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Some results on filters in residuated lattices

Somayeh Motamed and *Javad Moghaderi*

Abstract. We study the notions of nodal, associative, strong, integral filters in residuated lattices and then state and prove some theorems which determine the relationships of these filters and other filters in residuated lattices. Also we get some new properties of positive implicative filters in residuated lattices. And we also study the notions of strong, integral residuated lattices and investigate its properties.

1. Introduction

Nonclassical logic is closely related to logic algebraic systems. A number of researches have motivated to develop nonclassical logics, and also to enrich the content of algebra. Ward and Dilworth [19], introduced the concept of residuated lattices (it should be, however, noted that the motivation was by far not logical) as generalization of ideal lattices of rings. The residuated lattice plays the role of semantics for a multiple-valued logic called residuated logic. Residuated logic is a generalization of intuitionistic logic. Therefore it is weaker than classical logic. Important examples of residuated lattices related to logic are Boolean algebras corresponding to basic logic, BL-algebras corresponding to Hájek's basic logic, and MV-algebras corresponding to Lukasiewicz many valued logic. Hájek [11], introduced the idea of filters and prime filters in BL-algebras. In researchs of logic, theory of filters plays a very important role in proving completeness with respect to algebraic semantics. From logical point of view, filters correspond to sets of provable formulae. Buşneag, Piciu [8] and Bourmand Saeid, Pourkhatoun [6] and Ahadpanah, Torkzadeh [1] introduced the notion of (positive) implicative filters, fantastic filters, easy filters, obstinate filters and normal filters in residuated lattices. The aim of this paper is to develop the filter theory of residuated lattices. In this paper, we study the notions of nodal, associative, strong, integral filters in residuated lattices and the relations among them and other type of fiters in residuated lattice are investigated.

The motivation of this paper is to give the simple general principle of studying the relations among some filters on residuated lattices. In contrast to proofs of particular results for concrete special types of these filters, proofs of those general theorems in this paper are simple. And the general principle can be applied to all the subvarieties of residuated lattices.

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2. Preliminaries

We review the basic definitions of residuated lattice, with more details.

A *residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leqslant$ satisfying the following:

(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(L, \odot, 1)$ is a commutative ordered monoid,

(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e, $c \leqslant a \rightarrow b$ if and only if $a \odot c \leqslant b$, for all $a, b, c \in L$.

Proposition 2.1. (cf. [3, 4, 8, 9, 11, 14, 18, 19]) *Let L be a residuated lattice. Then for any $x, y, z, w \in L$ we have:*

(R_1) $1 \rightarrow x = x, \ \ x \rightarrow x = 1$;

(R_2) $x \odot y \leqslant x, y$ *hence* $x \odot y \leqslant x \wedge y, \ \ x \leqslant y \rightarrow x$ *and* $x \odot 0 = 0$;

(R_3) $x \odot y \leqslant x \rightarrow y$;

(R_4) $x \leqslant y$ *if and only if* $x \rightarrow y = 1$;

(R_5) $x \rightarrow y = y \rightarrow x = 1$ *if and only if* $x = y$;

(R_6) $x \rightarrow 1 = 1,\ 0 \rightarrow x = 1, \ \ 1 \rightarrow 0 = 0$;

(R_7) $x \leqslant y \rightarrow (x \odot y),\ x \odot (x \rightarrow y) \leqslant y$ (*so,* $x \odot (x \rightarrow y) \leqslant x \wedge y$), $x \leqslant (x \rightarrow y) \rightarrow y$ *and* $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;

(R_8) $x \rightarrow y \leqslant (x \odot z) \rightarrow (y \odot z)$;

(R_9) $x \rightarrow y \leqslant (z \rightarrow x) \rightarrow (z \rightarrow y) \leqslant z \rightarrow (x \rightarrow y)$;

(R_{10}) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$ (*so,* $x \rightarrow y \leqslant y^* \rightarrow x^*$) *and* $(x \rightarrow y) \odot (y \rightarrow z) \leqslant x \rightarrow z$;

(R_{11}) $x \leqslant y$, *then* $y \rightarrow z \leqslant x \rightarrow z, \ \ z \rightarrow x \leqslant z \rightarrow y, \ \ x \odot z \leqslant y \odot z, \ \ y^* \leqslant x^*$ *and* $x^{**} \leqslant y^{**}$;

(R_{12}) $x \leqslant y$ *and* $z \leqslant w$ *then* $x \odot z \leqslant y \odot w$;

