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Categorical equivalences in the theory of sharp transitivity

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Abstract. There are well-known correspondences between loops and regular permutation sets; neardomains and sharply 2-transitive groups; and KT-fields and sharply 3-transitive groups. Initially, these correspondences only considered isomorphisms. However, the first two correspondences were realized as more general categorical equivalences. In this note, we offer a simplified development of these equivalences and extend the results to a categorical equivalence between KT-fields and sharply 3-transitive groups. We then show how these three equivalences are related to one another via a diagram of functors.

1. Introduction and overview

Given a loop L, its set of left translations $T_1L = \{\lambda_a : x \mapsto ax \mid a \in L\}$ acts *regularly* on L, meaning for every $x, y \in L$, there is a unique $\lambda_a \in T_1L$ such that $\lambda_a(x) = y$. It turns out, not only is this construction *functorial*, it forms an equivalence between the category of loops (denoted **Loop**) and the category of regular permutation sets (denoted **RPS**) [1].

There is a related correspondence between neardomains and sharply 2-transitive groups. Here, a neardomain essentially consists of $(F, +, \cdot)$ where $(F, +)$ is a loop and $(F \setminus \{0\}, \cdot)$ is a group, while a sharply 2-transitive group is a group action that is regular on pairs of distinct points. Given a neardomain, its group of affine transformations $T_2F = \{x \mapsto a + bx \mid b \neq 0\}$ acts sharply 2-transitively on F (cf. [2] (6.1)). In [1], this correspondence was proven to be an equivalence between the category of neardomains (denoted **nDomain**) and a category of sharply 2-transitive groups (denoted **s2tGroup**). Proving that T_2 is a categorical equivalence hinges on the definition of **s2tGroup**; the crucial realization in [1] is that not every conceivable morphism of sharply 2-transitive groups corresponds to a neardomain morphism, so we must pare down our morphisms of sharply 2-transitive groups accordingly.

Finally, there is another correspondence between KT-fields and sharply 3-transitive groups, where KT-fields are neardomains with a distinguished invo-

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lution (thought of as a generalized *inversion*) and sharply 3-transitive groups are groups that act regularly on triples of distinct points. As shown in [2] (11.1), given a KT-field F, one can form the group of generalized fractional affine transformations T_3F which acts sharply 3-transitively on F. The name "generalized fractional affine transformations" comes from the fact that, if F is a field, then $T_3F = \left\{x \mapsto \frac{a+bx}{c+dx} \mid ad - bc \neq 0\right\}$. In general, T_3F is generated as a subgroup of the permutations on F by T_2F and the distinguished involution of F. Our main result is that, not only is T_3 functorial, the category of KT-fields (denoted **KTfield**) is equivalent to a category of sharply 3-transitive groups (denoted **s3tGroup**); cf. Theorem 3.13. Moreover the three equivalences T_1, T_2, and T_3 enjoy a particularly nice interdependence.

As is the case for T_2, proving that T_3 is an equivalence of categories largely depends on the morphisms we allow in **s3tGroup**. In particular, the construction of T_3F demands that morphisms in **s3tGroup** induce morphisms in **s2tGroup** on stabilizers. Once this is done, however, the argument proceeds swiftly. This is substantially a consequence of the general scheme in which T_1, T_2, and T_3 fit. As we will see:

- Given a neardomain F, the functor T_2 is essentially two applications of T_1 – once to $(F \setminus \{0\}, \cdot)$ and once to $(F, +)$.
- Given a KT-field F with distinguished involution σ, the functor T_3 is essentially T_2 after forgetting and then remembering σ.

So, overall, our categorical equivalences T_2 and T_3 are built in a very tangible way from the comparatively simple equivalence $T_1 \colon \mathbf{Loop} \xrightarrow{\simeq} \mathbf{RPS}$; this is formalized in Theorem 3.14.

Organization. Section 2 focuses on preliminary definitions and basic results from category theory, non-associative algebra, and sharply multiply transitive actions. In particular, we have the lemma:

Lemma 1.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and suppose there are functors* $F \colon \mathbf{C} \to \mathbf{D}$ *and* $G \colon \mathbf{D} \to \mathbf{C}$ *such that* $GF = id_{\mathbf{C}}$. *If* G *is faithful, then* F *is an equivalence of categories with inverse* G.

We also have the lemma:

Lemma 1.2. *Let* $\mathbf{C}$ *be one of the categories* **RPS**, **s2tGroup**, *or* **s3tGroup**. *Then two morphisms* (f, Φ) *and* (g, Ψ) *are equal in* $\mathbf{C}$ *if and only if* $\Phi = \Psi$.

These two lemmas combine in Section 3 to shorten the proofs of Theorems 3.4, 3.9, and 3.13 regarding the assortment of categorical equivalences mentioned above. In particular, in Section 3 we prove:

Theorem 1.3. *There is an equivalence of categories* $T_3\colon \mathbf{KTfield} \xrightarrow{\simeq} \mathbf{s3tGroup}$.

We also concisely exhibit the relationship between T_1, T_2, and T_3 in Section 3 via the following theorem:

Theorem 1.4. *There is a commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{KTfield} & \xrightarrow[T_3]{\simeq} & \mathbf{s3tGroup} \\ \downarrow & & \downarrow \\ \mathbf{nDomain} & \xrightarrow[T_2]{\simeq} & \mathbf{s2tGroup} \\ \downarrow & & \downarrow \\ \mathbf{Loop} & \xrightarrow[T_1]{\simeq} & \mathbf{RPS}. \end{array}$$

2. Preliminaries

This section is dedicated to some definitions and basic results for reference in the development to follow.

2.1. Category theory

We assume familiarity with the notions of category and functor; the standard reference is [4]. We recall some basic definitions here, namely that of *natural equivalence* and *categorical equivalence*. We also prove Lemma 2.1 that helps us provide succinct proofs of our main theorems in Section 3.

