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Characterization of obstinate $\mathsf{H}_v\mathsf{MV}$-ideals

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Abstract. One motivation to study obstinate ideals in any algebra of logic is that the induced quotient algebra by these ideals is the two-element Boolean algebra. In this paper, we introduce two types of obstinate ideals in $\mathsf{H}_v\mathsf{MV}$-algebras; obstinate $\mathsf{H}_v\mathsf{MV}$-ideals and obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals. Giving several theorems and examples we characterize these $\mathsf{H}_v\mathsf{MV}$-ideals. For example, we prove that an $\mathsf{H}_v\mathsf{MV}$-ideal (if exists) must be maximal, and any $\mathsf{H}_v\mathsf{MV}$-algebra with odd number of elements does not contatin an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. Also, we characterize these $\mathsf{H}_v\mathsf{MV}$-ideals in finite $\mathsf{H}_v\mathsf{MV}$-algebras with at most six elements; we investigate that which subsets can be an obstinate (weak) $\mathsf{H}_v\mathsf{MV}$-ideal. In the sequel, we investigate the relationships between obstinate (weak) $\mathsf{H}_v\mathsf{MV}$-ideals, and Boolean and prime $\mathsf{H}_v\mathsf{MV}$-ideals. Finally, we prove that in a commutative $\mathsf{H}_v\mathsf{MV}$-algebra, the quotient $\mathsf{H}_v\mathsf{MV}$-algebra induced by an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal must be a two-elements Boolean algebra.

1. Introduction

In 1958, Chang [8] introduced the concept of an MV-algebra as an algebraic proof of completeness theorem for $\aleph_0$-valued Łukasiewicz propositional calculus, see also [9]. Many mathematicians have worked on MV-algebras and obtained significant results. Mundici [21] proved that MV-algebras and Abelian ℓ-groups with strong unit are categorically equivalent. He also proved that MV-algebras and bounded commutative BCK-algebras are categorically equivalent (see [20]). The ideal theory have an important role in studying algebras of logics such as MV-algebras because they are correspond to the sets of provable formulas in the correspond logics. In this respect various researches have published by many authors (see for example [14, 15, 16, 17]).

The hyperstructure theory (called also multialgebras) was introduced in 1934 by Marty [19]. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructures have many applications to several sectors of both pure and applied sciences. A short review of the theory of hyperstructures appear in [10]. In [11] a wealth of applications can be found, too. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities.

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Borzooei et al. [6, 18] applied the hyperstructures to BCK-algebras and introduced the notion of a hyper BCK-algebra and a hyper K-algebra, which is a hyperstructure weaker than hyper BCK-algebras. Recently, Ghorbani et al. [13] applied the hyperstructures to MV-algebras and introduced the concept of hyper MV-algebra and investigated some related results, see also [22]. Particularly, they investigated the relationships between hyper MV-algebras and hyper K-algebras. They proved that any hyper MV-algebra together with suitable (hyper) operations is a hyper K-algebra, and any hyper K-algebra satisfying some conditions can be viewed as a hyper MV-algebra.

In 1995, Vougiouklis introduced a generalization of hyperstructures so-called H_v-structure (see [23, 24]). Indeed, H_v-structures are a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Since then the study of H_v-structure theory has been pursued in many directions by Vougiouklis, Davvaz, Spartalis and others. To investigate the relationships between H_v-structures such as H_v-groups and suitable generalizations of MV-algebras, the first author introduced $\mathsf{H}_v\mathsf{MV}$-algebras and gave various results. He introduced some types of ideals such as (fuzzy) $\mathsf{H}_v\mathsf{MV}$-ideals and (fuzzy) weak $\mathsf{H}_v\mathsf{MV}$-ideals and their generalizations (see [1, 2, 3, 4, 5]).

2. Preliminaries

This section is devoted to give some definitions and results from the literature. For more details we refer to the references.

Definition 2.1. An $\mathsf{H}_v\mathsf{MV}$-*algebra* is a nonempty set H endowed with a binary hyperoperation '$\oplus$', a unary operation '*' and a constant '0' satisfying the following conditions:

($\mathsf{H}_v\mathsf{MV}1$) $x \oplus (y \oplus z) \cap (x \oplus y) \oplus z \neq \emptyset$, (weak associativity)
($\mathsf{H}_v\mathsf{MV}2$) $(x \oplus y) \cap (y \oplus x) \neq \emptyset$, (weak commutativity)
($\mathsf{H}_v\mathsf{MV}3$) $(x^*)^* = x$,
($\mathsf{H}_v\mathsf{MV}4$) $(x^* \oplus y)^* \oplus y \cap (y^* \oplus x)^* \oplus x \neq \emptyset$,
($\mathsf{H}_v\mathsf{MV}5$) $0^* \in (x \oplus 0^*) \cap (0^* \oplus x)$,
($\mathsf{H}_v\mathsf{MV}6$) $0^* \in (x \oplus x^*) \cap (x^* \oplus x)$,
($\mathsf{H}_v\mathsf{MV}7$) $x \in (x \oplus 0) \cap (0 \oplus x)$,
($\mathsf{H}_v\mathsf{MV}8$) $0^* \in (x^* \oplus y) \cap (y \oplus x^*)$ and $0^* \in (y^* \oplus x) \cap (x \oplus y^*)$ imply $x = y$.

On any $\mathsf{H}_v\mathsf{MV}$-algebra H, the binary relation '$\preceq$' is defined as

$$x \preceq y \;\Leftrightarrow\; 0^* \in x^* \oplus y \cap y \oplus x^*.$$

Proposition 2.2. *In any* $\mathsf{H}_v\mathsf{MV}$-*algebra* H*, the following hold:* $\forall x, y \in H$ *and* $\forall A, B \subseteq H$*,*

(1) $A \preceq A$, $0 \preceq A \preceq 1$, *where* $1 = 0^*$,

(2) $A \preceq B$ *implies* $B^* \preceq A^*$,

(3) $(A^*)^* = A$,

(4) $A \cap B \neq \emptyset$ *implies that* $A \preceq B$,

(5) $x \odot (y \odot z) \cap (x \odot y) \odot z \neq \emptyset$, *where* $x \odot y = (x^* \oplus y^*)^*$,

(6) $(x \odot y) \cap (y \odot x) \neq \emptyset$,

(7) $0 \in (x \odot 0) \cap (0 \odot x)$,

(8) $0 \in (x \odot x^*) \cap (x^* \odot x)$,

(9) $x \in (x \odot 1) \cap (1 \odot x)$,

(10) $0 \in (x \wedge 0) \cap (0 \wedge x)$, *where* $x \wedge y = (x \oplus y^*) \odot y$,

(11) $x \preceq y$ *and* $y \preceq x$ *imply* $x = y$.

Definition 2.3. Let I be a nonempty subset of $\mathsf{H}_v\mathsf{MV}$-algebra H satisfying
(I_0) $x \preceq y$ and $y \in I$ imply $x \in I$.

I is called

(1) an $\mathsf{H}_v\mathsf{MV}$-*ideal* if $x \oplus y \subseteq I$, for all $x, y \in I$,

(2) a *weak* $\mathsf{H}_v\mathsf{MV}$-*ideal* if $x \oplus y \preceq I$, for all $x, y \in I$.