(R_{13}) $x \vee x^* = 1$ *implies* $x \wedge x^* = 0$;

(R_{14}) $x \odot (y \rightarrow z) \leqslant y \rightarrow (x \odot z) \leqslant (x \odot y) \rightarrow (x \odot z)$;

(R_{15}) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$ (*so,* $x \rightarrow y^* = y \rightarrow x^* = (x \odot y)^*$);

(R_{16}) $x \rightarrow y \leqslant (z \rightarrow w) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow w)]$;

(R_{17}) $x \leqslant x^{**}, \ \ x^{***} = x^*$ *and* $x \leqslant x^* \rightarrow y$;

(R_{18}) $x \odot x^* = 0, \ \ x \odot y = 0$ *if and only if* $x \leqslant y^*$;

(R_{19}) $x^* \odot y^* \leqslant (x \odot y)^*$ (*so,* $(x^*)^n \leqslant (x^n)^*$ *for every* $n \geqslant slant 1$);

(R_{20}) $x^{**} \odot y^{**} \leqslant (x \odot y)^{**}$ (*so,* $(x^{**})^n \leqslant (x^n)^{**}$ *for every* $n \geqslant slant 1$);

(R_{21}) $(x \vee y)^* = x^* \wedge y^*$;

(R_{22}) $(x \rightarrow y^{**})^{**} = x \rightarrow y^{**}$;

(R_{23}) $x \vee y = 1$ *implies* $x \odot y = x \wedge y$ *and* $x^n \odot y^n = 1$, *for every* $n \geqslant slant 1$;

(R_{24}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;

(R_{25}) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z), \ \ (x \rightarrow z) \vee (y \rightarrow z) \leqslant (x \wedge y) \rightarrow z$ *and* $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;

(R_{26}) $(x \vee y) \odot (x \vee z) \leqslant x \vee (y \odot z)$, *hence* $(x \vee y)^{mn} \leqslant x^n \vee y^m$; *for* $m, n \geqslant slant 1$;

(R_{27}) $x \vee y \leqslant ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$.

From now on, unless mentioned otherwise, $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ will be a residuated lattice, which will often be referred by its support set L.

The following definitions are stated from [1, 6, 8, 18]. Let $\phi \neq F \subseteq L$, and $x, y, z \in L$. For convenience, we enumerate some conditions which will be used.

(F_1) $x, y \in F$ implies $x \odot y \in F$ and $x \in F$, $x \leqslant y$ imply $y \in F$.
(F_1') $1 \in F$ and $x, x \rightarrow y \in F$ then $y \in F$.
(F_2) $x \vee y \in F$ implies $x \in F$ or $y \in F$.
(F_2') $x \rightarrow y \in F$ or $y \rightarrow x \in F$.
(F_3) $x \notin F$ if and only if there is $n \geqslant 1$ such that $(x^n)^* \in F$.
(F_4) $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$.
(F_4') $x \rightarrow x^2 \in F$.
(F_5) $(y \rightarrow z) \rightarrow y \in F$ implies $y \in F$.
(F_6) $y \rightarrow x \in F$ implies $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.
(F_7) $(y \rightarrow x) \rightarrow x \in F$ implies $(x \rightarrow y) \rightarrow y \in F$.
(F_7') $x^{**} \in F$ if and only if $x \in F$.
(F_8) $(x \odot y)^* \in F$ implies $(x^n)^* \in F$ or $(y^m)^* \in F$, for some $m, n \in N$.
(F_9) $x, y \notin F$ implies $x \rightarrow y \in F$ and $y \rightarrow x \in F$.
(F_9') $x \in F$ or $x^* \in F$.
(F_{10}) $x^{**} \rightarrow (y \rightarrow z) \in F$ and $x^{**} \rightarrow y \in F$ imply $x^{**} \rightarrow z \in F$.

Conditions (F_n) and (F_n') are equivalent.

A subset F of L is called
– a *filter* of L, if it satisfies (F_1),
– proper if $F \neq L$ (that is, $0 \notin F$),
– a *prime filter* of L, if and it satisfies (F_1), (F_2) and $0 \notin F$,
– a *maximal filter* of L, if it satisfies (F_1) and (F_3) and $0 \notin F$,
– an *implicative filter* of L, if it satisfies (F_1) and (F_4).
– a *positive implicative filter* of L, if it satisfies (F_1) and (F_5).
– a *fantastic filter* of L, if it satisfies (F_1) and (F_6).
– a *normal filter* of L, if it satisfies (F_1) and (F_7).
– a *primary filter* of L if it satisfies (F_1), (F_8) and $0 \notin F$.
– an *obstinate filter* of L, if it satisfies (F_1), (F_9) $0 \notin F$.
– an *easy filter* of L, if it satisfies (F_1) and (F_{10}).

