_3^2$, which together with (6) implies $m_2^2 = m_4\Delta$. This together with $m_2^2 = \Delta\theta$ gives $\theta = m_4 \in F_q$. Hence θ is the permutation representation of the action corresponding to the homomorphism α. □

Theorem 2.2. *The transformation $\bar{t}$ has fixed vertices in $D(\theta,q)$ if and only if $\theta(\theta-2)$ is a square in F_q.*

Proof. Let $\alpha\colon \Gamma^* \to G^{*3,4}(2,q)$ be a non-degenerate homomorphism that satisfies the relations $r\alpha = \bar{r}, s\alpha = \bar{s}$ and $t\alpha = \bar{t}$ and α' be its dual. Choose the matrices, $R = \begin{bmatrix} r_1 & kr_3 \\ r_3 & -r_1 \end{bmatrix}$, $S = \begin{bmatrix} s_1 & ks_3 \\ s_3 & -\sqrt{2}-s_1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$, representing $\bar{r}, \bar{s}$

and $\bar{t}$ respectively, where r_1, r_3, s_1, s_3, $k \in F_q$ and satisfies the equations (2) to (6). As we know that, $tr(RS) = 0$ if and only if $(\bar{r}\,\bar{s})^2 = 1$. Also, $\dfrac{tra(RST)}{k} = m_3 = 0$ if and only if $(\bar{r}\,\bar{s}\bar{t})^2 = 1$. Now $det(RS) = 1$, gives parameter of $\bar{r}\,\bar{s}$ as $m_2^2 = \theta$. Also $tr(RST) = km_3$ and $det(RST) = k$ [Since $det(R) = 1$, $det(S) = 1$ and $det(T) = k \Rightarrow det(RST) = k$], gives parameter of $\bar{r}\,\bar{s}\bar{t}$ as km_3^2. Let this parameter be denoted by ϕ. Therefore, $\theta + \phi = \frac{m_2^2 + km_3^2}{\Delta}$. Putting values from equation (6), $\theta + \phi = 2$. Hence, $\phi = \theta - 2$.

Since change from α to α' interchanges both $\bar{r}$ and $\bar{r}\bar{t}$ and θ and $\theta - 2$, so $\bar{r}\bar{t}$ maps to an element $\Delta^*(2,4,k)$ if and only if $\theta(\theta-2)$ is a square in F_q. Since $\bar{t}$ lies in $\Delta^*(2,4,k)$ if both of $\bar{r}$ and $\bar{r}\bar{t}$, so $\bar{t}$ belongs to $G^*(2,4,k)$ if and only if $\theta(\theta-2)$ is a square in F_q. Now $\bar{t}$ has fixed points in $PL(F_q)$ if either $\bar{t}$ belongs to $\Delta^*(2,4,k)$ and $q \equiv -1(mod4)$ or $\bar{t}$ does not belong to $\Delta^*(2,4,k)$ and $q \equiv 1(mod4)$, which means that -1 is a square in F_q. Hence it can be concluded that $\bar{t}$ has fixed vertices in $D(\theta, q)$ if and only if $-\theta(2-\theta) = \theta(\theta-2)$ is a square in F_q. $\square$

3. Action of $\Delta(2,4,k)$ on $PL(F_q)$ for $\theta = 2$

Following computer coding scheme generate parameterizations and coset diagrams for actions of $\Delta(2,4,k)$ over $PL(F_q)$, wherein q is a prime number $q+2$ gives perfect square.