Obviously, any $\mathsf{H}_v\mathsf{MV}$-ideal is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, but the converse is not true in general (see [1], for more details).

The set of all $\mathsf{H}_v\mathsf{MV}$-ideals of $\mathsf{H}_v\mathsf{MV}$-algebra H is denoted by $\mathbf{Id}(H)$.

From Proposition 2.2(4) it follows that

Theorem 2.4. *Every* $\mathsf{H}_v\mathsf{MV}$-*ideal is a weak* $\mathsf{H}_v\mathsf{MV}$-*ideal.*

From ($\mathsf{H}_v\mathsf{MV}$7) it follows that $0 \in 0 \oplus 0$, whence $\{0\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, in any $\mathsf{H}_v\mathsf{MV}$-algebra H. Generally $\{0\}$ is not an $\mathsf{H}_v\mathsf{MV}$-ideal, while H is itself an $\mathsf{H}_v\mathsf{MV}$-ideal (and so a weak $\mathsf{H}_v\mathsf{MV}$-ideal). Hence H is called trivial $\mathsf{H}_v\mathsf{MV}$-ideal, and $\{0\}$ and H are called the trivial weak $\mathsf{H}_v\mathsf{MV}$-ideals of H. Any (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H (except H itself) is called proper.

Definition 2.5. Let θ be an equivalence relation in $\mathsf{H}_v\mathsf{MV}$-algebra H.

- θ is called a *congruence* if
  (1) $x\theta y$ and $u\theta v$ imply that $x \oplus u\ \theta y \oplus v$, where $A\theta B$ means that for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$.
  (2) $x\theta y$ implies that $x^*\theta y^*$,

- θ is said to be *regular* if $x^* \oplus y \cap y \oplus x^* \theta_w \{0^*\}$ and $y^* \oplus x \cap x \oplus y^* \theta_w \{0^*\}$ imply $x\theta y$, where $A\theta_w B$ means that there exist $a \in A$ and $b \in B$ such $a\theta b$.
- The congruence class $0/\theta$ is called the congruence kernel of θ.

Throughout the paper, H will denotes an $\mathsf{H}_v\mathsf{MV}$-algebra, unless otherwise stated.

3. Main results

Definition 3.1. A proper $\mathsf{H}_v\mathsf{MV}$-ideal I of H is called an *obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal* if it satisfies (OI), where

$$\text{(OI)} \quad (\forall x, y \in H \setminus I) \qquad x \odot y^* \cup y^* \odot x \subseteq I \text{ and } x^* \odot y \cup y \odot x^* \subseteq I$$

Definition 3.2. A proper weak $\mathsf{H}_v\mathsf{MV}$-ideal I of H is called an *obstinate* weak $\mathsf{H}_v\mathsf{MV}$-*ideal* if it satisfies (WOI), where

$$\text{(WOI)} \quad (\forall x, y \in H \setminus I) \qquad x \odot y^* \cup y^* \odot x \preceq I \text{ and } x^* \odot y \cup y \odot x^* \preceq I$$

From the definition it immediately follows that every obstinate $\mathsf{H}_v\mathsf{MV}$-ideal is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, whereas the converse may not be true, in general.

Example 3.3. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$, where $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as given in Table 1. It is not difficult to check that $I = \{0, a\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, while it is not an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal because $b, 1 \in H \setminus I$ but $1^* \odot b \cup b \odot 1^* = \{0, a, 1\} \not\subseteq I$.

$\oplus$	0	a	b	1
0	$\{0, a, b\}$	$\{a, b\}$	$\{b\}$	$\{0, a, b, 1\}$
a	$\{a\}$	$\{a\}$	$\{1\}$	$\{1\}$
b	$\{b\}$	$\{1\}$	$\{a, b, 1\}$	$\{a, 1\}$
1	$\{0, a, b, 1\}$	$\{0, b, 1\}$	$\{0, b, 1\}$	$\{a, b, 1\}$
$*$	1	b	a	0

Table 1: Cayley table of Example 3.3

Example 3.4. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$, where $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as given in Table 2.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{a\}$	$\{1\}$	$\{1\}$
b	$\{b\}$	$\{1\}$	$\{b\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{b, 1\}$
$*$	1	b	a	0

Table 2: Cayley table of Example 3.4

It is not difficult to check that $I = \{0, a\}$ is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Theorem 3.5. *In an* $\mathsf{H}_v\mathsf{MV}$-*algebra with at least three elements, the singleton* $\{0\}$ *can not be an obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal.*

Proof. Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra with $|H| \geqslant 3$ and assume that $\{0\}$ is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H, by contrary. Then for $x \in H \setminus \{0,1\}$ we have $x^* \odot 1 \cup 1 \odot x^* \subseteq \{0\}$; i.e., $x^* \odot 1 = \{0\}$, whence $x \oplus 0 = \{1\}$. This contradicts ($\mathsf{H}_v\mathsf{MV7}$). Thus $\{0\}$ is not an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. $\square$

Theorem 3.6. *Any obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal* I *of* H *satisfies*

$$x \in I \text{ or } x^* \in I \quad (\forall x \in H). \tag{3.1}$$

Proof. Assume that I is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H and $x \in H \setminus I$. Since $1 \notin I$, so $x^* \in x^* \odot 1 \subseteq I$. $\square$

Example 3.7. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ in which $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as in Table 3. It is easily seen that $\{0, a\}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1), while it is not an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal because $b, 1 \notin \{0, a\}$, but $1^* \odot b \cup b \odot 1^* = \{0, b, 1\} \nsubseteq \{0, a\}$. This example shows that the converse of Theorem 3.6 is not true in general.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{0,b,1\}$	$\{0,1\}$
b	$\{b\}$	$\{0,1\}$	$\{b\}$	$\{0,1\}$
1	$\{0,1\}$	$\{a,1\}$	$\{0,b,1\}$	$\{0,1\}$
$*$	1	b	a	0

Table 3: Cayley table of Example 3.7

Theorem 3.8. *An* $\mathsf{H}_v\mathsf{MV}$-*algebra with* $2n+1$ *elements, where* n *is a positive integer, does not contain any obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal.*

Proof. Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra with $2n+1$ elements, where $n \geqslant 1$ is a positive integer, and let I be an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H (by contrary). Then there exists $x \in H$ such that $x^* = x$. On the other hand, by Theorem 3.6 we must have $x^* = x \in I$. Hence $0^* \in x^* \oplus x \subseteq I$, which a contradiction. Therefore, H can not contain any obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. $\square$

Theorem 3.9. *In an* $\mathsf{H}_v\mathsf{MV}$-*algebra, every obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal, if exists, is maximal.*

Proof. Let I be an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H and J be an $\mathsf{H}_v\mathsf{MV}$-ideal of H such that properly contains I. Let $a \in J \setminus I$. By Theorem 3.6, $a^* \in I \subset J$. Hence $1 \in a \oplus a^* \subseteq J$, whence $J = H$. Therefore I is a maximal $\mathsf{H}_v\mathsf{MV}$-ideal of H. $\square$

Theorem 3.10. (Extension Theorem) *Let* I *and* J *be* $\mathsf{H}_v\mathsf{MV}$-*ideals of* H *such that* $I \subseteq J$. *If* I *is an obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal,* J *is also an obstinate* $\mathsf{H}_v\mathsf{MV}$-*ideal.*

Proof. Assume that $x, y \notin J$, for $x, y \in H$. Then $x, y \notin I$ and so $x^* \odot y \cup y \odot x^* \subseteq I \subseteq J$. Similarly, $y^* \odot x \cup x \odot y^* \subseteq J$, proving J is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

Example 3.11 shows that the converse of Theorem 3.9 does not hold in general.

