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Semirings which are distributive lattices of weakly left k-Archimedean semirings

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Abstract. We introduce a binary relation $\xrightarrow{l}$ on a semiring S, and generalize the notion of left k-Archimedean semirings and introduce weakly left k-Archimedean semirings, via the relation $\xrightarrow{l}$. We also characterize the semirings which are distributive lattices of weakly left k-Archimedean semirings.

1. Introduction

The notion of the semirings was introduced by Vandiver [12] in 1934. The underlying algebra in idempotent analysis [6] is a semiring. Recently idempotent analysis have been used in theoretical physics, optimization etc., various applications in theoretical computer science and algorithm theory [5, 7]. Though the idempotent semirings have been studied by many authors like Monico [8], Sen and Bhuniya [11] and others as a $(2, 2)$ algebraic structure, idempotent semirings are far different from the semirings whose multiplicative reduct is just a semigroup and additive reduct is a semilattice. So for better understanding about the abstract features of the particular semirings $\mathbb{R}_{max}$(Maslov's dequantization semiring), Max-Plus algebra, syntactic semirings we need a separate attention to the semirings whose additive reduct is a semilattice. From the algebraic point of view while studying the structure of semigroups, semilattice decomposition of semigroups, an elegant technique, was first defined and studied by Clifford [4]. This motivated Bhuniya and Mondal to study on the structure of semirings whose additive reduct is a semilattice [1, 2, 9, 10]. In [1], Bhuniya and Mondal studied the structure of semirings with a semilattice additive reduct. There, the description of the least distributive lattice congruence on such semirings was given. In [10], Mondal and Bhuniya gave the distributive lattice decompositions of the semirings into left k-Archimedean semirings. In this paper we generalize the notion of left k-Archimedean semirings introducing weakly left k-Archimedean semirings, analogous to the notion of weakly left k-Archimedean semigroups [3] and characterize the semirings which

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are distributive lattices of weakly left k-Archimedean semirings.

The preliminaries and prerequisites for this article has been discussed in section 2. In section 3 we introduce the notion of weakly left k-Archimedean semirings. We give a sufficient condition for a semiring S to be weakly left k-Archimedean in terms of a binary relation $\xrightarrow{l}$ on S. We also give a condition under which a weakly left k-Archimedean semiring becomes a left k-Archimedean semiring. In section 4 we characterize the semirings which are distributive lattices of weakly left k-Archimedean semirings.

2. Preliminaries and prerequisites

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

Thus the semirings can be viewed as a common generalization of both rings and distributive lattices. A band is a semigroup F in which every element is an idempotent. Moreover if it is commutative, then F is called a semilattice. Throughout the paper, unless otherwise stated, S is always a semiring with semilattice additive reduct.

Every distributive lattice D can be regarded as a semiring $(D, +, \cdot)$ such that both the additive reduct $(D, +)$ and the multiplicative reduct $(D, \cdot)$ are semilattices together with the absorptive law:

$$x + xy = x \text{ for all } x, y \in S.$$

An equivalence relation ρ on S is called a congruence relation if it is compatible with both the addition and multiplication, i.e., for $a, b, c \in S, a\rho b$ implies $(a + c)\rho(b + c), ac\rho bc$ and $ca\rho cb$. A congruence relation ρ on S is called a distributive lattice congruence on S if the quotient semiring S/ρ is a distributive lattice. Let $\mathcal{C}$ be a class of semirings which we call C-semirings. A semiring S is called a distributive lattice of C-semirings if there exists a congruence ρ on S such that S/ρ is a distributive lattice and each ρ-class is a semiring in $\mathcal{C}$.

Let S be a semiring and $\phi \neq A \subseteq S$. Then the k-closure of A is defined by $\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_i \in A\} = \{x \in S \mid x + a = a \text{ for some } a \in A\}$, and the k-radical of A by $\sqrt{A} = \{x \in S \mid (\exists\, n \in \mathbb{N})\ x^n \in \overline{A}\}$. Then $\overline{A} \subseteq \sqrt{A}$ by definition, and $A \subseteq \overline{A}$ since $(S, +)$ is a semilattice. A non empty subset L of S is called a left (resp. right) ideal of S if $L + L \subseteq L$, and $SL \subseteq L$ (resp. $LS \subseteq L$). A non empty subset I of S is called an ideal of S if it is both left and a right ideal of S. An ideal (resp. left ideal) A of S is called a k-ideal (left k-ideal) of S if and only if $\overline{A} = A$.