We denote by $(F(L)$, $(Spec(L)$, $Max(L)$, $IF(L)$, $PIF(L)$, $FF(L)$, $NF(L)$, $PF(L)$, $OF(L)$ $EF(L))$ the set of all filters (resp., prime, maximal, implicative, positive implicative, fantastic, normal, primary, obstinate, easy filters) of L.

Theorem 2.2. (cf. [1, 6, 8, 18]) *The following statements hold.*
(i) $Spec(L) \subseteq PF(L)$.
(ii) $OF(L) \subseteq PIF(L) \subseteq IF(L)$.
(iii) $PIF(L) \subseteq FF(L)$.
(iv) $PIF(L) \subseteq NF(L)$.
(v) $IF(L) \subseteq EF(L)$.
(vi) $PIF(L) \cap IF(L) = NF(L)$.
(vii) $OF(L) \subseteq Spec(L)$.

The filter of L generated by $X \subseteq L$ is denoted by $\langle X \rangle$. We have $\langle \emptyset \rangle = \{1\}$ and $\langle X \rangle = \{a \in L : x_1 * x_2 * \ldots * x_n \leqslant a, \exists n \in N, \exists x_1, x_2, \ldots, x_n \in X\}$. The filter $F = \langle a \rangle$ is called *principal.* $\langle F \cup G \rangle = \{a \in L : a \geqslant f * g, \exists f \in F, \exists g \in G\}$ for any $F, G \in F(L)$.

Proposition 2.3. (cf. [7, 10]) *If $x, y \in L$, then we have*
- (i) $x \leqslant y$ *implies* $\langle y \rangle \subseteq \langle x \rangle$;
- (ii) $\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$;
- (iii) $\langle x \rangle \vee \langle y \rangle = \langle x \wedge y \rangle$.

Let $F \in F(L)$. Then the relation $\sim_F$ defined on L by $(x, y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on L. The quotient algebra $L/\sim_F$ denoted by L/F becomes a residuated lattice in a natural way, with the operations induced from those of L. So, the order relation on L/F is given by $x/F \leqslant y/F$ if and only if $x \to y \in F$. Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^* \in F$.

L is said to be *local residuated lattice* if and only if it has exactly one maximal filter. L/F is a local residuated lattice if and only if F is a primary filter of L.

Theorem 2.4. *Let $F \in F(L)$. Then $F \in NF(L)$ if and only if $x, (x \to y)^{**} \in F$ imply $y \in F$.*

Proof. We know that $F \in NF(L)$ if and only if $z^{**} \in F$ implies $z \in F$. If $x, (x \to y)^{**} \in F$, then $x \to y \in F$. Since $x \in F$ we get $y \in F$. Conversely, let $x^{**} \in F$. We have $(1 \to x)^{**} = x^{**} \in F$. So, $x \in F$. Therefore $F \in NF(L)$. □

3. Main results

From now on, unless mentioned otherwise, $(L, \wedge, \vee, \odot, \to, 0, 1)$ will be a residuated lattice, which will often be referred by its support set L.

Definition 3.1. (cf. [2]) *A* node *of a poset L is an element of L which is comparable with every element of L.*

The set of all node elements of a L is denoted by $nod(L)$. It is clear that $0, 1 \in nod(L)$.

Example 3.2. (i).Let $L = \{0, a, b, c, d, 1\}$, where $0 < a < b < 1$, $0 < a < d < 1$ and $0 < c < d < 1$. Define $\odot$ and $\to$ as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	0
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice and $nod(L) = \{0, 1\}$.

(ii). Let $L = \{0, a, b, c, 1\}$, where $0 < c < a, b < 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice and $nod(L) = \{0, c, 1\}$.

Remark 3.3. *The following facts are obvious.*

(i) $nod(L) = L$ *if and only if* L *is a linearly ordered residuated lattice.*

(ii) $a \in nod(L)$ *if and only if* $a \wedge x \in nod(L)$ *and* $a \vee x \in nod(L)$*, for all* $x \in L$.