Example 3.11. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H, \oplus, ^*, 0\rangle$, where $H = \{0, a, b, c, 1\}$ and $\oplus$ and * are defined as in Table 4. It is easy to verify that the only proper $\mathsf{H}_v\mathsf{MV}$-ideals of H are $\{0\}$ and $\{0, a\}$. Hence $\{0, a\}$ is a maximal $\mathsf{H}_v\mathsf{MV}$-ideal of H, while it is not obstinate because $b, c \in H \setminus \{0, a\}$ and $b^* \odot c \cup c \odot b^* = H \nsubseteq \{0, a\}$.

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$
a	$\{a\}$	$\{0, a\}$	$\{b, 1\}$	$\{0, a, c\}$	$\{1\}$
b	$\{b\}$	$\{b, 1\}$	$\{b, 1\}$	H	$\{1\}$
c	$\{c\}$	$\{0, a, c\}$	$H \setminus \{1\}$	$\{c, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$*$	1	b	a	c	0

Table 4: Cayley table of Example 3.11

Example 3.12. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H, \oplus, ^*, 0\rangle$, where $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as in Table 5. Routine calculations show that $\{0, a\}$ and $\{0, a, b\}$ are obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals of H. This example shows that Theorem 3.9 does not hold for obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals, in general.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{a, b\}$	H
a	$\{a\}$	$\{a, 1\}$	$\{a, b\}$	H
b	$\{0, b\}$	$\{0, a, b\}$	H	$\{1\}$
1	H	$\{0, a, 1\}$	$\{0, a, 1\}$	$\{b, 1\}$
$*$	1	a	b	0

Table 5: Cayley table of Example 3.12

Theorem 3.13. *Let $H = \{0, a, 1\}$ be an $\mathsf{H}_v\mathsf{MV}$-algebra.*

(i) *If $|a \oplus a| = 1$, H does not contain any obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

(ii) *If $|a \oplus a| > 1$, $\{0, a\}$ is the maximal obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. Let $H = \{0, a, 1\}$ be an $\mathsf{H}_v\mathsf{MV}$-algebra with three elements.

(i) We observe that $a^* = a$ and since $0^* \in a^* \oplus a = a \oplus a$, hence $a \oplus a = \{0^*\}$. This implies that $a \oplus a \not\preceq \{0, a\}$. Hence $\{0, a\}$ can not be a weak $\mathsf{H}_v\mathsf{MV}$-ideal and so is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.

(ii) We assume that $|a \oplus a| > 1$. Then

$$\{0, 1\} \subseteq a \oplus a \text{ or } \{a, 1\} \subseteq a \oplus a \text{ or both.} \tag{3.2}$$

We prove that $I = \{0, a\}$ is a maximal obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. From ($\mathsf{H}_v\mathsf{MV7}$) it follows that $0 \oplus 0 \preceq I$, $0 \oplus a \preceq I$ and $a \oplus 0 \preceq I$ and from (3.2) it follows that $a \oplus a \preceq I$. Obviously, I satisfies (I_0). Thus I is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Now, from $1 \notin I$ and that $0 \in 1^* \odot 1 \cup 1 \odot 1^*$ it follows that $1^* \odot 1 \cup 1 \odot 1^* \preceq I$. Hence I is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. It is obvious that I is maximal. □

Remark 3.14. We mention that the intersection of two $\mathsf{H}_v\mathsf{MV}$-ideals is again an $\mathsf{H}_v\mathsf{MV}$-ideal (see [1, Theorem 4.14]), while it is not true for obstinate $\mathsf{H}_v\mathsf{MV}$-ideals. To see this consider Example 3.4. It is easy to check that $\{0, a\}$ and $\{0, b\}$ are obstinate $\mathsf{H}_v\mathsf{MV}$-ideals of H, while their intersection, $\{0\}$, is not an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal because $a, b \in H \setminus \{0\}$ but $a \odot b^* \cup b^* \odot a = \{a\} \not\subseteq \{0\}$.

On the other hand, the union of two $\mathsf{H}_v\mathsf{MV}$-ideals may not be an $\mathsf{H}_v\mathsf{MV}$-ideal, in general (see Example 3.7 in which $\{0, a\}$ and $\{0, b\}$ are $\mathsf{H}_v\mathsf{MV}$-ideals of H but the union, $\{0, a, b\}$, is not an $\mathsf{H}_v\mathsf{MV}$-ideal because $a \oplus b = \{0, b, 1\} \not\subseteq \{0, a, b\}$). If this is true it is easily proved that the union of two obstinate $\mathsf{H}_v\mathsf{MV}$-ideals is again an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. Indeed we have

Theorem 3.15. *Assume that A is a nonempty family of obstinate $\mathsf{H}_v\mathsf{MV}$-ideals of H such that $\cup A$ is closed with respect to '$\oplus$'. If each member of A is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal, $\cup A$ is again an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

Proof. The proof is routine. We only observe that if $\cup A$ is closed with respect to $\oplus$, $\cup A$ satisfies Definition 2.3(1). □

Corollary 3.16. *If* $\mathbf{Id}(H)$ *is closed with respect to the union, then* $\mathbf{OId}(H)$*, the set of all obstinate $\mathsf{H}_v\mathsf{MV}$-ideals of H, is an upper semilattice with respect to set inclusion as the partial ordering.*

In the sequel, we give several characterizations of obstinate week $\mathsf{H}_v\mathsf{MV}$-ideals.

Definition 3.17. We say that an $\mathsf{H}_v\mathsf{MV}$-algebra H satisfies the condition (AP) if for all $n \in \mathbb{N}$ and for all $x, y_1, y_2, \ldots, y_n \in H$ we have

$$x \preceq (\cdots(x \oplus y_1) \oplus \cdots) \oplus y_n \text{ and } x \preceq (\cdots(y_1 \oplus y_2) \oplus \cdots \oplus y_n) \oplus x$$

Remark 3.18. We observe that if H satisfies (AP), then $x \preceq x \oplus y$ and $x \preceq y \oplus x$, for all $x, y \in H$ and so $x \odot y \preceq x$ and $x \odot y \preceq y$, by Proposition 2.2(2).

Example 3.19. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$, where $H = \{0, a, 1\}$ and $\oplus$ and * are defined as given in Table 6.

$\oplus$	0	a	1
0	$\{0, a\}$	$\{0, a\}$	$\{1\}$
a	$\{0, a\}$	$\{0, a, 1\}$	$\{1\}$
1	$\{1\}$	$\{a, 1\}$	$\{0, 1\}$
$*$	1	a	0

Table 6: Cayley table of Example 3.19

It is easy to verify that H satisfies (AP). This example shows that those $\mathsf{H}_v\mathsf{MV}$-algebras satisfying (AP) do exist.

