Quasigroups and Related Systems **27** (2019), 273 − 280

Congruences on nil-extension of a b-lattice of skew-rings

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Abstract. Let S be a nil-extension of a b-lattice of skew-rings K by a semiring Q. A congruence pair (δ, ω) on S consists of a congruence δ on Q and a congruence ω on K. In this paper, we establish that there is an order preserving bijection between the set of all congruences on S onto the set of all congruence pairs on S. It is also proved that if S is a nil-extension of a completely regular semiring, then every congruence on S can be uniquely represented by a congruence pair and there is an order preserving bijection from the set of all congruences on S onto the set of all congruence pairs on S.

1. Introduction

Nil-extensions of semigroups are precisely the ideal extensions by nil semigroups. Semigroups which are nil-extensions of completely simple semigroups was first studied by S. Bogdanović and S. Milić [2] in 1984. Decomposition of completely $\pi-$regular semigroups into a semilattice of Archimedean semigroups was studied by Bogdanović [1]. Nil-extensions of regular semigroups, regular poe-semigroups are special classes of semigroups that attracted many researchers. Moreover, nil-extension of Clifford semigroup was also a matter of interest.

The structure of semirings has been recently studied by many authors, for example, by F. Pastijin, Y. Q. Guo, M. K. Sen, K. P. Shum and others. Recently, in paper [9], the study of completely regular semirings have derived profilic results which were analouge properties as completely regular semigroups and it has also been derived that a completely regular semiring is a b-lattice of completely simple semirings. Many interesting results in completely regular semigroups and inverse semigroups have been extended to semirings by Sen, Maity and Shum in [9]. In [7], Maity, Ghosh and Chatterjee characterized b-lattice of quasi skew-rings. In [10], Ren and Wang studied the congruences on Clifford quasi-regular semigroups.

In this paper we study the congruences on nil-extension of a band-semilattice (shortly: b-lattice) of skew-rings and congruences on nil-extension of completely regular semiring.

2010 Mathematics Subject Classification: 16Y60, 20M10, 20M07.
Keywords: Quasi completely regular semiring, additive inverse semiring, b-lattice of skew-rings, nil-extension, bi-ideal, skew-ring, quasi skew-ring, congruence pair.

2. Preliminaries

A *semiring* $(S,+,\cdot)$ is a type (2, 2)-algebra whose semigroups $(S,+)$ and $(S,\cdot)$ are connected by ring like distributivity, i.e., $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in S$. An element a in a semiring S is said to be *infinite* [4] if and only if $a+x=a=x+a$ for all $x\in S$. Infinite element in a semiring is unique and is denoted by ∞. An infinite element ∞ in a semiring S having the property that $x\cdot\infty=\infty=\infty\cdot x$ for all $x(\neq 0)\in S$ is called *strongly infinite* [4]. A semiring $(S,+,\cdot)$ is called *additively regular* if for every element $a\in S$ there exists an element $x\in S$ such that $a+x+a=a$. In a semiring S, an element y satisfying $a+y+a=a$ and $y+a+y=y$ is said to be an *additive inverse* of an element $a\in S$. We call a semiring $(S,+,\cdot)$ *additively quasi regular* if for every element $a\in S$ there exists a positive integer n such that na is additively regular. An element a in a semiring $(S,+,\cdot)$ is said to be *completely regular* [9] if there exists an element $x\in S$ such that, $a=a+x+a$, $a+x=x+a$ and $a(a+x)=a+x$. We call a semiring S, a *completely regular semiring* if every element a of S is completely regular.