(iii) *If* $a, b \in nod(L)$ *then* $a \wedge b, a \vee b \in nod(L)$.

Definition 3.4. (cf. [2]) *A filter* F *of* L *is called a* nodal filter *if it is a node of* $F(L)$.

We denote by $nod(F(L))$ the set of all nodal filters of a residuated lattice L. Clearly, $\{L, \{1\}\} \subseteq nod(F(L))$. If $x \in nod(L)$, then $\langle x \rangle \in nod(F(L))$.

Example 3.5. In Example 3.2(i), $F(L) = \{\{1\}, \{b, 1\}, \{c, d, 1\}, L\}$ and $nod(F(L)) = \{\{1\}, L\}$. But in Example 3.2(ii) $\{\{1\}, \{a, b, c, 1\}, L\} = nod(F(L)) \neq F(L) = \{\{1\}, \{b, 1\}, \{a, 1\}, \{a, b, c, 1\}, L\}$. Moreover, in Example 3.2(i) we have $\langle a \rangle = L \in nod(F(L))$, while $a \notin nod(L)$.

Proposition 3.6. *Let* $x^2 = x$*, for each* $x \in L$*. If* $\langle x \rangle \in nod(F(L))$*, then* $x \in nod(L)$.

Proof. Let $x \notin nod(L)$. Then there exists $y \in L$ such that $x \nleqslant y$ and $y \nleqslant x$. We have $\langle x \rangle \in nod(F(L))$. If $\langle y \rangle \subseteq \langle x \rangle$, then $y \in \langle x \rangle$, i.e., $\exists n \in N$ such that $y \geqslant x^n$. We have $x^2 = x$, so $x^n = x$, for all $n \in N$. Thus $y \geqslant x$ that is a contradiction. So $\langle y \rangle \nsubseteq \langle x \rangle$. Similarly $\langle x \rangle \nsubseteq \langle y \rangle$. That is a contradiction with $\langle x \rangle \in nod(F(L))$, i.e., $x \in nod(L)$. $\square$

Corollary 3.7. *Let* $x^2 = x$ *and* $\langle x \rangle \in nod(F(L))$*, for each* $x \in L$*. Then*

(i) L *is a linearly ordered residuated lattice.*

(ii) L *is a local residuated lattice.*

Proof. (i). Let $x, y \in L$. We must show that $x \leqslant y$ or $y \leqslant x$. We have $\langle x \rangle, \langle y \rangle \in F(L)$, so $\langle x \rangle, \langle y \rangle \in nod(F(L))$. Hence by Proposition 3.6, $x, y \in nod(L)$, i.e., $x \leqslant y$ or $y \leqslant x$.

(ii). By part (i), L is a linearly ordered residuated lattice. We know that $L \cong L/\{1\}$, so $L/\{1\}$ is a linearly ordered residuated lattice. Hence $\{1\} \in Spec(L)$. By the fact that $Spec(L) \subseteq PF(L)$, we get $\{1\} \in PF(L)$. Hence $L/\{1\}$ is local residuated lattice. Therefore L is a local residuated lattice. $\square$

Lemma 3.8. *If L is a linearly ordered residuated lattice then $nod(F(L)) = F(L)$.*

Proof. We know $nod(F(L)) \subseteq F(L)$. Let $F(L) \nsubseteq nod(F(L))$. Hence there exists $H \in F(L)$ such taht $H \notin nod(F(L))$. So there exists $G \in F(L)$ such that $H \nsubseteq G$ and $G \nsubseteq H$. Thus there exist $g \in G - H$ and $h \in H - G$. Since L is a linearly ordered, $g < h$ or $h < g$. Therefore $h \in G$ and $g \in H$, that is a contradiction. Hence $F(L) \subseteq nod(F(L))$. Thus the proof is complete. □

Lemma 3.9. *If $nod(F(L)) = F(L)$ and $x^2 = x$, for all $x \in L$, then L is a linearly ordered residuated lattice.*

Proposition 3.10. *$\{1\} \in Spec(L)$ if and only if $nod(F(L)) = F(L)$.*

Proof. Let $\{1\} \in Spec(L)$. Then L is a linearly ordered residuated lattice and so by Lemma 3.8, $nod(F(L)) = F(L)$. Now assume that $nod(F(L)) = F(L)$ and $a \vee b = 1$, for $a, b \in L$. Then $< a > \subseteq < b >$ or $< b > \subseteq < a >$ and so $a \in < b >$ or $b \in < a >$. Thus $a \geqslant b^n$ or $b \geqslant a^m$, for some $n, m \in N$. Therefore $a \vee b^n = a$ or $b \vee a^m = b$. Since $a \vee b = 1$, we have $a^n \vee b^n = 1$ and $a^m \vee b^m = 1$. Then $a = 1$ or $b = 1$ and so $\{1\} \in Spec(L)$. □