Natural and categorical equivalences. Given two functors $F, G\colon \mathbf{C} \to \mathbf{D}$, a **natural transformation** $\eta : F \to G$ is an assignment for each object X of $\mathbf{C}$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathbf{D}$ such that, for every morphism $f\colon X \to Y$ in $\mathbf{C}$, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y). \end{array}$$

A natural transformation in which each η_X is an equivalence is called a **natural equivalence**.

We are primarily interested in exhibiting pairs of categories as equivalent, in a sense we will make precise immediately: A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an **equivalence of categories** if there is a functor $G\colon \mathbf{D} \to \mathbf{C}$ and natural equivalences $FG \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}$

and $GF \xrightarrow{\cong} \mathrm{id}_{\mathbf{C}}$. In this case, we write $F\colon \mathbf{C} \xrightarrow{\simeq} \mathbf{D}$ or simply $\mathbf{C} \simeq \mathbf{D}$, and we call G the **inverse** of F. Assuming the Axiom of Choice, $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if F is full, faithful, and essentially surjective ([4] Theorem IV.4.1).

The faithful retract lemma. We will call a category $\mathbf{C}$ a **retract** of a category $\mathbf{D}$ provided there are functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that $GF = \mathrm{id}_{\mathbf{C}}$. In this case, the functor G is called a **retraction**. If, additionally, there is a natural equivalence $FG \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}$, we call $\mathbf{C}$ a **deformation retract** of $\mathbf{D}$. We recognize deformation retract as a stronger notion of categorical equivalence. This next lemma is an adaptation of a familiar result from category theory that if a retraction has a left inverse, then it has a two sided inverse.

Lemma 2.1 (Faithful Retract Lemma). *Suppose $\mathbf{C}$ is a retract of $\mathbf{D}$ via $F\colon \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$. If G is faithful, then there exists a natural transformation $FG \xrightarrow{\cong} id_{\mathbf{D}}$. In particular, F is an equivalence of categories with inverse G.*

Proof. Suppose G is a retraction. We claim G is essentially surjective and full. To see this, let X be an object in $\mathbf{C}$. Then $GF(X) = X$ since $GF = \mathrm{id}_{\mathbf{C}}$, thus G is (essentially) surjective on objects. Now suppose $f : G(X) \to G(Y)$ is a morphism in $\mathbf{C}$ where X and Y are objects of $\mathbf{D}$. Then, $GFf = f\colon G(X) \to G(Y)$, hence G is full.

So, if G is also faithful, then G is an equivalence of categories. This means there exists a functor $H\colon \mathbf{C} \to \mathbf{D}$ and a natural equivalence $HG \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}$. Thus, there is a natural equivalence $HGF \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}F$, i.e. a natural equivalence $H \xrightarrow{\cong} F$. Therefore, we induce a natural equivalence $FG \xrightarrow{\cong} HG \xrightarrow{\cong} \mathrm{id}_{\mathbf{D}}$ as claimed. □

2.2. Loops, neardomains, and KT-fields

In this section, we briskly review some basic definitions from non-associative algebra and define categories of loops, neardomains, and KT-fields.

Loops. A **loop** is a nonempty set L with a binary operation $(a, b) \mapsto ab$ such that:

1. The operation has a two-sided identity element $e \in L$;
2. For every $a, b \in L$, there exist unique $x, y \in L$ such that $ax = b$ and $ya = b$.

A **morphism of loops** is a function $f\colon L \to L'$ that preserves the loop operation. The category whose objects are loops and whose arrows are loop homomorphisms will be denoted **Loop**.

Neardomains. A set F with operations $+$ and $\cdot$ is called a **neardomain** if:

1. F is a loop under $+$ with identity 0;
2. For all $a, b \in F$: $a + b = 0$ implies $b + a = 0$;
3. $F - \{0\}$ is a group under $\cdot$ with identity 1;
4. For all $a \in F$: $0 \cdot a = 0$;
5. For all $a, b, c \in F$: $a \cdot (b + c) = a \cdot b + a \cdot c$;
6. For all $a, b \in F$, there exists $d_{a,b} \in F - \{0\}$ such that, for all $x \in F$, $a + (b + x) = (a + b) + d_{a,b} \cdot x$.

A **morphism of neardomains** $f : F \to F'$ is a function that preserves both operations. The category of neardomains and neardomain morphisms is denoted **nDomain**. Note: F is a **nearfield** if and only if all of the $d_{a,b} = 1$ (i.e., if and only if F is a group under $+$). Every finite neardomain is a nearfield. Whether there exists a neardomain that is not a nearfield was a long-standing open question, but the construction provided in [5] confirms the existence of proper neardomains.

It can be shown (cf. [1] Property 3.2) that all neardomain morphisms are injective. Consequently, if there exists a neardomain morphism $f : F \to F'$, then char $F = 2$ if and only if char $F' = 2$. This fact turns out to be essential for defining the appropriate category of sharply 2-transitive groups in Section 3.

KT-fields. A **KT-field** is quadruple $(F, +, \cdot, \sigma)$ such that

1. $(F, +, \cdot)$ is a neardomain;
2. $\sigma : F \to F$ is an involutary automorphism of $(F \setminus \{0\}, \cdot)$ satisfying
$$\sigma(1 + \sigma(x)) = 1 - \sigma(1 + x) \ ,$$
for all $x \in F \setminus \{0, -1\}$.

The characteristic of F as a KT-field is defined to be the characteristic of the neardomain F. A **morphism** of KT-fields $(F, +, \cdot, \sigma)$ and $(F', +, \cdot, \sigma')$ is a neardomain morphism $f : F \to F'$ such that $\sigma' \circ f = f \circ \sigma$ on $F \setminus \{0\}$. KT-fields and KT-field morphisms constitute a category denoted **KTfield**. We note that if F is a field, then $\sigma(x) = x^{-1}$ ([2] Theorem 13.2).