We define an element a in a semiring $(S,+,\cdot)$ as *quasi completely regular* [6] if there exists a positive integer n such that na is completely regular. Naturally, a semiring S is said to be *quasi completely regular* if every element of S is quasi completely regular. A semiring $(S,+,\cdot)$ is a *b-lattice* [9] if $(S,\cdot)$ is a band and $(S,+)$ is a semilattice. Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S and $C(S)$ be the set of all congruences on the semiring S. Also we denote the set of all additive inverses of a, if it exists, in a semiring S by $V^+(a)$. We further denote the Green's relations on a completely π-regular semigroup as $\mathscr{L}^*$, $\mathscr{R}^*$, $\mathscr{H}^*$, $\mathscr{D}^*$ and $\mathscr{J}^*$. If $(S,+,\cdot)$ is an additively quasi regular semiring, the relations $\mathscr{L}^{*+},\mathscr{R}^{*+}$, $\mathscr{J}^{*+},\mathscr{H}^{*+}$ and $\mathscr{D}^{*+}$ [6] are defined by: for $a,b\in S$,

$$a\,\mathscr{L}^{*+}\,b \text{ if and only if } pa\,\mathcal{L}^+\,qb,$$
$$a\,\mathscr{R}^{*+}\,b \text{ if and only if } pa\,\mathscr{R}^+\,qb,$$
$$a\,\mathscr{J}^{*+}\,b \text{ if and only if } pa\,\mathscr{J}^+\,qb,$$
$$\mathscr{H}^{*+}=\mathscr{L}^{*+}\cap\mathscr{R}^{*+} \quad\text{and}\quad \mathscr{D}^{*+}=\mathscr{L}^{*+}\,o\,\mathscr{R}^{*+},$$

where p and q are the smallest positive integers such that pa and qb are additively regular.

A quasi completely regular semiring S is said to be *completely Archimedean* [6] if any two elements of S are $\mathscr{J}^{*+}$-related.

A congruence ρ on a semiring S is called a *b-lattice congruence* (*idempotent semiring congruence*) if S/ρ is a b-lattice (respectively, an idempotent semiring). A semiring S is called a *b-lattice* (*idempotent semiring*) Y of semirings S_α $(\alpha\in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ρ on S such that $Y=S/\rho$ and each S_α is a ρ-class mapped onto α by the natural epimorphism $\rho^{\#}\;:S\longrightarrow Y$.

A nonempty subset I of a semiring S is said to be a *bi-ideal* [3] of S if for all $a\in I$ and for all $x\in S$ implies $a+x$, $x+a$, ax, $xa\in I$. Let I be a bi-ideal of a semiring S. We define a relation ρ_I on S by $a\rho_I b$ if and only if either $a, b \in I$ or $a = b$ where $a, b \in S$. It is easy to verify that ρ_I is a congruence on S. This congruence is said to be the *Rees congruence* on S and the quotient semiring S/ρ_I contains a strongly infinite element, viz., I. This quotient semiring S/ρ_I is said to be the *Rees quotient semiring* and is denoted by S/I. In this case the semiring S is said to be an *ideal extension* or simply an *extension* of I by the semiring S/I. An ideal extension S of a semiring I is a *nil-extension* [5] of I if for any $a \in S$ there exists a positive integer n such that $na \in I$.

For other notations and terminologies see [1] and [4].

3. Nil-extensions of a b-lattice of skew-rings

In this section we establish the structure of an additively quasi regular semiring which is a nil-extension of a b-lattice of skew-rings.

Definition 3.1. A semiring $(S, +, \cdot)$ is called a *skew-ring* if $(S, +)$ is a group. If for every $a \in S$ there exists a positive integer n such that $na \in R$, where R is subskew-ring of S, then S is said to be a *quasi skew-ring*.

Theorem 3.2. (cf. [6]) *Let a be an element of a semiring S such that na lies in a subskew-ring R of S for some positive integer n. If e is the zero of R, then*

(i) $e + a = a + e \in R$;

(ii) $ma \in R$ *for any integer* $m > n$;

(iii) $ae = ea = e$.