Theorem 3.20. *Every $\mathsf{H}_v\mathsf{MV}$-algebra with three elements satisfies (AP).*

Proof. It follows from ($\mathsf{H}_v\mathsf{MV}5$)-($\mathsf{H}_v\mathsf{MV}7$) and Proposition 2.2(1). $\square$

Definition 3.21. An element $a \in H$ is said to be a *scalar* if $|x \oplus a| = |a \oplus x| = 1$, where the vertical lines means the cardinality.

Theorem 3.22. *Let I be a nonempty subset of H.*

(i) *Assume that H satisfies (AP). If I is a proper weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfying* (3.1)*, then it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

(ii) *If 0 is a scalar, every obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfies* (3.1).

Proof. (i) We assume that H satisfies (AP) and I is a proper weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1). For $x, y \in H \setminus I$ we have $x^*, y^* \in I$. On the other hand $x \odot y^* \preceq y^*$ and $y^* \odot x \preceq y^*$, whence $x \odot y^* \cup y^* \odot x \preceq I$. Similarly, it is proved that $x^* \odot y \cup y \odot x^* \preceq I$, completes the proof.

(ii) Assume that 0 is a scalar, I is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H and $x \in H \setminus I$. Since $1 \notin I$, so $\{x^*\} = x^* \odot 1 \cup 1 \odot x^* \preceq I$, whence $x^* \in I$. $\square$

The next corollary is immediately follows.

Corollary 3.23. *In an $\mathsf{H}_v\mathsf{MV}$-algebra satisfying (AP) and in which 0 is a scalar, a proper weak $\mathsf{H}_v\mathsf{MV}$-ideal is obstinate if and only if it satisfies* (3.1).

Example 3.24. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ with $H = \{0, a, b, c, d, 1\}$ and $\oplus$ and * are defined as in Table 7.

$\oplus$	0	a	b	c	d	1
0	$\{0,a,c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	H	$\{0,a,c\}$	$H\setminus\{1\}$	H
b	$\{b\}$	H	$H\setminus\{1\}$	$\{0,a,c\}$	$H\setminus\{1\}$	H
c	$\{c\}$	$\{0,a,c\}$	$\{0,a,c\}$	$H\setminus\{1\}$	$\{1\}$	H
d	$\{d\}$	$H\setminus\{1\}$	$H\setminus\{1\}$	$\{1\}$	$H\setminus\{1\}$	H
1	H	H	H	H	H	H
$*$	1	b	a	d	c	0

Table 7: Cayley table of Example 3.24

Then H does not satisfy (AP) because $b \not\preceq \{0, a, c\} = b \oplus c$. Moreover, $\{0, a, c\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfying (3.1), while it is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal because $b, d \notin \{0, a, c\}$ but $b^* \odot d \cup d \odot b^* = \{1, b, d\} \not\preceq \{0, a, c\}$. This example shows that the condition (AP) is necessary in Theorem 3.22(i).

Example 3.25. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ in which $H = \{0, a, b, c, 1\}$ and $\oplus$ and * are defined as in Table 8. Routine calculations show that H satisfies (AP). Moreover, $\{0, a\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which does not satisfy (3.1) because $c = c^* \notin \{0, a\}$. This example shows that the converse of Theorem 3.22(i) may not be true in general.

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	H
a	$\{a\}$	$\{0,a,b,c\}$	H	$\{0,a,b,c\}$	H
b	$\{0,a,b,c\}$	H	$\{0,a,b,c\}$	$\{0,a,b,c\}$	H
c	$\{0,a,b,c\}$	$\{0,a,b,c\}$	$\{0,a,b,c\}$	$\{1\}$	H
1	H	H	H	$\{0,a,c,1\}$	$\{0,b,1\}$
$*$	1	b	a	c	0

Table 8: Cayley table of Example 3.25

Example 3.26. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ in which $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as in Table 9. Obviously, 0 is not a scalar. Moreover, $\{0, a\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which does not satisfy (3.1) because $b = b^* \notin \{0, a\}$. This example shows that if 0 is not a scalar, Theorem 3.22(ii) may not be true.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{a,b\}$	H
a	$\{a\}$	$\{a,1\}$	$\{b\}$	H
b	$\{0,b\}$	$\{0,a,b\}$	H	$\{1\}$
1	H	$\{0,a,1\}$	$\{a,1\}$	H
$*$	1	a	b	0

Table 9: Cayley table of Example 3.26

Example 3.27. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.12. It is not difficult to check that H satisfies (AP) and $\{0, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal because $a, 1 \notin \{0, b\}$, while $a^* \odot 1 \cup 1 \odot a^* = \{a\} \not\preceq \{0, b\}$. We observe that $a, a^* \notin \{0, b\}$. This example shows that the condition (3.1) is necessary in Theorem 3.22(i).

Lemma 3.28. *For $a \in H \setminus \{0\}$, if $H \setminus \{a, 1\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying* (3.1)*, it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, too.*

Proof. Let $I = H \setminus \{a, 1\}$ (with $a \neq 0$) be a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H which satisfies (3.1). Now, we prove that I satisfies (WOI), for $x, y \in \{a, 1\}$. If $a = 1$, from $0 \in 1^* \odot 1 \cup 1 \odot 1^*$, the proof is complete. Assume that $a \neq 1$. Again from $0 \in 1^* \odot 1 \cup 1 \odot 1^*$ and that $0 \in a^* \odot a \cup a \odot a^*$ and $0 \in 1^* \odot a \cup a \odot 1^*$ it follows that $1^* \odot 1 \cup 1 \odot 1^* \preceq I$, $a^* \odot a \cup a \odot a^* \preceq I$ and $1^* \odot a \cup a \odot 1^* \preceq I$. Also, since $a^* \in a^* \odot 1 \cup 1 \odot a^*$ and $a^* \in I$, so $a^* \odot 1 \cup 1 \odot a^* \preceq I$, completes the proof. $\square$

Now, we give more general case than Lemma 3.28.

Lemma 3.29. *Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \dots, a_n, a_{n+1} = 1 \in H$ be such that*

$$(\exists k \in \{1,2,\dots,n,n+1\})\ a_k^* \in a_i^* \odot a_j \cup a_j \odot a_i^*,\ \forall i,j \in \{1,2,\dots,n,n+1\}. \quad (3.3)$$

If $H \setminus \{a_1, a_2, \dots, a_n, 1\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1)*, it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, too.*

Proof. Let $I = H \setminus \{a_1, a_2, \dots, a_n, a_{n+1} = 1\}$ be a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. We know that $0 \in 1^* \odot a_i \cup a_i \odot 1^*$ and $0 \in a_i^* \odot a_i \cup a_i \odot a_i^*$, whence $1^* \odot a_i \cup a_i \odot 1^* \preceq I$ and $a_i^* \odot a_i \cup a_i \odot a_i^* \preceq I$, for all $i \in \{1,2,\dots,n+1\}$. From (3.1) it follows that $a_i^* \in I$, for all $i \in \{1,2,\dots,n+1\}$, whence combining $a_i^* \in 1 \odot a_i^* \cup a_i^* \odot 1$ we get $1 \odot a_i^* \cup a_i^* \odot 1 \preceq I$. Moreover, from (3.3) and that $a_k^* \in I$ for $k \in \{1,2,\dots,n+1\}$, it follows that $a_i^* \odot a_j \cup a_j \odot a_i^* \preceq I$, completes the proof. □

Example 3.30. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ in which $H = \{0, a, b, c, 1\}$ and $\oplus$ and * are defined as in Table 10.

