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On (m, n)-regular and intra-regular ordered semigroups

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Abstract. Let m, n be non-negative integers. A subsemigroup A of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *(m, n)-ideal* of S if $A^mSA^n \subseteq A$, and if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$. In this paper, various types of such (m, n)-ideals are described.

1. Introduction

The notion of (m, n)-ideal was introduced by S. Lajos in [4] as a generaization of left ideals, right ideals and bi-ideals and was used to a characterization of regular semigroups [5]. J. Sanborisoot and T. Changphas used in [7] (m, n)-ideals to various characterizations of (m, n)-regular ordered semigroups. T. Changphas, P. Luangchaisri and R. Mazurek studied an interval of completely prime ideals in right chain ordered semigroups [2]. Recently, Ze Gu investigated an ordered semigroup which is regular and intra-regular using various types of bi-ideals [8]. The purpose of this paper is to generalize the results of Ze Gu based on the notion of (m, n)-ideals.

An *ordered semigroup* $(S, \cdot, \leqslant)$ is a semigroup $(S, \cdot)$ together with a partially order that is compatible with the semigroup operation, that is,

$$x \leqslant y \Rightarrow zx \leqslant zy, \quad xz \leqslant yz$$

for any $x, y, z \in S$. For non-empty sets A, B of an ordered semigroup $(S, \cdot, \leqslant)$, the multiplication between A and B is defined by $AB = \{ab \mid a \in A, b \in B\}$. And the set $(A]$ is defined to be the set of all elements x of S such that $x \leqslant a$ for some a in A, that is,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

It is clear that for nonempty subsets A, B of S, (1) $A \subseteq (A]$; (2) $((A]] = (A]$; (3) $A \subseteq B \Rightarrow (A] \subseteq (B]$; (4) $(A](B] \subseteq (AB]$.

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2. Main results

Hereafter, let m and n be any two positive integers.

Definition 2.1. Let $(S, \leqslant, \cdot)$ be an ordered semigroup. A subsemigroup A of S is called an *(m,n)-ideal* of S if A satisfies the following:

(i) $A^m S A^n \subseteq A$

(ii) $(A] \subseteq A$, equivalently, if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Definition 2.2. An (m,n)-ideal A of an ordered semigroup $(S, \leqslant, \cdot)$ is said to be

- *quasi-prime* if $A_1A_2 \subseteq A \Rightarrow A_1 \subseteq A$ or $A_2 \subseteq A$,
- *strongly quasi-prime* if $(A_1A_2] \cap (A_2A_1] \subseteq A \Rightarrow A_1 \subseteq A$ or $A_2 \subseteq A$,
- *quasi-semiprime* if $(A_1)^2 \subseteq A \Rightarrow A_1 \subseteq A$

for all (m,n)-ideals A_1, A_2 of S.

It is clear that the following implications are valid:

strongly quasi-prime $\Rightarrow$ quasi-prime $\Rightarrow$ quasi-semiprime

Example 2.3. Let $S = \{0, a, b, c\}$. Define a binary operation and a partial order $\leqslant$ on S as follows:

	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

$$\leqslant := \{(0,0),(0,a),(0,b),(0,c),(a,a),(a,b),(a,c),(b,b),(c,c)\}.$$

Then $(S, \cdot, \leqslant)$ is an ordered semigroup and $P = \{0, a, b\}$ is its strongly quasi-prime $(1,1)$-ideal. Thus, P is quasi-prime and quasi-semiprime as well.

Example 2.4. Let $S = \{a, b, c, d, e\}$. Define a binary operation on S by $xy = x$ for all $x \in S$ and define a partial order $\leqslant$ on S by

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(b,c)\}.$$

Then $(S, \cdot, \leqslant)$ is an ordered semigroup and $P = \{a, b, c\}$ is its quasi-prime $(1,1)$-ideal, but it is not strongly quasi-prime.

Definition 2.5. An (m,n)-ideal A of an ordered semigroup $(S, \leqslant, \cdot)$ is said to be

- *irreducible* if $A_1 \cap A_2 = A$ implies $A_1 = A$ or $A_2 = A$,
- *strongly irreducible* if $A_1 \cap A_2 \subseteq A$ implies $A_1 \subseteq A$ or $A_2 \subseteq A$

for all (m,n)-ideals A_1, A_2 of S.

A strongly irreducible (m,n)-ideal is irreducible.