Lemma 2.1. (cf. [1]) *Let S be a semiring.*

(a) *For $a, b \in S$ the following statements are equivalent*

(i) *There are $s_i, t_i \in S$ such that $b + s_1 a t_1 = s_2 a t_2$.*

(ii) *There are $s, t \in S$ such that $b + sat = sat$.*

(iii) *There is $x \in S$ such that $b + xax = xax$.*

(b) *If $a, b, c \in S$ such that $b + xax = xax$ and $c + yay = yay$ for some $x, y \in S$, then there is $z \in S$ such that $b + zaz = zaz = c + zaz$.*

(c) *If $a, b, c \in S$ such that $c + xax = xax$ and $c + yby = yby$ for some $x, y \in S$, then there is $z \in S$ such that $c + zaz = zaz$ and $c + zbz = zbz$.*

Lemma 2.2. (cf. [1]) *For a semiring S and $a, b \in S$ the following statements hold.*

1. *$\overline{SaS}$ is a k-ideal of S.*

2. *$\sqrt{SaS} = \sqrt{\overline{SaS}}$.*

3. *$b^m \in \sqrt{SaS}$ for some $m \in \mathbb{N} \Leftrightarrow b^k \in \sqrt{SaS}$ for all $k \in \mathbb{N}$.*

Lemma 2.3. (cf. [10]) *Let S be a semiring.*

(a) *For $a, b \in S$ the following statements are equivalent:*

(i) *there are $s_i \in S$ such that $b + s_1 a = s_2 a$,*

(ii) *there are $s \in S$ such that $b + sa = sa$.*

(b) *If $a, b, c \in S$ such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.*

Theorem 2.4. (cf. [10]) *The following conditions on a semiring S are equivalent:*

1. *S is a distributive lattice of left k-Archimedean semirings,*

2. *for all $a, b \in S$, $b \in \overline{SaS}$ implies that $b \in \sqrt{Sa}$,*

3. *for all $a, b \in S$, $ab \in \sqrt{Sa}$,*

4. *$\sqrt{L}$ is a k-ideal of S, for every left k-ideal L of S,*

5. *$\sqrt{Sa}$ is a k-ideal of S, for all $a \in S$,*

6. *for all $a, b \in S$, $\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$.*

3. Weakly left k-Archimedean semirings

In [1], Bhuniya and Mondal studied the structure of semirings, and during this they gave the description of the least distributive lattice congruence on a semiring S stem from the divisibility relation defined by: for $a, b \in S$, $a|b \Longleftrightarrow b \in \overline{SaS}$,

$$a \longrightarrow b \iff b \in \sqrt{SaS} \iff b^n \in \overline{SaS} \quad \text{for some } n \in \mathbb{N}.$$

Thus it follows from the Lemma 2.1, $a \longrightarrow b \Longleftrightarrow b^n + xax = xax$, for some $n \in \mathbb{N}$ and $x \in S$.

In this section we introduce the relation $\overset{l}{\longrightarrow}$(left analogue of $\longrightarrow$) on a semiring S, the notion of weakly left k-Archimedean semirings and study them.

Proposition 3.1. *Let S be a semiring. Then $\overline{Sa}$ is a left k-ideal of S for every $a \in S$.*

Proof. For $b, c \in \overline{Sa}$, there is $x \in S$ such that $b + xa = xa = c + xa$, by Lemma 2.3. This implies $(b + c) + xa = xa$, i.e., $b + c \in \overline{Sa}$. Moreover, for any $s \in S$ we get $sb + sxa = sxa$, and so $sb \in \overline{Sa}$. For $u \in \overline{\overline{Sa}}$ there is some $b \in \overline{Sa}$ such that $u + b = b$. Using again $b + xa = xa$ for some $x \in S$, we get $u + xa = u + b + xa = b + xa = xa$, i.e., $u \in \overline{Sa}$. So $\overline{Sa} = \overline{\overline{Sa}}$ is a left k-ideal of S. □

Now we introduce the relation $\overset{l}{\longrightarrow}$ on a semiring S as a generalization of the division relation $|_l$, and they are given by: for $a, b \in S$, $a \mid_l b \Longleftrightarrow b \in \overline{Sa}$,

$$a \overset{l}{\longrightarrow} b \iff b \in \sqrt{Sa} \iff b^n \in \overline{Sa} \quad \text{for some } n \in \mathbb{N}.$$

Thus $a \overset{l}{\longrightarrow} b$ if there exist some $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$, by Lemma 2.3.

In [10], Mondal and Bhuniya defined *left k-Archimedean semirings* as: A semiring S is called left k-Archimedean if for all $a \in S$, $S = \sqrt{Sa}$. For example, let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, define $+$ and $\cdot$ on $S = A \times A$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\max\{a, c\}, \max\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, b).$$

Then $(S, +, \cdot)$ is a left k-Archimedean semiring.