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$
a	$\{a\}$	$\{0,a,b,c\}$	H	$\{0,a,b,c\}$	H
b	$\{b\}$	H	$\{0,a,b,c\}$	$\{0,a,b,c\}$	H
c	$\{c\}$	$\{0,a,b,c\}$	$\{0,a,b,c\}$	H	H
1	$\{1\}$	H	H	$\{0,a,1\}$	$\{0,a,b,1\}$
$*$	1	b	a	c	0

Table 10: Cayley table of Example 3.30

(i) It is obvious that $H \setminus \{b, 1\} = \{0, a, c\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1). This example shows that those weak $\mathsf{H}_v\mathsf{MV}$-ideals satisfying the conditions of Lemma 3.28 do exist.

(ii) It is not difficult to check that $I = \{0, a, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c, 1 \notin I$, while $c^* \odot 1 \cup 1 \odot c^* = \{c\} \not\preceq I$. Also, obviously $\{0, a, b\}$ does not satisfy (3.1). Hence the condition (3.1) is necessary in Lemma 3.28.

(iii) Routine calculations show that $J = \{0, a\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c, 1 \notin J$, while $c^* \odot 1 \cup 1 \odot c^* = \{c\} \not\preceq J$. We observe that J satisfies (3.3) but does not satisfy (3.1) because $c = c^* \notin \{0, a\}$. This example shows that the condition (3.1) is necessary in Lemma 3.29.

Theorem 3.31. *Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra with $|H| < 6$. Then every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1) is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. Assume that H is an $\mathsf{H}_v\mathsf{MV}$-algebra with at most five elements. We consider the following cases.

Case 1: $|H| = 2$ or 3. If $H = \{0, 1\}$ or $H = \{0, a, 1\}$, then the only possible proper weak $\mathsf{H}_v\mathsf{MV}$-ideals of H satisfying (3.1) are $\{0\}$ and $\{0, a\}$, whence by Lemma 3.28, they are obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals.

Case 2: Assume that $H = \{0, a, b, 1\}$ with four elements. If $a^* = a$ and $b^* = b$, the only possible proper weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1) is $\{0, a, b\}$, whence by Lemma 3.28, it follows that it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. If $a^* = b$ (whence $b^* = a$), the only possible proper weak $\mathsf{H}_v\mathsf{MV}$-ideals satisfying (3.1) are $\{0, a\}$ and $\{0, b\}$, whence by Lemma 3.28 it follows that they are obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals of H.

Case 3: Assume that $H = \{0, a, b, c, 1\}$ with five elements. We first assume that $a^* = a$, $b^* = b$ and $c^* = c$. Then the only possible proper weak $\mathsf{H}_v\mathsf{MV}$-ideal of H satisfying (3.1) is $\{0, a, b, c\}$, whence by Lemma 3.28 it follows that it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. Let $a^* = b$, $b^* = a$ and $c^* = c$. Then $\{0, a, c\}$ and $\{0, b, c\}$ can be the only proper weak $\mathsf{H}_v\mathsf{MV}$-ideals of H satisfying (3.1), whence by Lemma 3.28 it follows that they are obstinate $\mathsf{H}_v\mathsf{MV}$-ideals. □

Now, we give some conditions under which those weak $\mathsf{H}_v\mathsf{MV}$-ideals mentioned in Lemma 3.28 there exist.

Theorem 3.32. *Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra. Then $H \setminus \{1\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal if and only if*

$$(\forall x, y \in H \setminus \{1\}) \quad x \oplus y \neq \{1\}. \tag{3.4}$$

Proof. Assume that $I = H \setminus \{1\}$ satisfies (3.4) and let $x \preceq y$ and $y \in I$, for some $x, y \in I$. It is clear that $x \neq 1$, whence $x \in I$. Now, let $x, y \in I$. Since $x \oplus y \neq \{1\}$, so there exists $a \in x \oplus y$ such that $a \neq 1$. This implies that $a \in I$. Hence $x \oplus y \preceq I$, proving I is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

The converse is obvious. □

Corollary 3.33. *In an $\mathsf{H}_v\mathsf{MV}$-algebra H, $H \setminus \{1\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal if and only if $x \oplus y \neq \{1\}$, for all $x, y \in H \setminus \{1\}$.*

Proof. Assume that $x \oplus y \neq \{1\}$, for all $x, y \in H \setminus \{1\}$. We must prove that $0 \odot 1 \cup 1 \odot 0 \preceq I$. But this follows from the fact that $0 \in 0 \odot 1 \cup 1 \odot 0$. Considering Theorem 3.32, we conclude that $H \setminus \{1\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. The converse follows from Theorem 3.32 and the fact that any obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal is a weak $\mathsf{H}_v\mathsf{MV}$-ideal. □

Example 3.34. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.30. It is easy to check that $H \setminus \{1\} = \{0, a, b, c\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfying (3.4). This example shows that those weak $\mathsf{H}_v\mathsf{MV}$-ideals satisfying the conditions of Theorem 3.32 do exist.

Example 3.35. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.4. Then $H \setminus \{1\} = \{0, a, b\}$ is not a weak $\mathsf{H}_v\mathsf{MV}$-ideal because $a \oplus b = \{1\} \not\preceq H \setminus \{1\}$. This example shows that the condition (3.4) is necessary in Theorem 3.32.

Definition 3.36. An element $a \in H$ is called a *coatom* if there is not any element $b \in H \setminus \{1\}$ such that $a \prec b$.

Theorem 3.37. *Let $a \in H$ be a coatom. Then $H \setminus \{a, 1\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H if and only if*

$$(\forall x, y \in H \setminus \{a, 1\}) \quad x \oplus y \not\subseteq \{a, 1\}. \tag{3.5}$$

Proof. Assume that (3.5) holds and let $I = H \setminus \{a, 1\}$. Also, let $x \preceq y$ and $y \in I$, for $x, y \in H$. Since $y \notin \{a, 1\}$ and a and 1 are coatoms, then $x \notin \{a, 1\}$, whence $x \in I$. Now, let $x, y \in I$. By hypothesis, there exists $z \in x \oplus y$ such that $z \notin \{a, 1\}$, whence $z \in I$. Hence $x \oplus y \preceq I$.

The converse follows from the fact that a and 1 are coatoms. □

Corollary 3.38. *Assume that $a \in H$ is a coatom with $a^* \neq a$. If $H \setminus \{a, 1\}$ satisfies* (3.1) *and* (3.5)*, then it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

Proof. It follows from Lemma 3.28 and Theorem 3.37. □

Example 3.39. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.11.

(i) Obviously, c is a coatom with $c^* = c$. Moreover, $\{0, a, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which is not obstinate because $c^* \odot 1 \cup 1 \odot c^* = \{c\} \not\preceq \{0, a, b\}$. Hence the condition '$a^* \neq a$, for all coatoms a' is necessary in Corollary 3.38.