2.3. Sharply multiply transitive actions

We now review some basic definitions and results from the theory of sharply transitive actions. We then define categories of regular permutations sets, sharply 2-transitive groups, and sharply 3-transitive groups, utilizing the crucial insight from [1] regarding which morphisms to allow between sharply 2-transitive groups. The section ends with a proof of Lemma 2.4 that allows us to more easily identify when two morphisms in these categories are equal.

Regular permutation sets. A **regular permutation set** is a triple $(M, \Omega, *)$ where Ω is a set, $* \in \Omega$ is a chosen base point, and M is a subset of the set of all permutations on Ω such that:

1. The identity map $\mathrm{id}_\Omega : \alpha \mapsto \alpha$ is in M;
2. M **acts regularly** on Ω: for every $\alpha, \beta \in \Omega$, there is a unique $m \in M$ such that $m(\alpha) = \beta$.

A **morphism of regular permutation sets** $(M, \Omega, *)$ and $(N, \Sigma, *)$ is a pair of functions (f, Φ) where $f : M \to N$ and $\Phi : \Omega \to \Sigma$ such that $\Phi(*) = *$, and the following diagram commutes:

$$\begin{CD} M \times \Omega @>{\mathrm{ev}}>> \Omega \\ @V{f \times \Phi}VV @VV{\Phi}V \\ N \times \Sigma @>>{\mathrm{ev}}> \Sigma, \end{CD}$$

where the horizontal maps are the evaluation maps. Regular permutation sets and morphisms of regular permutation sets (with composition defined componentwise) assemble into a category which we denote **RPS**.

An action of a group G on a set Ω is said to be **sharply transitive** provided for every $\alpha, \beta \in \Omega$, there exists a unique $g \in G$ such that $g\alpha = \beta$. A sharply transitive group action of a group G on a set Ω corresponds (via the standard identification of $G \leq \mathrm{SYM}\Omega$) to a regular permutation set $(G, \Omega, *)$ for a chosen base point $* \in \Omega$. In this way, we can define the category **s1tGroup** as the full subcategory of **RPS** whose objects are sharply transitive group actions.

Sharply 2-transitive groups. Suppose a group G acts on a set Ω. The action is **sharply 2-transitive** provided: for every $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Omega \times \Omega$ where $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, there is a unique $g \in G$ with $g\alpha_1 = \beta_1$ and $g\alpha_2 = \beta_2$.

It is tempting at this point to try to define a category of sharply 2-transitive groups analogous to the category **RPS**. To start, we might say the objects of this category are of the form $(G, \Omega, 0, 1)$ where G acts sharply 2-transitively on Ω and $0, 1 \in \Omega$ are distinct base points. Then, we could define a morphism to be a pair of functions $(f, \Phi) : (G, \Omega, 0, 1) \to (H, \Sigma, 0, 1)$ with $f : G \to H$ a group homomorphism, $\Phi : \Omega \to \Sigma$ a function sending $0 \mapsto 0$ and $1 \mapsto 1$, such that the following diagram commutes:

$$\begin{CD} G \times \Omega @>>> \Omega \\ @V{f \times \Phi}VV @VV{\Phi}V \\ H \times \Sigma @>>> \Sigma. \end{CD}$$

This defines a perfectly reasonable category. However, as shown in [1], this category has "too many" morphisms to be equivalent to the category of neardomains!

So, we must eliminate morphisms between sharply 2-transitive groups that do not correspond to neardomain morphisms. Recall: all neardomain morphisms are injective and, consequently, the existence of a neardomain morphism $f: F \to F'$ implies char $F = 2$ if and only if char $F' = 2$. The characterestic of a neardomain turns out to be related to what is called the *type* of its corresponding sharply 2-transitive group, which we explore forthwith.

Suppose a group G acting sharply 2-transitively on a set Ω and consider the involutions of G, $\mathrm{INV}G = \{g \in G \mid g \neq \mathrm{id}_\Omega,\ g^2 = \mathrm{id}_\Omega\}$. Note: $\mathrm{INV}G$ is never empty if Ω has at least two elements. It can be shown that exactly one of the following conditions holds: (1) every $g \in \mathrm{INV}G$ has a unique fixed point; or (2) no $g \in \mathrm{INV}G$ has a fixed point ([2] Section 2).

If each $g \in \mathrm{INV}G$ has a unique fixed point we say G **is of type** 1, and if no $g \in \mathrm{INV}G$ has a fixed point, we say G **is of type** 0. We can now define the appropriate category of sharply 2-transitive groups as follows: The category **s2tGroup** has objects $(G, \Omega, 0, 1)$ where G is a group that acts sharply 2-transitively on a set Ω, and $0, 1 \in \Omega$ are distinct base points. The morphisms in **s2tGroup** are pairs of maps $(f, \Phi)\colon (G, \Omega, 0, 1) \to (H, \Sigma, 0, 1)$ where

1. G and H are of the same type;
2. $f\colon G \to H$ is a group homomorphism;
3. $\Phi\colon \Omega \to \Sigma$ is injective, mapping $0 \mapsto 0$ and $1 \mapsto 1$; and
4. the following diagram commutes:

$$\begin{array}{ccc} G \times \Omega & \longrightarrow & \Omega \\ {\scriptstyle f\times\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ H \times \Sigma & \longrightarrow & \Sigma\,, \end{array}$$

   where the horizontal maps are evaluation.