Theorem 3.3. (cf. [6]) *A semiring S is additively quasi regular with exactly one additive idempotent if and only if S is a quasi skew-ring.*

Theorem 3.4. (cf. [5]) *A semiring S is a quasi skew-ring if and only if S is a nil-extension of a skew-ring.*

Theorem 3.5. (cf. [6]) *For a semiring S the following conditions are equivalent:*

(i) *S is a quasi completely regular semiring,*

(ii) *Every $\mathscr{H}^{*+}$- class is a quasi skew-ring,*

(iii) *S is (disjoint) union of quasi skew-rings,*

(iv) *S is a b-lattice of completely Archimedean semirings,*

(v) *S is an idempotent semiring of quasi skew-rings.*

Since each $\mathscr{H}^{*+}$- class in a quasi completely regular semiring S is a quasi skew-ring, it follows from Theorem 3.3 that each $\mathscr{H}^{*+}$-contains a unique additive idempotent. The unique additive idempotent in the $\mathscr{H}^{*+}$- containing an element $x \in S$ is denoted by 0_x.

Theorem 3.6. (cf. [5]) *The following conditions on a semiring are equivalent:*

(i) *S is a completely Archimedean semiring;*

(ii) *S is a nil extension of a completely simple semiring;*

(iii) *S is Archimedean and quasi completely regular.*

Definition 3.7. (cf. [8]) A subsemiring T of a semiring S is a *retract* of S if there exists a homomorphism $\varphi\colon S \longrightarrow T$ such that $\varphi(t) = t$ for all $t \in T$. Such a homomorphism is called a *retraction*. A nil-extension S of T is said to be a *retractive nil-extension* of T if T is a retract of S.

Theorem 3.8. (cf. [8]) *The following conditions on a semiring are equivalent:*
(*i*) *S is a completely Archimedean semiring;*
(*ii*) *S is a nil extension of a completely simple semiring;*
(*iii*) *S is retractive nil-extension of a completely simple semiring.*

Theorem 3.9. (cf. [8]) *The following conditions on a semiring S are equivalent:*
(*i*) *S is a nil-extension of a b-lattice of skew-rings.*
(*ii*) *S is a retractive nil-extension of a b-lattice of skew-rings.*

Theorem 3.10. *The following conditions on a semiring S are equivalent:*
(*i*) *S is a nil-extension of a b-lattice of skew-rings;*
(*ii*) *S is a quasi completely regular semiring such that $Reg^+(S)$ is a bi-ideal of S and $a + e = e + a$ for all $a \in S$ and for all $e \in E^+(S)$;*
(*iii*) *S is a b-lattice of quasi skew-rings and $Reg^+(S)$ is a bi-ideal of S.*

Proof. $(i) \Rightarrow (ii)$: Let S be a nil-extension of a b-lattice of skew-rings T. Then clearly S is a quasi completely regular semiring and $\mathrm{Reg}^+(S) = T$ is a bi-ideal of S. Let $a \in S$ and $e \in E^+(S)$. Then $a + e, e \in T$ and thus $a + e = a + (e + e) = (a + e) + e = e + (a + e) = (e + a) + e = e + (e + a) = e + a$.

$(ii) \Rightarrow (iii)$: Follows from [7, Theorem 3.5].

$(iii) \Rightarrow (i)$: Let S be a b-lattice Y of quasi skew-rings S_α $(\alpha \in Y)$ and $\mathrm{Reg}^+(S)$ is a bi-ideal of S. For each $\alpha \in Y$, let S_α be the nil-extension of a skew-ring R_α. Clearly, $\mathrm{Reg}^+(S) = \bigcup_{\alpha \in Y} R_\alpha$ is a completely regular semiring and S is a nil-extension of $\mathrm{Reg}^+(S)$. Since S is a b-lattice of quasi skew-rings, it follows by [7, Theorem 3.5] that S is a quasi completely inverse semiring and hence by [7, Theorem 3.6], it follows that every additively regular element possesses a unique additive inverse. Thus every element of $\mathrm{Reg}^+(S)$ possesses a unique additive inverse, i.e., $\mathrm{Reg}^+(S)$ is an additive inverse semiring. Thus, $\mathrm{Reg}^+(S)$ is a completely regular semiring as well as an additive inverse semiring. Hence $\mathrm{Reg}^+(S)$ is a b-lattice of skew-rings. Consequently, S is a nil-extension of a b-lattice of skew-rings. □

4. Congruences on nil-extensions

In this section we introduce congruence pair on an additively quasi regular semiring which is a nil-extension of a b-lattice of skew-rings.