(ii) Obviously, b is also a coatom with $b^* = a \neq b$. It is easily checked that $H \setminus \{b, 1\} = \{0, a, c\}$ satisfies (3.1) and (3.5). Hence it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Theorem 3.40. *Assume that $a_1, \ldots, a_n$ be coatoms of H. Then $H \setminus \{a_1, \ldots, a_n, 1\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H if and only if*

$$(\forall x, y \in H \setminus \{a_1, \ldots, a_n, 1\}) \quad x \oplus y \not\subseteq \{a_1, \ldots, a_n, 1\}. \tag{3.6}$$

Proof. It is similar to the proof of Theorem 3.37. □

Corollary 3.41. *Let $a_1, \ldots, a_n$ be coatoms of H which satisfy the conditions of Lemma* 3.29. *If $H \setminus \{a_1, \ldots, a_n, 1\}$ satisfies* (3.1) *and* (3.6)*, then it is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. It follows from Lemma 3.29 and Theorem 3.40. □

Example 3.42. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.25. It is easy to check that a, b and c are coatoms of H. Moreover, $H \setminus \{b, c, 1\} = \{0, a\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H and (3.6) satisfied. This example shows that those weak $\mathsf{H}_v\mathsf{MV}$-ideals satisfying (3.6) do exist.

4. Boolean, prime and obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals

In this section, the notions of Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals and prime weak $\mathsf{H}_v\mathsf{MV}$-ideals are introduced and the relationships between them and obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals are investigated.

Definition 4.1. Let I be a proper weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. I is called a

(i) *prime* weak $\mathsf{H}_v\mathsf{MV}$-ideal if $x \wedge y \preceq I$ implies that $x \in I$ or $y \in I$, for all $x, y \in H$,

(ii) *Boolean* weak $\mathsf{H}_v\mathsf{MV}$-ideal if $x \wedge x^* \cup x^* \wedge x \preceq I$, for all $x, y \in H$.

Example 4.2. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$ in which $\oplus$ and * are defined as in Table 11. It is not difficult to check that $\{0, a, b, c\}$ is prime weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal because $a \wedge b \cup b \wedge a = \{1\} \not\preceq \{0, a, b, c\}$.

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$
a	$\{a\}$	$\{c\}$	H	$\{0, a, b, c\}$	H
b	$\{b\}$	H	$\{c\}$	$\{0, a, b, c\}$	H
c	$\{c\}$	$\{0\}$	$\{0\}$	H	H
1	$\{1\}$	H	H	$\{0, a, c, 1\}$	$\{0, a, c, 1\}$
$*$	1	b	a	c	0

Table 11: Cayley table of Example 4.2

Example 4.3. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$, where $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as in Table 12. It is not difficult to check that $I = \{0, b\}$ is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, while it is not a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal because $a \wedge a = H \preceq I$ but $a \notin I$.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, a, b\}$	$\{0, a, b, 1\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$
1	$\{0, 1\}$	$\{a, 1\}$	$\{b, 1\}$	$\{a, b, 1\}$
$*$	1	b	a	0

Table 12: Cayley table of Example 4.3

Theorem 4.4. *Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra with the property (AP) and assume that 0 is a scalar. Then every obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. Let I be an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. By Theorem 3.22(ii), we have $x \in I$ or $x^* \in I$, for all $x \in H$. On the other hand, since H satisfies (AP), so $x \wedge x^* \preceq x^*$ and $x^* \wedge x \preceq x$, whence $x \wedge x^* \cup x^* \wedge x \preceq I$. Hence I is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

Example 4.5. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 4.2. Obviously, 0 is a scalar. Also, H does not satisfy (AP) because $a \not\preceq \{c\} = a \oplus a$. It is not difficult to check that $\{0, a, b\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal because $a \wedge b = \{1\} \not\preceq \{0, a, b\}$. This example shows that the condition (AP) is necessary in Theorem 4.4.

Example 4.6. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.25. Obviously, 0 is not a scalar. Routine calculations show that H satisfies (AP). Moreover, $\{0, b\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c \wedge c = \{c\} \not\preceq \{0, b\}$. This example shows that if 0 is not a scalar, Theorem 4.4 may not be true in general.

Example 4.7. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.30. Obviously, 0 is a scalar. Also, it is not difficult to check that H satisfies (AP). Moreover $\{0, a, b\}$ is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c, 1 \notin \{0, a, b\}$, while $c^* \odot 1 \cup 1 \odot c^* = \{c\} \not\preceq \{0, a, b\}$. This example shows that the converse of Theorem 4.4 does not true in general.

Theorem 4.8. *In an $\mathsf{H}_v\mathsf{MV}$-algebra with the property (AP), every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal which is both Boolean and prime is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. Let H be an $\mathsf{H}_v\mathsf{MV}$-algebra with the property (AP) and let I be a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal and a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Then $x \wedge x^* \cup x^* \wedge x \preceq I$, for all $x \in H$. This implies that $x \wedge x^* \preceq I$ or $x^* \wedge x \preceq I$, for all $x \in H$. In any case, we get $x \in I$ or $x^* \in I$. Now, by Theorem 3.22(i) the proof is complete. □

Example 4.9. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra $\langle H; \oplus, ^*, 0\rangle$, where $H = \{0, a, b, 1\}$ and $\oplus$ and * are defined as given in Table 13. It is not difficult to check that H satisfies (AP). Also, $I = \{0, a\}$ is a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, while it is neither a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal nor an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal because $b \wedge b^* = b^* \wedge b = \{b\} \not\preceq I$ and $b, 1 \notin I$, while $b^* \odot 1 \cup 1 \odot b^* = \{b\} \not\preceq I$, respectively. This example shows that the condition 'Boolean' is necessary in Theorem 4.8.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a, b\}$	$\{b\}$	H
a	$\{a\}$	H	$\{a, b\}$	H
b	$\{b\}$	$\{a, b\}$	$\{1\}$	$\{1\}$
1	$\{b, 1\}$	$\{0, 1\}$	$\{1\}$	$\{a, b, 1\}$
$*$	1	a	b	0

Table 13: Cayley table of Example 4.9

Example 4.10. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.30. It is easily seen that H satisfies (AP). Also, it is not difficult to check that $\{0, b\}$ is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal, while it is neither an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal nor a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c, 1 \notin \{0, b\}$ but $c^* \odot 1 \cup 1 \odot c^* = \{c\} \not\preceq \{0, b\}$ and $a \wedge c = H \preceq \{0, b\}$, while $a, c \notin \{0, b\}$. This example shows that the condition 'prime' is necessary in Theorem 4.8.

Example 4.11. Consider Example 3.3. Routine calculations show that H satisfies (AP). Moreover, $I = \{0, a\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal because $1 \wedge b = \{0, a, b, 1\} \preceq I$, while $b, 1 \notin I$. Also, it is not a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal because $b \wedge b = \{a, b\} \preceq I$, while $b \notin I$. Hence the converse of Theorem 4.8 is not true in general.