Theorem 2.6. *The intersection of quasi-semiprime (m,n)-ideals of an ordered semigroup $(S,\leqslant,\cdot)$, if it is non-empty, is a quasi-semiprime (m,n)-ideal of S.*

Theorem 2.7. *Let A be an (m,n)-ideal of an ordered semigroup $(S,\cdot,\leqslant)$. If A is strongly irreducible and quasi-semiprime, then A is strongly quasi-prime.*

Proof. Assume that A is strongly irreducible and quasi-semiprime. Let A_1 and A_2 be (m,n)-ideals of S such that

$$(A_1A_2] \cap (A_2A_1] \subseteq A.$$

Since

$$(A_1 \cap A_2)^2 \subseteq A_1A_2 \quad \text{and} \quad (A_1 \cap A_2)^2 \subseteq A_2A_1,$$

it follows that

$$(A_1 \cap A_2)^2 \subseteq A_1A_2 \cap A_2A_1 \subseteq (A_1A_2] \cap (A_2A_1] \subseteq A.$$

Now, there are two cases to consider:

Case 1: $A_1 \cap A_2 = \emptyset$. This implies $A_1 \cap A_2 \subseteq A$.

Case 2: $A_1 \cap A_2 \neq \emptyset$. Then $A_1 \cap A_2$ is an (m,n)-ideal of S. Since A is quasi-semiprime, it follows that $A_1 \cap A_2 \subseteq A$.

By the above two cases, we conclude that $A_1 \cap A_2 \subseteq A$. Since A is strongly irreducible, $A_1 \subseteq A$ or $A_2 \subseteq A$. Hence, A is strongly quasi-prime. □

Definition 2.8. (cf. ([7]) An ordered semigroup $(S,\cdot,\leqslant)$ is said to be *(m,n)-regular* if every element $a \in S$ is *(m,n)-regular*, i.e., $a \in (a^mSa^n]$.

Definition 2.9. (cf. [3]) An ordered semigroup $(S,\cdot,\leqslant)$ is said to be *intra-regular* if every element $a \in S$ is *intra-regular*, i.e., $a \in (Sa^2S]$.

Lemma 2.10. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup. Then S is both (m,n)-regular and intra-regular if and only if $(A^2] = A$ for every (m,n)-ideal A of S.*

Proof. Assume that S is both (m,n)-regular and intra-regular. Let A be an (m,n)-ideal of S. Then

$$(A^2] \subseteq (A] = A.$$

There are four cases to consider:

Case 1: $m = 1$ and $n = 1$. We can prove this case as the proof of Theorem 3.1 in [8].

Case 2: $m = 1$ and $n > 1$. Since S is $(1,n)$-regular, it follows that

$$A \subseteq (ASA^n] \quad \text{and} \quad A \subseteq (SA^2S].$$

Then

$$A \subseteq (ASA^n] \subseteq (ASA^{n-1}ASA^n] \subseteq (ASAASA^n] \subseteq (ASASA^nASA^n]$$
$$\subseteq (ASA^nASA^n] \subseteq (A^2].$$

Thus, $A = (A^2]$.

Case 3: $m > 1$ and $n = 1$. It can be proved similarly to Case 2.

Case 4: $m > 1$ and $n > 1$. Since S is (m, n)-regular and intra-regular, we obtain that

$$A \subseteq (A^mSA^n] \text{ and } A \subseteq (SA^2S].$$

Then

$$A \subseteq (A^mSA^n] \subseteq (A^mSA^{n-1}A^mSA^n] \subseteq (A^mSAASA^n]$$
$$\subseteq (A^mSA^mSA^nA^mSA^nSA^n] \subseteq (A^mSA^nA^mSA^n] \subseteq (A^2].$$

Thus, $(A^2] = A$. By these cases, we infer that $(A^2] = A$ for all (m, n)-ideals of S.

Conversely, let $a \in S$. By assumption, we obtain that

$$\left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right] = \left(\left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right]^2\right] = \left(\left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right)^2\right].$$

Continue in the same manner, we have that

$$a \in \left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right] = \left(\left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right)^{m+n+1}\right] \subseteq (a^mSa^n].$$

Thus, a is (m, n)-regular. In the same way, we also have

$$a \in \left(\left(\bigcup_{i=1}^{m+n} a^i \bigcup a^mSa^n\right)^4\right] \subseteq (Sa^2S].$$

Thus, a is intra-regular. Hence, S is both (m, n)-regular and intra-regular. $\square$

Lemma 2.11. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then the following statements are equivalent:*

(1) $(A^2] = A$ *for every (m, n)-ideal A of S;*

(2) $A_1 \cap A_2 = (A_1A_2] \cap (A_2A_1]$ *for all (m, n)-ideals A_1, A_2 of S;*

(3) *every (m, n)-ideal of S is quasi-semiprime.*

Proof. (1) $\Rightarrow$ (2): Let A_1, A_2 be (m,n)-ideal of S. Then we have two cases to consider:

Case 1: $A_1 \cap A_2 = \emptyset$. By assumption, we have that

$$(A_1A_2]^m S(A_1A_2]^n \subseteq ((A_1A_2)^m S(A_1A_2)^n] \subseteq (A_1SA_1A_2] = (A_1^m SA_1^n A_2] \subseteq (A_1A_2]$$

and $((A_1A_2]] = (A_1A_2]$. Thus, $(A_1A_2]$ is an (m,n)-ideal of S. Similarly, we obtain that $(A_2A_1]$ is (m,n)-ideal of S. Suppose $(A_1A_2] \cap (A_2A_1] \neq \emptyset$. Then $(A_1A_2] \cap (A_2A_1]$ is an (m,n)-ideal of S. This implies that

$$\begin{aligned}(A_1A_2] \cap (A_2A_1] &= \big(((A_1A_2] \cap (A_2A_1])^2\big] \subseteq ((A_1A_2)(A_2A_1)] \subseteq (A_1SA_1] \\ &= (A_1^m SA_1^n] \subseteq (A_1] = A_1.\end{aligned}$$

Similarly, we have that $(A_1A_2] \cap (A_2A_1] \subseteq A_2$. Thus,

$$(A_1A_2] \cap (A_2A_1] \subseteq A_1 \cap A_2 = \emptyset.$$

This is a contradiction. Hence, $(A_1A_2] \cap (A_2A_1] = \emptyset = A_1 \cap A_2$.

Case 2: $A_1 \cap A_2 \neq \emptyset$. Then $A_1 \cap A_2$ is an (m,n)-ideal of S. This implies that

$$\begin{aligned}A_1 \cap A_2 &= (A_1 \cap A_2) \cap (A_1 \cap A_2) = \big((A_1 \cap A_2)^2\big] \cap \big((A_1 \cap A_2)^2\big] \\ &\subseteq (A_1A_2] \cap (A_2A_1].\end{aligned}$$

Thus, $(A_1A_2] \cap (A_2A_1] \neq \emptyset$. We can prove similarly the above case that

$$(A_1A_2] \cap (A_2A_1] \subseteq A_1 \cap A_2.$$

Hence, $(A_1A_2] \cap (A_2A_1] = A_1 \cap A_2$.

(2) $\Rightarrow$ (3): Let A and A_1 be (m,n)-ideals of S such that $A_1^2 \subseteq A$. By hypothesis, we have that

$$A_1 = A_1 \cap A_1 = (A_1A_1] \cap (A_1A_1] = (A_1A_1] \subseteq (A] = A.$$

Thus, A is a quasi-semiprime (m,n)-ideal of S.

(3) $\Rightarrow$ (1): Let A be an (m,n)-ideal of S. Then $(A^2] \subseteq A$. Since

$$(A^2]^m S(A^2]^n \subseteq (A^{2m} SA^{2n}] \subseteq (A^m SA^n A] \subseteq (A^2]$$

and $((A^2]] = (A^2]$, it follows that $(A^2]$ is an (m,n)-ideal of S. This implies that $(A^2]$ is quasi-semiprime. Since $A^2 \subseteq (A^2]$, we have that $A \subseteq (A^2]$. Hence, $(A^2] = A$. $\square$

Consequently,

Corollary 2.12. *Let $(S, \cdot, \leqslant)$ be an (m,n)-regular and intra-regular ordered semigroup. Then an (m,n)-ideal A of S is strongly irreducible if and only if A is strongly quasi-prime.*

Lemma 2.13. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then the following statements are equivalent:*

(1) *The set of all (m, n)-ideals of S is totally ordered under inclusion.*

(2) *Every (m, n)-ideal of S is strongly irreducible and $A_1 \cap A_2 \neq \emptyset$ for all (m, n)-ideals A_1, A_2 of S.*

(3) *Every (m, n)-ideal of S is irreducible and $A_1 \cap A_2 \neq \emptyset$ for all (m, n)-ideals A_1, A_2 of S.*

Proof. (1) $\Rightarrow$ (2): Assume that (1) holds. Then we have immediately that the finite intersection of (m, n)-ideals of S is not empty and so, it is an (m, n)-ideal of S. Let A, A_1, A_2 be (m, n)-ideals of S such that $A_1 \cap A_2 \subseteq A$. By assumption, we can suppose that $A_1 \subseteq A_2$ and then $A_1 = A_1 \cap A_2 \subseteq A$. Thus, A is a strongly irreducible (m, n)-ideal of S.

(2) $\Rightarrow$ (3): This direction is obvious.

(3) $\Rightarrow$ (1): Assume that (3) holds. Let A_1, A_2 be (m, n)-ideals of S. Since $A_1 \cap A_2 \neq \emptyset$, it follows that $A_1 \cap A_2$ is an (m, n)-ideal of S. By hypothesis, we have that $A_1 = A_1 \cap A_2$ or $A_2 = A_1 \cap A_2$. Then $A_1 = A_1 \cap A_2 \subseteq A_2$ or $A_2 = A_1 \cap A_2 \subseteq A_1$. □

Theorem 2.14. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then every (m, n)-ideal of S is strongly quasi-prime and $A_1 \cap A_2 \neq \emptyset$ for all (m, n)-ideals A_1, A_2 of S if and only if S is (m, n)-regular, intra-regular and the set of all (m, n)-ideal of S is totally ordered under inclusion.*

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