Let $(G, \Omega, 0, 1)$ be a sharply 2-transitive group on a set Ω. In the case G has type 1, call ι the unique (by [2] (3.1)) involution fixing the base point $0 \in \Omega$. We define a subset $\mathrm{A}G \subseteq G$ by $\mathrm{A}G = \mathrm{INV}G \circ \iota$ if G has type 1, and $\mathrm{A}G = \mathrm{INV}G \cup \{\mathrm{id}_\Omega\}$ if G has type 0. It is shown in [2] that $(\mathrm{A}G, \Omega, 0)$ is a regular permutation set. Moreover, this construction is functorial:

Proposition 2.2. *There is a functor* $\mathrm{A}\colon$ **s2tGroup** $\to$ **RPS**.

Proof. For a sharply 2-transitive group $(G, \Omega, 0, 1)$ define $\mathrm{A}(G, \Omega, 0, 1) = (\mathrm{A}G, \Omega, 0)$ as above. The fact that $(\mathrm{A}G, \Omega, 0)$ is a regular permutation set is shown in [2] Theorem 3.3. To show that A is functorial, let $(f, \Phi): (G, \Omega, 0, 1) \to (H, \Sigma, 0, 1)$ be a morphism in **s2tGroup**. By Lemma 3.8 in [1], we have $f(\mathrm{A}G) \subseteq \mathrm{A}H$. Thus, we define $\mathrm{A}f$ as the restriction $f|_{\mathrm{A}G}\colon \mathrm{A}G \to \mathrm{A}H$. We are left to show that

$(\mathrm{A}f, \Phi) : (\mathrm{A}G, \Omega, 0) \to (\mathrm{A}H, \Sigma, 0)$ is a morphism in **RPS**, as verifying that A is functorial is then routine.

First, we note that $\Phi(0) = 0$ by assumption. Now, consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{A}G \times \Omega & \hookrightarrow & G \times \Omega & \longrightarrow & \Omega \\ \downarrow{\scriptstyle \mathrm{A}f\times\Phi} & & \downarrow{\scriptstyle f\times\Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathrm{A}H \times \Sigma & \hookrightarrow & H \times \Sigma & \longrightarrow & \Sigma. \end{array}$$

The left square commutes by definition of $\mathrm{A}f$, while the right square commutes since (f, Φ) is a morphism in **s2tGroup**. Thus, the total rectangle commutes, verifying that $(\mathrm{A}f, \Phi)$ is a morphism in **RPS**. □

Sharply 3-transitive groups. A group action of G on a set Ω is said to be **sharply 3-transitive** if, for all 3-tuples of distinct elements of Ω, $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$, there is a unique $g \in G$ such that $g\alpha_i = \beta_i$ for $i = 1, 2, 3$. We can recognize sharply 3-transitive groups in terms of sharply 2-transitive groups as follows:

Proposition 2.3. *A group G acts sharply 3-transitively on a set Ω if and only if for every $\alpha \in \Omega$ the stabilizer G_α acts sharply 2-transitively on $\Omega \setminus \{\alpha\}$.*

Proof. See [2] (1.1)(a). □

As with the category of sharply 2-transtive groups, we must be careful to avoid excess morphisms in our category of sharply 3-transitive groups. Let (G, Ω) be a sharply 3-transitive group. As shown in [2] Section 2, each stabilizer $(G_a, \Omega \setminus \{a\})$ is a sharply 2-transitive group of the same type. This allows us to define the appropriate category of sharply 3-transitive groups, as follows: The category **s3tGroup** has objects $(G, \Omega, 0, 1, \infty)$ where G is a group that acts sharply 3-transitively on a set Ω, and $0, 1, \infty \in \Omega$ are distinct base points. The morphisms in **s3tGroup** are pairs of maps $(f, \Phi) \colon (G, \Omega, 0, 1, \infty) \to (H, \Sigma, 0, 1, \infty)$ where:

1. The stabilizers G_∞ and H_∞ have the same type as sharply 2-transitive groups;
2. $f \colon G \to H$ is a group homomorphism;
3. $\Phi \colon \Omega \to \Sigma$ is injective, mapping $0 \mapsto 0$, $1 \mapsto 1$, and $\infty \mapsto \infty$; and
4. the following diagram commutes:

$$\begin{array}{ccc} G \times \Omega & \longrightarrow & \Omega \\ \downarrow{\scriptstyle f\times\Phi} & & \downarrow{\scriptstyle \Phi} \\ H \times \Sigma & \longrightarrow & \Sigma\,, \end{array}$$

where the horizontal maps are evaluation.

The sharp morphism lemma. We now prove a lemma that allows us to more easily recognize when two morphisms in **RPS**, **s2tGroup**, and **s3tGroup** are equal. Together with Lemma 2.1, this result greatly expedites the proofs of Theorems 3.4, 3.9, and 3.13.

Lemma 2.4 (Sharp Morphism Lemma). *Let* $\mathbf{C}$ *be one of the categories* **RPS**, **s2tGroup**, *or* **s3tGroup**. *Then two morphisms* (f, Φ) *and* (g, Ψ) *are equal in* $\mathbf{C}$ *if and only if* $\Phi = \Psi$.

Proof. We will prove the nontrivial assertion for $\mathbf{C} = \mathbf{s3tGroup}$. The proofs for the other options of $\mathbf{C}$ are similar.