Definition 4.1. Let S be a nil-extension of b-lattice of skew-rings K by the semiring Q with a strongly infinite element ∞ and σ be a congruence on S. We define $K\sigma$ by $K\sigma = \{a \in S : (a, k) \in \sigma \text{ for some } k \in K\}$. Also, we define two relations on K and Q, respectively denoted by σ_K and σ_Q, by $\sigma_K = \sigma|_K$ and $\sigma_Q = (\sigma \vee \rho_K)/\rho_K$, where ρ_K is the Rees congruence on S induced by the bi-ideal K.

Definition 4.2. Let S be a nil-extension of a b-lattice of skew-rings K by a semiring Q with strongly infinite element ∞, δ be a congruence on the semiring Q and ω be a congruence on K. Then a pair $(\delta, \omega) \in C(Q) \times C(K)$ is called a *congruence pair* on S if it satisfies the following conditions.

(M_1) If $(e, f) \in \omega$ for some additive idempotents $e, f \in E^+(S)$, then $(p+e, p+f) \in \omega$ and $(e+p, f+p) \in \omega$ for any $p \in Q$.

(M$_2$) If $(p, q) \in \delta|_{Q\setminus\infty\delta}$, then $(p+e, q+e) \in \omega$ and $(e+p, e+q) \in \omega$ for any $e \in E^+(S)$.

(M$_3$) (a) If $(p, q) \in \delta|_{Q\setminus\infty\delta}$, then $(0_{p+c}, 0_{q+c}) \in \omega$ and $(0_{c+p}, 0_{c+q}) \in \omega$ for any $c \in S$.

(b) If $(p, q) \in \delta|_{Q\setminus\infty\delta}$, then $(0_{pc}, 0_{qc}) \in \omega$ and $(0_{cp}, 0_{cq}) \in \omega$ for any $c \in S$.

(M$_4$) If $a (\neq \infty) \in \infty\delta$, then $(a+0_a+c, a+c+0_{a+c}) \in \omega$ and $(c+a+0_a, c+a+0_{c+a}) \in \omega$ for any $c \in S$.

We need two results similar to Proposition 2.2 and Proposition 2.3 from [10].

Lemma 4.3. *Let S be a nil-extension of b-lattice of skew-rings K by the semiring Q with a strongly infinite element ∞ and σ be a congruence on S. Then $a \in K\sigma$ if and only if $(a, a+0_a) \in \sigma$ and $(a, 0_a + a) \in \sigma$.*

Lemma 4.4. *Let S be a nil-extension of b-lattice of skew-rings K by the semiring Q with a strongly infinite element ∞ and σ be a congruence on S. Then $(0_a, 0_b) \in \sigma$ for any $(a, b) \in \sigma$.*

Lemma 4.5. *Let S be a nil-extension of b-lattice of skew-rings K by the semiring Q with a strongly infinite element ∞ and σ be a congruence on S. Then $\sigma \subseteq \tau$ if and only if $\sigma_Q \subseteq \tau_Q$ and $\sigma_K \subseteq \tau_K$ for any $\sigma, \tau \in C(S)$.*

Proof. Now, $\sigma_Q = (\sigma \vee \rho_K)/\rho_K, \tau_Q = (\tau \vee \rho_K)/\rho_K, \sigma_K = \sigma|_K$ and $\tau_K = \tau|_K$.