In $\mathsf{H}_v\mathsf{MV}$-algebras with at most five elements we have more strong result:

Theorem 4.12. *In any $\mathsf{H}_v\mathsf{MV}$-algebra H with $|H| < 6$, every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal which is both Boolean and prime is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. It is obvious that every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal which is both Boolean and prime satisfies (3.1). The remains follows from Theorem 3.31. $\square$

Theorem 4.13. *Let $H = \{0, a, 1\}$ be an $\mathsf{H}_v\mathsf{MV}$-algebra satisfying*

$$a \in a \oplus a \quad \text{or} \quad 0 \in a \oplus a. \tag{4.7}$$

Then $\{0, a\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Proof. Let $I = \{0, a\}$. Obviously, (I_0) is satisfied. From $0 \in 0 \oplus 0$ and $a \in a \oplus 0 \cap 0 \oplus a$ it follows that $0 \oplus 0 \preceq I$, $a \oplus 0 \preceq I$ and $0 \oplus a \preceq I$. Under condition (4.7) it is obvious that $a \oplus a \preceq I$, as well. Hence I is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. $\square$

Theorem 4.14. *Let $H = \{0, a, 1\}$ be an $\mathsf{H}_v\mathsf{MV}$-algebra satisfying (4.7). Then $\{0\}$ and $\{0, a\}$ are Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals of H.*

Proof. We know that $\{0\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, in any $\mathsf{H}_v\mathsf{MV}$-algebra. From Proposition 2.2(10), it follows that $1 \wedge 1^* \cup 1^* \wedge 1 \preceq \{0\}$. If $a \in a \oplus a$, then $0 \in (a^* \oplus a)^* \subseteq ((a \oplus a)^* \oplus a)^* = a \wedge a$, whence $a^* \wedge a = a \wedge a \preceq \{0\}$. Similarly, if $0 \in a \oplus a$, then $0 \in (0^* \oplus a)^* \subseteq ((a \oplus a)^* \oplus a)^* = a \wedge a = a^* \wedge a$. Hence $a^* \wedge a \preceq \{0\}$. Thus, $\{0\}$ is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Now, from $0^* \in a^* \oplus a = a \oplus a$ it follows that $a \in 0 \oplus a \subseteq (a \oplus a)^* \oplus a$, whence $a = a^* \in ((a \oplus a)^* \oplus a)^* = a^* \wedge a$. Hence $a^* \wedge a \preceq I$. Therefore I is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. $\square$

Lemma 4.15. *In an $\mathsf{H}_v\mathsf{MV}$-algebra, every two distinct elements a, b with $a^* = a$ and $b^* = b$ are incomparable.*

Proof. Let a, b be two distinct elements of H. Then

$$a \preceq b \;\Leftrightarrow\; 0^* \in a^* \oplus b \cap b \oplus a^* = a \oplus b \cap b \oplus a = a \oplus b^* \cap b^* \oplus a \;\Leftrightarrow\; b \preceq a,$$

which is a contradiction. $\square$

Lemma 4.16. *In any $\mathsf{H}_v\mathsf{MV}$-algebra H, for every $x \in H$ the following hold:*

(i) *if $x^* = x$, then $x \in x \wedge x$,*

(ii) *if* $x \in x \oplus x$ *or* $0 \in x \oplus x$, *then* $0 \in x \wedge x^*$,

(iii) *if* $x^* \in x \oplus x$, *then* $x \in x \wedge x^*$,

(iv) *if* $0^* \in x \oplus x$, *then* $x^* \in x \wedge x^*$,

Proof. (i) Assume that $x^* = x$, for $x \in H$. From $0^* \in x^* \oplus x$ and $x \in 0 \oplus x$ it follows that $x = x^* \subseteq (0 \oplus x)^* \subseteq ((x^* \oplus x)^* \oplus x)^* = x \wedge x$.

(ii) Assume that $x \in x \oplus x$. Then $0 \in x \odot x^* \subseteq (x \oplus x) \odot x^* = x \wedge x^*$. Similarly, if $0 \in x \oplus x$, then $0 \in 0 \odot x^* \subseteq (x \oplus x) \odot x^* = x \wedge x^*$.

(iii) If $x^* \in x \oplus x$, then $x \in (x \oplus x)^* \subseteq ((x \oplus x)^* \oplus x)^* = x \wedge x^*$.

(iv) If $0^* \in x \oplus x$, then $x^* \in 0^* \odot x^* \subseteq (x \oplus x) \odot x^* = x \wedge x^*$. $\square$

Theorem 4.17. *Let* $H = \{0, a, b, 1\}$ *be an* $\mathsf{H}_v\mathsf{MV}$*-algebra.*

(i) *If* $a^* = a$ *and* $b^* = b$, $\{0, a\}$ *and* $\{0, b\}$ *can not be simultaneously a prime weak* $\mathsf{H}_v\mathsf{MV}$*-ideal and an obstinate weak* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

(ii) *Let* $a^* = b$. *Then* $\{0, a\}$ *and* $\{0, b\}$*) are weak* $\mathsf{H}_v\mathsf{MV}$*-ideals of* H *if and only if they are Boolean weak* $\mathsf{H}_v\mathsf{MV}$*-ideals of* H.

(iii) $\{0, a, b\}$ *is a weak* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H *if and only if it is a Boolean weak* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

Proof. (i) By contrary, we assume that $I = \{0, a\}$ is a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal and an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. From $b, 1 \notin I$ it follows that $(b \oplus 0)^* \cup (0 \oplus b)^* = b^* \odot 1 \cup 1 \odot b^* \preceq I$, whence $(b \oplus 0)^* \preceq I$ or $(0 \oplus b)^* \preceq I$. Considering Lemma 4.15, it follows that $a \in b \oplus 0$ or $0^* \in b \oplus 0$ or $a \in 0 \oplus b$ or $0^* \in 0 \oplus b$. From the two first cases it follows that $b \wedge b \preceq I$ and from the two second cases it follows that $b \wedge 1 \preceq I$. This contradicts the hypothesis that I is a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Similarly, it is proved that $\{0, b\}$ can not be simultaneously a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal and an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal.

(ii) Assume that $a^* = b$ (whence $b^* = a$) and $I = \{0, a\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. From $0 \in 0 \wedge 1 \cup 1 \wedge 0$ it follows that $0 \wedge 1 \cup 1 \wedge 0 \preceq I$. It remains that to show that $a \wedge b \cup b \wedge a \preceq I$. Since $a \in I$ and I is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, so $a \oplus a \preceq I$, whence $0 \in a \oplus a$ or $a \in a \oplus a$, or $b \in a \oplus a$ and $b \preceq a$. In the first two cases it follows that $0 \in a \wedge b$, whence $a \wedge b \preceq I$. In the last case, we have $0^* \in b^* \oplus a = a \oplus a$ and so $0 \in (a \oplus a)^* \subseteq ((a \oplus a)^* \oplus a)^* = a \wedge b$. Hence $a \wedge b \preceq I$, proving I is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Similar argument shows that if $\{0, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, it is also a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal.

The converse is obvious.

(iii) Assume that$I = \{0, a, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Obviously, $0 \wedge 0^* \cup 0^* \wedge 0 = 1^* \wedge 1 \cup 1 \wedge 1^* \preceq I$. Now, if $a^* = a$ and $b^* = b$, from Lemma 4.16(i) it follows that $a \in a \wedge a^* \cup a^* \wedge a$ and $b \in b \wedge b^* \cup b^* \wedge b$, whence $a \wedge a^* \cup a^* \wedge a \preceq I$ and $b \wedge b^* \cup b^* \wedge b \preceq I$. Otherwise, since I is a weak $\mathsf{H}_v\mathsf{MV}$-ideal, so we must have

$a \oplus a \preceq I$ and $b \oplus b \preceq I$, whence $\{0, a, b\} \subseteq a \oplus a$ and similarly $\{0, a, b\} \subseteq b \oplus b$. If $a^* = b \in a \oplus a$, then $a \in a \wedge a$, by Lemma 4.16(iii), otherwise $0 \in a \wedge a^*$, by Lemma 4.16(ii). In any case $a \wedge a^* \preceq I$. Similarly, we can show that $b \wedge b^* \preceq I$. Hence I is a Boolean $\mathsf{H}_v\mathsf{MV}$-ideal.