Let $(f, \Phi), (g, \Psi) \colon (G, \Omega, 0, 1, \infty) \to (H, \Sigma, 0, 1, \infty)$ be two morphisms in **s3tGroup**, and suppose $\Phi = \Psi$. Now, let $x \in G$ and $\alpha \in \Omega$. Since (f, Φ) and (g, Φ) are morphisms in **s3tGroup**, we have $f(x)\Phi(\alpha) = \Phi(x(\alpha)) = g(x)\Phi(\alpha)$. For $\alpha \in \{0, 1, \infty\}$, this implies we must have $f(x)(0) = g(x)(0)$, $f(x)(1) = g(x)(1)$, and $f(x)(\infty) = g(x)(\infty)$. Now, each of $g(x)(0)$, $g(x)(1)$, and $g(x)(\infty)$ must be distinct since the base points are distinct. By the sharp 3-transitivity of H on Σ, $g(x)$ is the unique element of H such that $0 \mapsto g(x)(0)$, $1 \mapsto g(x)(1)$, and $\infty \mapsto g(x)(\infty)$. Thus, we must have $f(x) = g(x)$, hence $f = g$. □

3. The categorical equivalences

In this section, we show that there are categorical equivalences (1) $\mathbf{RPS} \simeq \mathbf{Loop}$ (Theorem 3.4); (2) $\mathbf{s2tGroup} \simeq \mathbf{nDomain}$ (Theorem 3.9); and (3) $\mathbf{s3tGroup} \simeq \mathbf{KTfield}$ (Theorem 3.13). The first two equivalences were first proved in [1]. We review their development for completeness (especially since we understand the third equivalence in terms of the first two), using Lemmas 2.1 and 2.4 to provide alternate, more concise proofs. We then exhibit the close relationship between these equivalences with a diagram of functors in Theorem 3.14.

3.1. Regular permutation sets and loops

Given a loop L, constructing an object of **RPS** is relatively straightforward. Denote $T_1L = \{\lambda_a : x \mapsto ax \mid x, a \in L\}$ the set of left translations of L. Then (T_1L, L, e) is a regular permutation set ([1] Property 2.5). As we see in the next proposition, this construction is functorial.

Proposition 3.1. *There is a functor* $T_1 \colon \mathbf{Loop} \to \mathbf{RPS}$.

Proof. For L a loop, define T_1L as above. For a loop homomorphsim $f \colon L \to L'$, if we define $T_1f \colon \lambda_a \mapsto \lambda_{f(a)}$, it can be shown that T_1 is a functor. □

Constructing a loop out of a regular permutation set is a little more subtle. Let $(M, \Omega, *)$ be a regular permutation set. Ultimately, we would like to find a loop structure on Ω. We do so by first building a loop out of M (which, being a set of permutations, comes with a little more structure than Ω), and importing the structure on M to Ω.

While composition of functions is the obvious operation on $\text{SYM}\Omega$, there is no guarantee that the subset $M \subseteq \text{SYM}\Omega$ is closed under this operation. However, since the action of M on Ω is regular, the map $\mu\colon M \to \Omega$ defined by $m \mapsto m(*)$ is a bijection. We define the operation $\otimes_* : M \times M \to M$ so that the following diagram commutes:

$$\begin{array}{ccccc} M \times M & & \xhookrightarrow{i \times i} & & \text{SYM}\Omega \times \text{SYM}\Omega \\ \Big\downarrow{\scriptstyle \otimes_*} & & & & \Big\downarrow{\scriptstyle \circ} \\ M & \xleftarrow[\mu^{-1}]{\cong} & \Omega & \xleftarrow[\text{ev}(-,*)]{} & \text{SYM}\Omega. \end{array}$$

Explicitly, we have $\otimes_* : (m, n) \mapsto \mu^{-1}(m \circ n(*))$. It can be shown that M is a loop under $\otimes_*$ with identity id_Ω. Furthermore, if M is a subgroup of $\text{SYM}\Omega$, then $(M, \otimes_*) = (M, \circ)$ (cf. [1] Property 2.1).

We use μ to define an operation $\cdot_*$ on Ω by the following commutative diagram:

$$\begin{array}{ccc} \Omega \times \Omega & \xrightarrow[\cong]{\mu^{-1} \times \mu^{-1}} & M \times M \\ \Big\downarrow{\scriptstyle \cdot_*} & & \Big\downarrow{\scriptstyle \otimes_*} \\ \Omega & \xleftarrow[\mu]{\cong} & M. \end{array}$$

Explicitly, we have $\cdot_*\colon (\alpha, \beta) \mapsto \mu^{-1}(\alpha)(\beta)$.

Proposition 3.2. *For Ω and $\cdot_*$ as above, Ω is a loop under $\cdot_*$ with identity $*$, and $\mu : M \to \Omega$ is a loop isomorphism. In particular, if M is a subgroup of $\text{SYM}\Omega$, then $(\Omega, \cdot_*)$ is a group. Furthermore, if $(f, \Phi)\colon (M, \Omega, *) \to (N, \Sigma, *)$ is a morphism in* **RPS**, *then $\Phi\colon \Omega \to \Sigma$ is a homomorphism of loops.*

Proof. See [1] Property 2.2 and Corollary 2.4. □

This lets us define two functors **RPS** $\to$ **Loop**:

- $P_\otimes\colon$ **RPS** $\to$ **Loop** sends $(M, \Omega, *) \mapsto (M, \otimes_*)$ and $(f, \Phi) \mapsto f$.
- $P\colon$ **RPS** $\to$ **Loop** sends $(M, \Omega, *) \mapsto (\Omega, \cdot_*)$ and $(f, \Phi) \mapsto \Phi$.

These functors are naturally equivalent, as shown in the following proposition:

Proposition 3.3. *There is a natural equivalence $\mu\colon P_\otimes \xrightarrow{\cong} P$.*

Proof. For any regular permutation set $(M, \Omega, *)$, define $\mu_M : M \to \Omega$ as the composite $M \xrightarrow{(\text{id}, *)} M \times \Omega \xrightarrow{\text{ev}} \Omega$. As we have seen in Proposition 3.2, $\mu_M : (M, \otimes_*) \xrightarrow{\cong} (\Omega, \cdot_*)$ is a loop isomorphism.

We are left to show that this construction is natural. Consider a morphism of regular permutation sets $(f, \Phi) : (M, \Omega, *) \to (N, \Sigma, *)$. By [1] Property 2.3, we have $\Phi \circ \mu_M = \mu_N \circ f$ witnessing the naturality of μ. □

Finally, we have the following theorem:

Theorem 3.4 (Cara–Kieboom–Vervloet [1]). *The functors $P\colon \mathbf{RPS} \rightleftarrows \mathbf{Loop} : T_1$ constitute an equivalence of categories. Moreover,* **Loop** *is a deformation retract of* **RPS**.