First we assume that $\sigma \subseteq \tau$. Suppose $a\rho_K, b\rho_K \in Q = S/\rho_K$ such that $a\rho_K \, \sigma_Q \, b\rho_K$. Then $a\rho_K \, (\sigma \vee \rho_K)/\rho_K \, b\rho_K$. This implies that $a \, (\sigma \vee \rho_K) \, b$, i.e., there exists a sequence of elements $c_1, c_2, \ldots, c_n \in S$ with $a = c_1, b = c_n$ such that $(c_i, c_i{+}1) \in \sigma$ or $(c_i, c_{i+1}) \in \rho_K$. This implies there exists a sequence of elements $c_1, c_2, \ldots, c_n \in S$ with $a = c_1, b = c_n$ such that $(c_i, c_i{+}1) \in \tau$ or $(c_i, c_{i+1}) \in \rho_K$, i.e, $a \, (\tau \vee \rho_K) \, b$ and thus $a\rho_K \, \tau_Q \, b\rho_K$.

To show $\sigma_K \subseteq \tau_K$, let $c, d \in K$ such that $c \, \sigma_K \, d$. This implies $c \, \sigma \, d$ and hence $c \, \tau \, d$ with $c, d \in K$. Therefore, $c \, \tau_K \, d$ and consequently $\sigma_K \subseteq \tau_K$.

Conversely, suppose that $\sigma_Q \subseteq \tau_Q$ and $\sigma_K \subseteq \tau_K$. To show $\sigma \subseteq \tau$, let $p \, \sigma \, q$ for some $p, q \in S$. If both $p, q \in K$, then $p \, \sigma_K \, q$. Now $\sigma_K \subseteq \tau_K$ implies $p \, \tau_K \, q$ and hence $p \, \tau \, q$. So we consider the cases when one of p, q does not belong to K, or both do not belong to K. Now $p\ \sigma\ q$ implies $p\ (\sigma \vee \rho_K)\ q$, i.e., $p\rho_K\ \sigma_Q\ q\rho_K$. This implies $p\rho_K\ \tau_Q\ q\rho_K$, i.e., $p\rho_K\ (\tau \vee \rho_K)/\rho_K\ q\rho_K$, i.e., $p\ (\tau \vee \rho_K)\ q$ and therefore there exists a sequence $x_1, x_2, \ldots, x_m \in S$ with $x_1 = p$ and $x_m = q$ such that either $(x_i, x_{i+1}) \in \tau$ or $(x_i, x_{i+1}) \in \rho_K$.

If $(x_i, x_{i+1}) \in \tau$ for all $i = 1, 2, \ldots, m$, then clearly $(x_1, x_m) \in \tau$ and hence $(p, q) \in \tau$. Therefore, $\sigma \subseteq \tau$.

On the other hand, if $(x_i, x_{i+1}) \in \rho_K$ for at least one $i = 1, 2, \ldots, m$, then we have $p, q \in K\tau$. Then by Lemma 4.3, we have $(p, p + 0_p) \in \tau$ and $(q, q + 0_q) \in \tau$. Again, $p\ \sigma\ q$ implies $0_p\ \sigma\ 0_q$ by Lemma 4.4 and hence $(p + 0_p, q + 0_q) \in \sigma$. Since $p + 0_p, q + 0_q \in K$, we must have $(p + 0_p, q + 0_q) \in \sigma_K$. Since $\sigma_K \subseteq \tau_K$, it follows that $(p+0_p, q+0_q) \in \tau_K$, i.e., $(p+0_p, q+0_q) \in \tau$. Therefore, $p\,\tau\,(p+0_p)\,\tau\,(q+0_q)\,\tau\,q$ and thus $p\,\tau\,q$. Consequently, $\sigma \subseteq \tau$. □