We now introduce a more general notion:

A semiring S will be called *weakly left k-Archimedean* if $ab \overset{l}{\longrightarrow} b$, for all $a, b \in S$.

Example 3.2. Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, define $+$ and $\cdot$ on $S = A \times A$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\max\{a, c\}, \max\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, d).$$

Then $(S,+,\cdot)$ is a weakly left k-Archimedean semiring. Now let $(a,\frac{1}{2}),(c,\frac{1}{3}) \in S$. If possible, let there exist $n \in \mathbb{N}$ and $(x,y) \in S$ satisfying $(a,\frac{1}{2})^n + (x,y)\cdot(c,\frac{1}{3}) = (x,y)\cdot(c,\frac{1}{3})$. This implies $(a^n,\frac{1}{2}) + (xc,\frac{1}{3}) = (xc,\frac{1}{3})$ so that $max\{a^n, xc\} = xc$, $max\{\frac{1}{2},\frac{1}{3}\} = \frac{1}{3}$, which is not possible. Consequently, $(S,+,\cdot)$ is not a left k-Archimedean semiring.

Here we see that the relation $\xrightarrow{l}$ is not symmetric on a semiring S in general. For, consider the Example 3.2, there $(a,\frac{1}{2}) \xrightarrow{l} (c,\frac{1}{3})$ but not $(c,\frac{1}{3}) \xrightarrow{l} (a,\frac{1}{2})$. Although, the semiring S is weakly left k-Archimedean. Now, in the following proposition we show that if the relation $\xrightarrow{l}$ is symmetric on a semiring S, then S is weakly left k-Archimedean.

Proposition 3.3. *A semiring S is weakly left k-Archimedean if the relation $\xrightarrow{l}$ is symmetric on S.*

Proof. Let $\xrightarrow{l}$ is a symmetric relation on S and $a,b \in S$. Now $ab \in \overline{Sb}$ implies that $b \xrightarrow{l} ab$ and so $ab \xrightarrow{l} b$, by symmetry of $\xrightarrow{l}$ on S. Thus S is weakly left k-Archimedean. □

Thus the condition of symmetry of $\xrightarrow{l}$ is only sufficient for a semiring S to be weakly left k-Archimedean, not necessary. Let S be a left k-Archimedean semiring, and $a,b \in S$. Then $b \in \sqrt{Sa}$ implies that $b^n + sa = sa$ for some $n \in \mathbb{N}$ and $s \in S$. Multiplying b on both sides on the right we get $b^{n+1} + sab = sab$. This yields $ab \xrightarrow{l} b$ so that S is a weakly left k-Archimedean semiring. Thus we have the following proposition:

Proposition 3.4. *Every left k-Archimedean semiring S is a weakly left k-Archimedean semiring.*

Here in the following proposition we find a condition for which the converse holds:

Proposition 3.5. *Let S be a semiring, and $ab \in \sqrt{Sa}$, for all $a,b \in S$ hold. Then S is left k-Archimedean semiring if it is weakly left k-Archimedean.*

Proof. Let $a,b \in S$. Then $ba \xrightarrow{l} a$, whence by Lemma 2.3, there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + sba = sba$. Again by hypothesis, there are $m \in \mathbb{N}$ and $t \in S$ such that $(sba)^m + tsb = tsb$. Now $a^m + sba = sba$ implies that $a^{nm} + (sba)^m = (sba)^m$. Adding tsb on both sides we get $a^{nm} + [(sba)^m + tsb] = [(sba)^m + tsb]$, i.e. $a^{nm} + tsb = tsb \in \overline{Sb}$. So $a \in \sqrt{Sb}$. Thus S is a left k-Archimedean semiring. □

Now, by Theorem 2.4, we see that a weakly left k-Archimedean semiring will be a left k-Archimedean semiring if it is a distributive lattice of left k-Archimedean semirings.

4. Lattices of weakly left k-Archimedean semirings

In this section we characterize the semirings which are distributive lattices of weakly left k-Archimedean semirings. A semiring S is called a *distributive lattice of weakly left k-Archimedean semirings* if there exists a congruence ρ on S such that S/ρ is a distributive lattice and each ρ-class is a weakly left k-Archimedean semiring.