Proof. See [1] Theorem 2.6 for a proof that $PT_1 = \mathrm{id}_{\mathbf{Loop}}$. The fact that P is faithful follows from Lemma 2.4, and the full result then follows from Lemma 2.1. □

In light of Proposition 3.2, we have the following corollary:

Corollary 3.5. *The equivalence of categories $P\colon \mathbf{RPS} \rightleftarrows \mathbf{Loop} : T_1$ restricts to an equivalence of categories* **s1tGroup** $\rightleftarrows$ **Group**. *Thus, we have the following commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{Group} & \xrightarrow{\simeq} & \mathbf{s1tGroup} \\ \downarrow & & \downarrow \\ \mathbf{Loop} & \xrightarrow[T_1]{\simeq} & \mathbf{RPS}, \end{array}$$

where the vertical functors are inclusions.

3.2. Sharply 2-transitive groups and neardomains

Given a neardomain F, the set of affine transformations is

$$T_2F = \{\langle a, b\rangle : x \mapsto a + bx \mid a, x \in F, b \in F \setminus \{0\}\}.$$

In [2], it is shown that T_2F is a subgroup of $\mathrm{SYM}F$ that acts sharply 2-transitively on F. This construction turns out to be functorial. Compare this with Proposition 3.1.

Proposition 3.6. *There is a functor $T_2\colon$* **nDomain** $\to$ **s2tGroup**.

Proof. See [2] (6.1) for a proof that $(T_2F, F, 0, 1)$ is a sharply 2-transitive group. To show that T_2 is functorial, consider a morphism of near domains $f\colon F \to F'$. Now define $T_2 f\colon T_2F \to T_2F'$ by $\langle a, b\rangle \mapsto \langle f(a), f(b)\rangle$. Then $(T_2 f, f)$ is verified to be a morphism in **s2tGroup** in the proof of Theorem 4.1 of [1], and this assignment is easily seen to be functorial. □

Given a neardomain F, the next proposition lets us identify the composite $\mathrm{A}T_2F$ as the set of left translations of the loop $(F, +)$, i.e. $\mathrm{A}T_2F = T_1(F, +)$.

Proposition 3.7. *Let F be a neardomain. Then $\mathrm{A}T_2F = \{\langle a, 1\rangle \mid a \in F\} = T_1(F, +)$. Thus, there is a commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{nDomain} & \xrightarrow{T_2} & \mathbf{s2tGroup} \\ {\scriptstyle (+)}\downarrow & & \downarrow{\scriptstyle \mathrm{A}} \\ \mathbf{Loop} & \xrightarrow[T_1]{} & \mathbf{RPS}, \end{array}$$

where $(+)\colon \mathbf{nDomain} \to \mathbf{Loop}$ is defined by $(F, +, \cdot) \mapsto (F, +)$ and $f \mapsto f$.

Proof. See [2] (6.5). □

Now consider a sharply 2-transitive group action of a group G on a set Ω (with at least two elements). For two distinct base points $0, 1 \in \Omega$, consider the regular permutation sets $(\mathrm{A}G, \Omega, 0)$ and $(G_0, \Omega \setminus \{0\}, 1)$ (by [2] (1.1)(a)). The functor $P\colon \mathbf{RPS} \to \mathbf{Loop}$ can be used to define two loops, namely $(\Omega, +_0) = P(\mathrm{A}G, \Omega, 0)$ and $(\Omega\setminus\{0\}, \cdot_1) = P(G_0, \Omega\setminus\{0\}, 1)$. Note that, since G_0 is a group, so is $(\Omega\setminus\{0\}, \cdot_1)$ by Proposition 3.2. We can further define $\alpha \cdot_1 \beta = 0$ if either $\alpha = 0$ or $\beta = 0$ to extend the operation $\cdot_1$ to all of Ω. These two operations constitute a neardomain structure on Ω, and with our carefully curated morphisms in **s2tGroup**, we have the following result:

Proposition 3.8. *For a sharply 2-transitive group $(G, \Omega, 0, 1)$, the set Ω equipped with addition $+_0$ and multiplication $\cdot_1$ forms a neardomain $F = (\Omega, +_0, \cdot_1)$. Furthermore, G is of type 0 if and only if $\operatorname{char} F = 2$, and if (f, Φ) is a morphism in the category of sharply 2-transitive groups, then Φ is a morphism of neardomains.*

Proof. See [2] (6.2), [3] Section 7.10, and [1] Property 3.9. □

This means we can define a functor $Q\colon \mathbf{s2tGroup} \to \mathbf{nDomain}$ by $(G, \Omega, 0, 1) \mapsto (\Omega, +_0, \cdot_1)$ and $(f, \Phi) \mapsto \Phi$. We now have the following theorem:

Theorem 3.9 (Cara–Kieboom–Vervloet [1])**.** *The functors $Q\colon \mathbf{s2tGroup} \rightleftarrows \mathbf{nDomain} : T_2$ constitute an equivalence of categories. Moreover,* **nDomain** *is a deformation retract of* **s2tGroup**.