Theorem 4.6. *If $\sigma \in C(S)$, then (σ_Q, σ_K) is a congruence pair on S.*

Proof. For any $\sigma \in C(S)$, clearly it follow that $\sigma_Q \in C(Q)$ and $\sigma_K \in C(K)$. By [10, Lemma 3.3], it follows that (σ_Q, σ_K) satisfies all the conditions in Definition 4.2 except $M_3(b)$. To complete the proof, we only prove that the pair (σ_Q, σ_K) satisfies the condition $M_3(b)$ in Definition 4.2. For this, let $(p, q) \in \sigma_Q|_{Q \setminus \infty\sigma_Q}$. Since σ is a congruence on $(S, \cdot)$, then for any $c \in S$, we have $(pc, qc) \in \sigma$ and hence by $(0_{pc}, 0_{qc}) \in \sigma$. As $0_{pc}, 0_{qc} \in K$, we have $(0_{pc}, 0_{qc}) \in \sigma_K$. Similarly, we have $(0_{cp}, 0_{cq}) \in \sigma_K$. This shows that (σ_Q, σ_K) satisfies the condition $M_3(b)$. Consequently, (σ_Q, σ_K) is a congruence pair on S. □

Theorem 4.7. *Let S be a nil-extension of a b-lattice of skew-rings K by a semiring Q with a strongly infinite element ∞. Let $(\delta, \omega) \in C(Q) \times C(K)$ be a congruence pair on S. Define a relation σ on S by : for $a, b \in S$, $a\ \sigma\ b$ if and only if*

(i) $(a, b) \in \delta$ for any $a, b \in S \setminus R$,

(ii) $(a + 0_a, b + 0_b) \in \omega$ for any $a, b \in R$ where $R = K \cup (\infty\delta \setminus \{\infty\})$.

Then σ is a congruence on S such that $K\sigma = R$.

Proof. By Lemma [10, Lemma 3.4], we have σ is a congruence on $(S, +)$ such that $K\sigma = R$. To complete the proof, it remains to prove that σ is a congruence on $(S, \cdot)$. For this let $a, b \in S$ such that $a\,\sigma\,b$ and $c \in S$.

Case - I : We assume that $a, b \in S \setminus R$. Then $a\,\delta\,b$. It is easy to verify that $ac \in S \setminus R$ if and only if $bc \in S \setminus R$ and in this case clearly $c \notin K$, i.e., $c \in Q$. Since $a\,\delta\,b$ and $c \in Q$, we must have $ac\,\delta\,bc$.

We now show that whether $c \in K$ or not, $ac\,\delta\,bc$ when both $ac, bc \in R$. Since $a, b \in S \setminus R$, we have $a, b \in Q \setminus \infty\delta$. So by condition $M_3(b)$, we have $(0_{ac}, 0_{bc}) \in \omega$. Since $\omega \in C(K)$ and $a + 0_a \in K$, we have $0_{ac}(a + 0_a)\ \omega\ 0_{bc}(a + 0_a)$, i.e, $0_{ac}a\ \omega\ 0_{bc}a$. Now, $(a, b) \in \delta|_{Q \setminus \infty\delta}$ implies $0_{bca}\ \omega\ 0_{bcb}$, i.e., $0_{bc}a\ \omega\ 0_{bc}b$. Therefore, $0_{ac}a\ \omega\ 0_{bc}b$. Since $0_{ac}a, 0_{bc}b \in E^+(S)$ and $0_{ac}a\ \omega\ 0_{bc}b$, so by condition M_1, we have $a + 0_{ac}a\ \omega\ a + 0_{bc}b$. Again, $a\ \delta\ b$ and $0_{bc}b \in E^+(S)$ imply $a + 0_{bc}b\ \omega\ b + 0_{bc}b$ [by the condition M_2]. So by transitivity of ω, we have $a + 0_{ac}a\ \omega\ b + 0_{bc}b$. Since