Lemma 4.1. *Suppose S is a distributive lattice $\mathcal{D}$ of subsemirings $S_\alpha, \alpha \in \mathcal{D}$. Then $a, b \in S_\alpha,\ \alpha \in \mathcal{D}$, then $a \stackrel{l}{\longrightarrow} b$ in S implies that $a \stackrel{l}{\longrightarrow} b$ in S_α.*

Proof. Let ρ be a distributive lattice congruence on S so that S is a distributive lattice $\mathcal{D}$ of subsemirings $S_\alpha, \alpha \in \mathcal{D}$. Let $a \stackrel{l}{\longrightarrow} b$. Then $b^n + xa = xa$ for some $n \in \mathbb{N}, x \in S$. Let $x \in S_\beta, \beta \in \mathcal{D}$. Now $b^{n+1} + bxa = bxa$, and so $b\rho(b + bxa)\rho(b^{n+1} + bxa) = bxa\rho abx$, i.e., $b\rho abx$. This implies $\alpha = \alpha\alpha\beta = \alpha\beta$, since $\mathcal{D}$ is a distributive lattice. Now $b^{n+1} + bxa = bxa \in S_{\alpha\beta}a = S_\alpha a$ so that $b^{n+1} \in \overline{S_\alpha a}$. Consequently, $a \stackrel{l}{\longrightarrow} b$ in S_α. □

Now we are in a position to present the main result of this paper. Here we characterize the semirings which are distributive lattices of weakly left k-Archimedean semirings.

Theorem 4.2. *The following conditions are equivalent on a semiring S:*

(1) *S is a distributive lattice of weakly left k-Archimedean semirings,*

(2) *for all $a, b \in S,\ a \longrightarrow b \Rightarrow ab \stackrel{l}{\longrightarrow} b$.*

Proof. (1) $\Rightarrow$ (2). Let S be a distributive lattice $D = S/\rho$ of weakly left k-Archimedean semirings $S_\alpha, \alpha \in D$, ρ being the corresponding distributive lattice congruence. Let $a, b \in S$ such that $a \longrightarrow b$ so that there are $n \in \mathbb{N}$ and $s \in S$ such that $b^n + sas = sas$, by Lemma 2.1. Also there are $\alpha, \beta \in D$ such that $a \in S_\alpha, b \in S_\beta$. Now $(b + sas)\rho(b^n + sas) = sas\rho as^2$. So $b\rho(b^2 + bsas)\rho bas^2$, which implies $b\rho(b + ba)\rho(bas^2 + ba)\rho ba$ and thus $ba \in S_\beta$. Since S_β is a weakly left k-Archimedean semiring, $b^n \in \overline{S_\beta bab} \subseteq \overline{Sab}$ for some $n \in \mathbb{N}$ yielding $ab \stackrel{l}{\longrightarrow} b$.

(2) $\Rightarrow$ (1). By Lemma 2.2, for $a, b \in S, (ab)^2 \in \overline{SaS}$ implies that $a \longrightarrow ab$. So by hypothesis, $a^2b = a(ab) \stackrel{l}{\longrightarrow} (ab)$. This shows that $(ab)^n \in \overline{Sa^2b} \subseteq \overline{Sa^2S}$, for some $n \in \mathbb{N}$. Then by Theorem 4.3[1], S is a distributive lattice($D = S/\eta$) of k-Archimedean semirings $S_\alpha, \alpha \in D$, where η is the least distributive lattice congruence on S. Let $a, b \in S_\alpha$. Then $a \longrightarrow b$ and so $ab \stackrel{l}{\longrightarrow} b$ in S. Then by Lemma 4.1, one gets $ab \stackrel{l}{\longrightarrow} b$ in S_α. Thus S_α is weakly left k-Archimedean. □

Now we give an example of a semiring which is a distributive lattice of left k-Archimedean semirings, whence a distributive lattice of weakly left k-Archimedean semirings.

Example 4.3. Consider the set $\mathbb{N}$ of all natural numbers, and define $+$ and $\cdot$ on $S = \mathbb{N} \times \mathbb{N}$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\min\{a, c\}, \min\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, b).$$

Then S is a distributive lattice of left k-Archimedean semirings.

Example 4.4. Consider the set $\mathbb{N}$ of all natural numbers, and define $+$ and $\cdot$ on $S = \mathbb{N} \times \mathbb{N}$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\min\{a, c\}, \min\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, d).$$

Then S is a distributive lattice of weakly left k-Archimedean semirings. But S is not a distributive lattice of left k-Archimedean semirings. Indeed, for $(1, 2), (2, 2) \in S$ suppose there exist $n \in \mathbb{N}$ and $(x, y) \in S$ satisfying $[(1, 2) \cdot (2, 1)]^n + (x, y) \cdot (1, 2) = (x, y) \cdot (1, 2)$. This implies $(2^n, 1) + (x, 2) = (x, 2)$, i.e. $\min\{2^n, x\} = x$, $\min\{1, 2\} = 2$. The last equality is absurd.

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