Proof. See [1] Theorem 4.1 for a proof that $QT_2 = \mathrm{id}_{\mathbf{nDomain}}$. The fact that Q is faithful follows from Lemma 2.4, and the full result then follows from Lemma 2.1. □

Given a sharply 2-transitive group $(G, \Omega, 0, 1)$, the associated neardomain $(\Omega, +_0, \cdot_1)$ is a nearfield if and only if $\mathrm{A}G$ is a subgroup of G (see the proof of Theorem 4.3 in [1]). Call $\mathbf{s2tGroup}_{\mathrm{A}}$ the full subcategory of **s2tGroup** whose objects $(G, \Omega, 0, 1)$ have $\mathrm{A}G$ a subgroup of G, and call **nField** the full subcategory of **nDomain** whose objects are nearfields. We have the following corollary:

Corollary 3.10 (Cara–Kieboom–Vervloet [1])**.** *The functors* $Q\colon \mathbf{s2tGroup} \rightleftarrows \mathbf{nDomain} : T_2$ *restrict to an equivalence of cateogries* $\mathbf{s2tGroup}_{\mathrm{A}} \rightleftarrows \mathbf{nField}$. *Thus, we have a commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{nField} & \xrightarrow{\simeq} & \mathbf{s2tGroup}_{\mathrm{A}} \\ \downarrow & & \downarrow \\ \mathbf{nDomain} & \xrightarrow[T_2]{\simeq} & \mathbf{s2tGroup}, \end{array}$$

where the vertical functors are inclusions.

Proof. See [1] Theorem 4.3. □

3.3. Sharply 3-transitive groups and KT-fields

Conisder a KT-field F. Let $\infty \notin F$, and call $\overline{F} = F \cup \{\infty\}$. Extend $+$ and $\cdot$ to $\overline{F}$ by $a + \infty = \infty$ and $b \cdot \infty = \infty$ for all $a \in F$ and $b \in F \setminus \{0\}$. Extend σ to an involution on $\overline{F}$ by $\sigma(0) = \infty$ and $\sigma(\infty) = 0$.

For $a \in F$ and $b \in F \setminus \{0\}$, extend $\langle a, b\rangle : F \to F$ to a permutation of $\overline{F}$ by asserting $\infty \mapsto \infty$. Then, by Proposition 3.6, $T_2F = \{\langle a, b\rangle \mid a \in F, b \in F \setminus \{0\}\}$ is a group that acts sharply 2-transitively on $\overline{F} \setminus \{\infty\} = F$.

Now, by [2] Theorems 10.21 and 1.1(b), and [6] Theorem 10.6.16, T_2F is the stabilizer G_∞ of a sharply 3-transitive group G on $\overline{F}$, where $G = \langle T_2F, \sigma\rangle \subseteq \mathrm{SYM}\overline{F}$. Thus, we assign $T_3F = \langle T_2F, \sigma\rangle$. We think of T_3F as generalized fractional affine transformations on F. This construction turns out to be functorial, as shown in the next proposition. Compare this with Propositions 3.1 and 3.6.

Proposition 3.11. *There is a functor* $T_3\colon \mathbf{KTfield} \to \mathbf{s3tGroup}$

Proof. See [2] Theorem 11.1 for a proof that T_3F acts sharply 3-transitively on $\overline{F}$. To verify that T_3 is functorial, consider a morphism of KT-fields $f\colon (F, +, \cdot, \sigma) \to (F', +, \cdot, \sigma')$. Define $T_3f : T_3F \to T_3F'$ by extending $T_2f : \langle a, b\rangle \mapsto \langle f(a), f(b)\rangle$ and $\sigma \mapsto \sigma'$ to a group homomorphism $\langle T_2F, \sigma\rangle \to \langle T_2F', \sigma'\rangle$. Extend f to $\overline{F}$ by saying $f(\infty) = \infty$. We now verify that (T_3f, f) is indeed a morphism in **s3tGroup** between $(T_3F, \overline{F}, 0, 1, \infty)$ and $(T_3F', \overline{F'}, 0, 1, \infty)$. That this assignment is functorial is readily verified.

Notice, since f is a morphism of neardomains, f is necessarily injective and satisfies $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. Furthermore, char $F = 2$ if and only if char $F' = 2$. Thus, the sharply 2-transitive stabilizers of ∞, T_2F and T_2F', are of the same type.

It remains to show that the following diagram commutes:

$$\begin{array}{ccc} T_3F \times \overline{F} & \xrightarrow{T_3 f \times f} & T_3F' \times \overline{F'} \\ \downarrow & & \downarrow \\ \overline{F} & \xrightarrow[f]{} & \overline{F'}. \end{array}$$

This follows from Theorem 3.9 and the fact that, as a KT-field morphism, $\sigma' \circ f = f \circ \sigma$. □

Suppose, now, we have a group G acting sharply 3-transitively on a set Ω with distinct base points $0, 1, \infty \in \Omega$. We know G_∞ acts sharply 2-transitively on $\Omega \setminus \{\infty\}$ by Proposition 2.3, and we can construct the corresponding neardomain $Q(G_\infty, \Omega \setminus \{\infty\}, 0, 1, \infty) = (\Omega \setminus \{\infty\}, +_0, \cdot_1)$. This neardomain can then be fixed up with an appropriate involution σ producing a KT-field as follows: σ is the restriction to $\Omega \setminus \{\infty, 0\}$ of the unique involution $\tau \in G$ sending $1 \mapsto 1$ and $0 \mapsto \infty$ (see [2] (11.2)). The next proposition shows that this construction is functorial.

Proposition 3.12. *There is a functor* R: **s3tGroup** $\to$ **KTfield**.