$c+0_c \in K$ and ω is a congruence on K, then $(a+0_{ac}a)(c+0_c)\,\omega\,(b+0_{bc}b)(c+0_c)$, i.e., $(ac+0_{ac})\,\omega\,(bc+0_{bc})$. Hence $ac\,\sigma\,bc$.
Case - II : We now assume that $a, b \in R$. In this case $ac, bc \in R$ for any $c \in S$. Again, $a\,\sigma\,b$ implies $(a+0_a, b+0_b) \in \omega$. Since $c+0_c \in K$ and ω is a congruence on K, it follows that $(a+0_a)(c+0_c)\ \omega\ (b+0_b)(c+0_c)$, i.e., $ac+0_{ac}\,\omega\,bc+0_{bc}$. Since both $ac, bc \in R$, hence we have $(ac, bc) \in \sigma$. Thus σ is a right congruence on $(S, \cdot)$. Similarly, we can show that σ is also a left congruence on $(S, \cdot)$ and hence σ is a congruence on the semiring S. □

Theorem 4.8. *Let S be a nil-extension of a b-lattice of skew rings K by a semiring Q with strongly infinite element and let (δ, ω) be a congruence pair on S. Then the congruence σ given in Theorem 4.7 is the unique congruence on S satisfying $\sigma_Q = \delta$ and $\sigma_K = \omega$.*

Proof. The proof is similar to [10, Lemma 3.5]. □

Combining Lemma 4.5, Theorem 4.6, Theorem 4.7 and Theorem 4.8 we get the following result.

Theorem 4.9. *Let S be a nil-extension of a b-lattice of skew-rings K by a semiring Q with strongly infinite element. Then a mapping $\Gamma : C(S) \longrightarrow C(Q) \times C(K)$ such that $\sigma \mapsto (\sigma_Q, \sigma_K)$ is an order preserving bijection from the set of all congruences on S onto the set of all congruence pairs on S.*

To give a description of congruences on a nil-extension of a completely regular semiring S, we introduce the following definition.

Definition 4.10. Let S a quasi completely regular semiring which is a nil-extension of a completely regular semiring K by a semiring Q with strongly infinite element ∞. Let $\delta \in C(Q)$ and $w \in C(K)$. Then the pair (δ, ω) is said to be a *congruence pair* if it satisfies all the conditions $(M_1), (M_2), (M_3), (M_4)$ in Definition 4.2 together with two additional conditions given by

(M_5) If $(p+e, q+f) \in \omega$ for some additive idempotents $e, f \in E^+(S)$ and any $p, q \in S$, then $(p+e, f+q) \in \omega$. Dually if $(e+p, f+q) \in \omega$, then $(e+p, q+f) \in \omega$.

(M_6) If $(p+e, f+q) \in \omega$ for some additive idempotents $e, f \in E^+(S)$ and any $p, q \in S$, then $(e+p, q+f) \in \omega$. Dually if $(e+p, q+f) \in w$, then $(p+e, f+q) \in \omega$.

Lemma 4.11. *Let S be a nil-extension of a completely regular semiring K by a semiring Q with strongly infinite element ∞. Let $(\delta, \omega) \in C(Q) \times C(K)$ be a congruence pair on S. Define a relation σ on S by: for $a, b \in S$, $a\sigma b$ if and only if*
(i) $(a, b) \in \delta$ for any $a, b \in S \setminus R$,
(ii) $(a+0_a, b+0_b) \in \omega$ for any $a, b \in R$ where $R = K \cup \{\infty\delta \setminus \{\infty\}\}$.
Then σ is a congruence on S such that $K\sigma = R$.

Proof. The proof follows similar to Theorem 4.7. □

Theorem 4.12. *Let S be a nil-extension of a completely regular semiring K by a semiring Q with strongly infinite element ∞. Then a mapping $\Gamma : C(S) \longrightarrow C(Q) \times C(K)$ such that $\sigma \mapsto (\sigma_Q, \sigma_K)$ is an order preserving bijection from the set of all congruences on S onto the set of all congruence pairs on S.*

Proof. The proof follows similar to Theorem 4.9. □

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Received March 19, 2019
Revised July 28, 2019

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