The converse is obvious. □

Example 4.18. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.26. Obviously, H satisfies the conditions of Theorem 4.17(i). Also, it is easily checked that $\{0, a\}$ is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, which is not a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal because $b \wedge b = H \preceq \{0, a\}$, while $b \notin \{0, a\}$.

Example 4.19. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 4.9. Then $a^* = a$ and $b^* = b$ and $\{0, a\}$ is a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal, while it is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. This example shows that those $\mathsf{H}_v\mathsf{MV}$-algebras satisfying the conditions of Theorem 4.17 do exist.

Example 4.20. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.3. Obviously, $\{0, a\}$ and $\{0, b\}$ are weak $\mathsf{H}_v\mathsf{MV}$-ideals of H and so by Theorem 4.17 are Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals of H.

Example 4.21. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.7. Then $\{0, a, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H and so by Theorem 4.17, it is a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Example 4.22. As Example 4.9 shows $\{0, a\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, while it is not a Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal. We observe that $a^* \neq a$ does not hold in H. So, this condition is necessary in Theorem 4.17(ii).

In connection with quotient $\mathsf{H}_v\mathsf{MV}$-algebras induced by obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals we have the following result. Before, we state it we observe that an $\mathsf{H}_v\mathsf{MV}$-algebra H is said to be *commutative* if $x \oplus y = y \oplus x$, for all $x, y \in H$.

Theorem 4.23. *Assume that H is commutative and let I be an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. If there exists a regular congruence θ in H such that $0/\theta = I$, then*

(i) *H/θ is the two-elements Boolean algebra,*

(ii) *I is an $\mathsf{H}_v\mathsf{MV}$-ideal,*

(iii) *$x^* \neq x$, for all $x \in H$,*

(iv) *$|H|$ is an even positive integer.*

Proof. Let I be an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal of H and θ be a regular congruence in H such that $0/\theta = I$.

(i) Let $x, y \in H$ be such that $x/\theta, y/\theta \neq 0/\theta$. Then $x, y \notin I$, whence $x^* \odot y = y \odot x^* \preceq I$ and $y^* \odot x = x \odot y^* \preceq I$. This implies that $(x \oplus y^*)^* = x^* \odot y \cap I \neq \emptyset$ and

$(y \oplus x^*)^* = y^* \odot x \cap I \neq \emptyset$. Hence there exist $a \in x \oplus y^*$ and $b \in y \oplus x^*$ such that $a^*, b^* \in I = 0/\theta$, whence $a, b \in 0^*/\theta$. This means that $x \oplus y^* \theta \{0^*\}$ and $y \oplus x^* \theta \{0^*\}$. Since θ is regular, so $x\theta y$; i.e., $x/\theta = y/\theta$. Therefore, $H/\theta = \{0/\theta, 0^*/\theta\}$.

(ii) We observe that in an $\mathsf{H}_v\mathsf{MV}$-algebra $0^* \notin 0 \oplus 0$, otherwise we must have $0^* \preceq 0$, which is impossible. Hence in H/θ we have $I \oplus I = 0/\theta \oplus 0/\theta = \{0/\theta\}$. This implies that for every $x, y \in I$, $x \oplus y \subseteq I$, which implies that I is an $\mathsf{H}_v\mathsf{MV}$-ideal.

(iii) Assume that $x^* = x$, for some $x \in H$. Considering (i) we have $x \in 0/\theta$ or $x \in 0^*/\theta$. In the first case we have $x = x^*\theta 0^*$, whence $0\theta 0^*$, which is a contradiction. Similarly, if $x \in 0^*/\theta$ we get $0\theta 0^*$, which is a contradiction.

(iv) Considering (iii), the proof is obvious. □

Remark 4.24. We notice that Theorem 4.23 does not state that an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal which is the kernel of a congruence is an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. It just states that, as a weak $\mathsf{H}_v\mathsf{MV}$-ideal, it must be an $\mathsf{H}_v\mathsf{MV}$-ideal. To see this consider the $\mathsf{H}_v\mathsf{MV}$-algebra given in Table 14. It is not difficult to check that H is a commutative $\mathsf{H}_v\mathsf{MV}$-algebra in which $I = \{0, a\}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal (and so a weak $\mathsf{H}_v\mathsf{MV}$-ideal) of H which is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, while it is not an obstinate $\mathsf{H}_v\mathsf{MV}$-ideal because $1 \notin I$ but $1^* \odot 1 = \{0, b\} \nsubseteq I$. It is not difficult to verify that the relation $\theta = \triangle_H \cup \{(0, a), (a, 0), (b, 1), (1, b)\}$ is a regular congruence in H such that $0/\theta = I$.

$\oplus$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{a, 1\}$
a	$\{a\}$	$\{0\}$	$\{1\}$	$\{b, 1\}$
b	$\{b\}$	$\{1\}$	$\{a, 1\}$	H
1	$\{a, 1\}$	$\{b, 1\}$	H	$\{0, a, 1\}$
$*$	1	a	b	0

Table 14: A commutative $\mathsf{H}_v\mathsf{MV}$-algebra

5. Conclusions

We introduced a new type of $\mathsf{H}_v\mathsf{MV}$-ideals (obstinate $\mathsf{H}_v\mathsf{MV}$-ideals and obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals) and gave a deep characterization of them. We proved that in any $\mathsf{H}_v\mathsf{MV}$-algebra with odd number of elements there does not exist any obstinate $\mathsf{H}_v\mathsf{MV}$-ideal. Especially, in an $\mathsf{H}_v\mathsf{MV}$-algebra with at least three elements, the singleton $\{0\}$ is not an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. Moreover, obstinate $\mathsf{H}_v\mathsf{MV}$-ideals are maximal (if exist). Next, we studied the properties of obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals. We proved that every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfying suitable conditions is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal. In the sequel, we introduced the notions of prime weak $\mathsf{H}_v\mathsf{MV}$-ideals and Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals and gave some basic properties. Furthermore, we investigated the relationships between obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals, prime weak $\mathsf{H}_v\mathsf{MV}$-ideals and Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals. We proved that every proper weak $\mathsf{H}_v\mathsf{MV}$-ideal which is both Boolean and prime is an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, under suitable conditions, but the converse may not be true. We also characterized obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideals and the relationships between prime weak $\mathsf{H}_v\mathsf{MV}$-ideals and Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideals in $\mathsf{H}_v\mathsf{MV}$-algebras with at most five elements and investigated what subsets can be a suitable candidate to be an obstinate weak $\mathsf{H}_v\mathsf{MV}$-ideal, Boolean weak $\mathsf{H}_v\mathsf{MV}$-ideal or a prime weak $\mathsf{H}_v\mathsf{MV}$-ideal.

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