3.1. Computer program to parameterize action

```
m4 = Input["m4"];
delta = Input["Delta"];
m2sq = delta*m4;
While[! (Element[Sqrt[m2sq], Integers]), m2sq += q];
m2 = Sqrt[m2sq];
m3sq = ((2*delta ) - (m2sq))/k;
While[m3sq < 0, m3sq += q;];
m3 = Sqrt[m3sq];
s3sq = (-1 - s1^2 - (Sqrt[2 + q]*s1))/k;
While[s3sq < 0, s3sq += q;];
While[! (Element[Sqrt[s3sq], Integers]), s3sq += q];
s3 = Sqrt[s3sq];
{c, d} = {a, b} /.
   First@Solve[{2*a* s1 + 2*k*b*s3 + (Sqrt[2 + q])*a == m2,
      2*a*s3 - 2*b*s1 - (Sqrt[2 + q])*b == m3}, {a, b}];
nom = Numerator[c];
denom = Denominator[c];
While[! (Element[nom/denom, Integers]), nom += q];
r1 = nom/denom;
```

```
nom = Numerator[d];
denom = Denominator[d];
While[! (Element[nom/denom, Integers]), nom += q];
r3 = nom/denom;
r11 = r1;
r12 = k*r3;
r13 = r3;
r14 = -r1;
s11 = s1;
s12 = k*s3;
s13 = s3;
s14 = -s1 - (Sqrt[2 + q]);
t2 = -k;
While[t2 < 0, t2 += q];
matrix_X = MatrixForm[{{r11, r12}, {r13, r14}}]
matrix_Y = MatrixForm[{{s11, s12}, {s13, s14}}]
matrix_T = MatrixForm[{{0, t2}, {1, 0}}]
```

3.2. Computer program to draw coset diagrams

Following coding scheme using java programming language to draw coset diagrams with respect to the primes q for the action of $\Delta(2,4,k)$ has been developed. The code given below will generate the permutations for R. Similar code is used for generating the permutations for S and T.

```
List<Integer> tmp=new ArrayList<Integer>();
     int count=R_values.get(0);
     tmp.add(count);
     while(cycle==true)
     {
     int permut_temp=(int) calculateFunc_R(count,a,b,c,d);
     count=permut_temp;
     if(!(tmp.contains(permut_temp))&& tmp.size()<2)
     {
     tmp.add((int) permut_temp);
     }
     else
     {
     Permutation_R.add(tmp);
     cycle=false;
     }
     }
```

Following code separates the fix points from permutation of S.

```
for(List<Integer> innerList : Permutation_S) {
        if(innerList.size()<4)
        {
         fixPointS.add(innerList);
        }
        }
```

The code given below will make the nodes symmetrical basing on the permutations of T.

```
for(List<Integer> innerList : Permutation_T) {
        if(innerList.size()==1)
        {
         fix=(Integer) Permutation_T.get(Permutation_T.indexOf(innerList)).get(0);
         for(List<Integer> innerSList : Permutation_S)
         {
         if(innerSList.contains(fix))
         {
         if(!PermutationS_toDrawCenter.contains(innerSList))
         {
         PermutationS_toDrawCenter.add(innerSList);
         toremove_S.add(innerSList);
         }
         toremove_T.add(innerList);
         }
         }
        }
        }
```

The symmetrical nodes will then be drawn by using the code given below:

```
public Node(Point p,int n_v, int r, Color color, Kind kind,int pos) {
            this.p = p;
            this.r = r;
            this.node_value=n_v;
            this.color = color;
            this.kind = kind;
            this.pos=pos;
            setBoundary(b);
        }
public void draw(Graphics g) {
         int x,y,r=5;
         if(this.pos==0)
         {
         x=b.x;
```

```
        y=b.y-r;
        }
        else if(this.pos==1)
        {x=b.x-r-8;
    y=b.y;}
        else if(this.pos==2)
        {x=b.x;
    y=b.y+r+15;}
        else
        {
        x=b.x+r;
        y=b.y;
        }
           g.setColor(this.color);
           if (this.kind == Kind.Circular) {
               g.fillOval(b.x, b.y, b.width, b.height);
           } else if (this.kind == Kind.Rounded) {
               g.fillRoundRect(b.x, b.y, b.width, b.height, r, r);
           } else if (this.kind == Kind.Square) {
               g.fillRect(b.x, b.y, b.width, b.height);
           }
           g.setColor(Color.BLACK);
           g.setFont(g.getFont().deriveFont(18.0f));
           g.drawString(Integer.toString(this.node_value), x, y);
       }
```

Example 3.1. Consider $q = 7$. Then $m_2^2 = m_4\Delta$. Also, $m_4 = \theta = 2$, $m_2^2 = 2\triangle$. Considering $\triangle = k = s_1 = 1$, and then by using the code given in section 2.3, corresponding matrices R, S, and T thus obtained are:

$R = \begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}, S = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}, T = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix}.$

Therefore, linear-fractional transformations are,

$r : z \mapsto \dfrac{3z+5}{5z+4}, \quad s : z \mapsto \dfrac{z+3}{3z+3}, \quad t : z \mapsto \dfrac{6}{z}.$

Applying r, s and t transformations on the elements of $PL(F_7)$, the permutations will be: r act as: $(0\ 3)(1\ 4)(2\ \infty)(5\ 6)$, s act as: $(0\ 1\ 3\ 4)(2\ 6\ \infty\ 5)$, t act as: $(0\ \infty)(1\ 6)(2\ 3)(4\ 5)$.

Obtained coset diagram is as follows.

This diagram is disconnected and consisting of two diagrams each having 4 vertices. Also note that each vertex of these diagrams is fixed by $(rs)^4$ and the group $\Delta(2,4,4) =< r, s : r^2 = s^4 = (rs)^4 = 1 >$. So G is is abelian and cyclic.

Example 3.2. Consider $q = 23$. Then $m_2^2 = m_4\Delta$. Also, $m_4 = \theta = 2$, $m_2^2 = 2\triangle$. Considering $\triangle = k = s_1 = 1$, and then by using the code given in sections 3.1 and 3.2, corresponding matrices R, S, and T thus obtained are:

$R = \begin{bmatrix} 17 & 3 \\ 3 & 6 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 4 \\ 4 & 17 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 22 \\ 1 & 0 \end{bmatrix}$.

Therefore, linear-fractional transformations are $r : z \mapsto \dfrac{17z+3}{3z+6}$, $s : z \mapsto \dfrac{z+4}{4z+17}$, $t : z \mapsto \dfrac{22}{z}$.

Applying r, s and t transformations on the elements of $PL(F_{23})$, the permutations will be,
r act as: $(0\ 21)(1\ 3)(2\ 9)(4\ 14)(5\ 11)(6\ 7)(8\ 16)(10\ 20)(12\ \infty)(13\ 19)(15\ 22)(17\ 18)$
s act as: $(0\ 7\ 12\ 19)(1\ 9\ 15\ 11)(2\ 3\ 5\ 22)(4\ 10\ 18\ 8)(8\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18)$
t act as: $(0\ \infty)(1\ 22)(2\ 11)(3\ 15)(4\ 17)(5\ 9)(6\ 19)(7\ 13)(8\ 20)(10\ 16)(12\ 21)(14\ 18)$.

The coset diagram generated by using code in section 2.3 is shown in Figure 2,

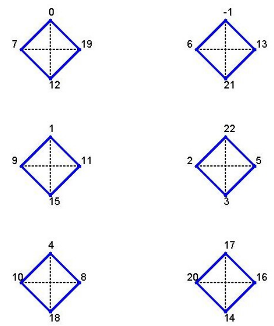

Figure 2: Intransitive action of $\Delta(2,4,k)$ on $PL(F_{23})$

This diagram is disconnected and has six diagrams each consisting of 4 vertices. Also note that each vertex of these diagrams is fixed by $(rs)^4$ and the group

$$\Delta(2,4,4) =< r, s : r^2 = s^4 = (rs)^4 = 1 > .$$

So G is an abelian and cyclic.

In Table 1, we have listed few primes and the number of diagrams corresponding to each prime. Here it can be observed that for each prime q, the coset diagram is disconnected. So the action of $\Delta(2,4,k)$ is intransitive on $PL(F_q)$.

Table 1: Number of disconnected diagrams

Primes	Diagrams of 4 Vertices
7	2
23	6
47	12
79	20

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T. Imran and M.A. Zaighum
Department of Mathematics and Statistics
Riphah International University
Islamabad
Pakistan
E-mails: tahirimran_78@yahoo.com, asadzaighum@gmail.com

M. Ashiq
Department of Humanities and Basic Sciences
MCS Capmus, National University of Sciences and Technology
Islamabad,Pakistan
E-mail: m.ashiq@mcs.edu.pk