Proof. Let $(G, \Omega, 0, 1, \infty)$ be a sharply 3-transitive group. Define $R(G, \Omega, 0, 1, \infty)$ to be $(\Omega \setminus \{\infty\}, +_0, \cdot_1, \sigma)$ as described above. Now, consider a morphism $(f, \Phi)\colon (G, \Omega, 0, 1, \infty) \to (H, \Sigma, 0, 1, \infty)$ in the category **s3tGroup**. Then G_∞ and H_∞ have the same type, and

$$(f|_{G_\infty}, \Phi|_{\Omega\setminus\{\infty\}}) : (G_\infty, \Omega \setminus \{\infty\}, 0, 1) \to (H_\infty, \Sigma \setminus \{\infty\}, 0, 1)$$

can be verified to be a morphism in **s2tGroup**. Thus,

$$Q(f|_{G_\infty}, \Phi|_{\Omega\setminus\{\infty\}}) = \Phi|_{\Omega\setminus\{\infty\}}\colon \Omega \setminus \{\infty\} \to \Sigma \setminus \{\infty\}$$

is a morphism of neardomains by Proposition 3.8. To show that $\Phi|_{\Omega\setminus\{\infty\}}$ is a morphism of KT-fields, we must verify that the following diagram commutes:

$$\begin{array}{ccc} \Omega \setminus \{\infty, 0\} & \xrightarrow{\Phi|_{\Omega\setminus\{\infty,0\}}} & \Sigma \setminus \{\infty, 0\} \\ \sigma \downarrow & & \downarrow \sigma' \\ \Omega \setminus \{\infty, 0\} & \xrightarrow[\Phi|_{\Omega\setminus\{\infty,0\}}]{} & \Sigma \setminus \{\infty, 0\}. \end{array}$$

To start, recall that (f, Φ) is a morphism in **s3tGroup**, so in particular, for every $g \in G$ and $\alpha \in \Omega$, we have $f(g)(\Phi(\alpha)) = \Phi(g(\alpha))$. Now, let $\tau\colon \Omega \to \Omega$ and $\tau'\colon \Sigma \to \Sigma$ be the involutions of G and H, respectively, that restrict to σ and σ', respectively. We note that $f(\tau) = \tau'$ since:

- $f(\tau)f(\tau) = f(\tau^2) = f(\mathrm{id}_\Omega) = \mathrm{id}_\Sigma$, as f is a group homomorphism;
- for $\alpha \in \{0, 1, \infty\}$, $f(\tau)(\alpha) = f(\tau)(\Phi(\alpha)) = \Phi(\tau(\alpha)) = \tau'(\alpha)$.

So, by uniqueness of the choice of τ', we are forced to conclude $f(\tau) = \tau'$.

We then find that for every $\alpha \in \Omega$, we have $f(\tau)(\Phi(\alpha)) = \Phi(\tau(\alpha))$, i.e. $\tau' \circ \Phi = \Phi \circ \tau$. This implies the desired result on the restrictions of these functions to $\Omega \setminus \{\infty, 0\}$. Therefore, we define $R(f, \Phi) = Q(f\,|_{G_\infty}, \Phi\,|_{\Omega\setminus\{\infty\}}) = \Phi\,|_{\Omega\setminus\{\infty\}}$, which is easily verified to be functorial. □

We now prove the main result of this section:

Theorem 3.13. *The functors* R: **s3tGroup** $\rightleftarrows$ **KTfield** :T_3 *constitute an equivalence of categories. Moreover,* **KTfield** *is a deformation retract of* **s3tGroup**.

Proof. Since $QT_2 = \mathrm{id}_{\mathbf{nDomain}}$, one can readily verify that RT_3 is $\mathrm{id}_{\mathbf{KTfield}}$. Suppose we have $R(f, \Phi) = R(g, \Psi)$ for two morphisms of sharply 3-transitive groups (f, Φ) and (g, Ψ). Then we have $\Phi\,|_{\Omega\setminus\{\infty\}} = \Psi\,|_{\Omega\setminus\{\infty\}}$, and since $\Phi(\infty) = \infty = \Psi(\infty)$, we conclude $\Phi = \Psi$. Lemma 2.4 implies $f = g$, thus R is faithful and the desired result follows from Lemma 2.1. □

3.4. A diagram of categorical equivalences

Far from being ad hoc, the categorical equivalences described above are very closely related to one another. In addition to our functors from the preceding development, we define:

- U : **KTfield** $\to$ **nDomain** is the forgetful functor, sending $(F, +, \cdot, \sigma) \mapsto (F, +, \cdot)$ and $f \mapsto f$;
- $(-)_\infty$: **s3tGroup** $\to$ **s2tGroup** sends $(G, \Omega, 0, 1, \infty) \mapsto (G_\infty, \Omega \setminus \{\infty\}, 0, 1)$ and $(f, \Phi) \mapsto (f\,|_{G_\infty}, \Phi\,|_{\Omega\setminus\{\infty\}})$; and
- $(+)$: **nDomain** $\to$ **Loop** sends $(F, +, \cdot) \mapsto (F, +)$ and $f \mapsto f$.

Now, call $\mathcal{A}$ the diagram **KTfield** $\xrightarrow{U}$ **nDomain** $\xrightarrow{(+)}$ **Loop** and call $\mathcal{S}$ the diagram **s3tGroup** $\xrightarrow{(-)_\infty}$ **s2tGroup** $\xrightarrow{\mathrm{A}}$ **RPS**. We have the following theorem:

Theorem 3.14. *There is a (pointwise) natural equivalence* $T\colon \mathcal{A} \xrightarrow{\simeq} \mathcal{S}$.

Proof. The components of T are T_1, T_2, and T_3, as in the commutative diagram of functors:

$$\begin{array}{ccc}
\mathbf{KTfield} & \xrightarrow[T_3]{\simeq} & \mathbf{s3tGroup} \\
{\scriptstyle U}\big\downarrow & (1) & \big\downarrow{\scriptstyle (-)_\infty} \\
\mathbf{nDomain} & \xrightarrow[T_2]{\simeq} & \mathbf{s2tGroup} \\
{\scriptstyle (+)}\big\downarrow & (2) & \big\downarrow{\scriptstyle \mathrm{A}} \\
\mathbf{Loop} & \xrightarrow[T_1]{\simeq} & \mathbf{RPS},
\end{array}$$

where square (1) commutes by definition of T_3, and square (2) commutes by Proposition 3.7. □

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