Corollary 3.11. *Let F be a filter of L. Then*
 (i) *$F \in Spec(L)$ if and only if $nod(F(L/F)) = F(L/F)$.*
 (ii) *If $F \in OF(L)$ then $nod(F(L/F)) = F(L/F)$.*

Proof. (i). Let $F \in Spec(L)$. Then L/F is a linearly residuated lattice. So by Lemma 3.8, $nod(F(L/F)) = F(L/F)$. Now assume that $nod(F(L/F)) = F(L/F)$ and $a \vee b \in F$, for $a, b \in L$. Hence $a/F \vee b/F = 1/F \in \{1/F\}$. By Proposition 3.10, $\{1/F\} \in Spec(L/F)$ and so $a/F \in \{1/F\}$ or $b/F \in \{1/F\}$. Thus $a \in F$ or $b \in F$ and therefore $F \in Spec(L)$.

(ii). By Theorem 2.2(vii) and part (i) the proof is clear. □

The following example shows that the converse of (ii) is not true.

Example 3.12. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice. Consider $F = \{b, 1\}$ and $G = \{a, b, 1\}$. Then $F \notin OF(L)$ (as $a, a^* = 0 \notin F$); but $F(L/F) = \{\{1/F\}, G/F\} = nod(F(L/F))$.

Theorem 3.13. *Let $F \in nod(F(L))$ be a non principal filter of L. Then $F \in Spec(L)$.*

Proof. Let $x \vee y \in F$ and $x, y \notin F$. We get $\langle x \vee y \rangle \subseteq F$ and $\langle y \rangle, \langle x \rangle \nsubseteq F$. Since $F \in nod(F(L))$ we get $F \subset \langle x \rangle$ and $F \subset \langle y \rangle$, thus $F \subseteq \langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$. Thus $F = \langle x \vee y \rangle$, which is a contradiction, and so $x \in F$ or $y \in F$. Therefore $F \in Spec(L)$. □

Theorem 3.14. *Let $F \in nod(F(L))$ be a proper filter of L and for each $x \in L$, $x^2 = x$. Then $N(F) = \{a \in F : \ a^{**} = 1\} = F$.*

Proof. Let $N(F) \neq F$. Hence there exists $x \in F$ such that $x \notin N(F)$, i.e., $x^{**} \neq 1$ so $x^* \neq 0$. Since $F \in nod(F(L))$ we get $F \subseteq \langle x^* \rangle$ or $\langle x^* \rangle \subseteq F$.

If $F \subseteq \langle x^* \rangle$, then $x \in \langle x^* \rangle$. So $x \geqslant (x^*)^n$ for some $n \in N$. Then by hypothesis, $x \geqslant x^*$ and so $x^* \to 0 = x^{**} \geqslant x^*$. Hence, by (LR_3), $x^* \odot x^* = 0$ and so $x^* = 0$, that is a contradiction.

If $\langle x^* \rangle \subseteq F$, then $0 = x \odot x^* \in F$, since $x \in F$. Thus $F = L$, which is a contradiction. Therefore $N(F) = F$. □

Example 3.15. In Example 3.2(ii), $N(F) = F$, for $F = \{b, 1\}$, and $x^2 = x$, for each $x \in L$. While $F \notin nod(F(L))$

Theorem 3.16. *Let $F \in F(L)$. Then $F \in nod(F(L))$ if and only if for each $x \in F$ and $y \notin F$, the relation $x > y^n$, for some $n \in N$, is satisfied.*

Proof. Let $F \in nod(F(L))$, $x \in F$ and $y \notin F$. Then $\langle x \rangle \subseteq F$ and $\langle y \rangle \nsubseteq F$. Since $F \in nod(F(L))$, we get $F \subset \langle y \rangle$. So, $x \in \langle y \rangle$, i.e., $x > y^n$, for some $n \in N$.

Conversely, let $F \notin nod(F(L))$. Then there exists $G \in F(L)$ such that $F \nsubseteq G$ and $G \nsubseteq F$. Hence there are $x, y \in L$ such that $x \in F - G$ and $y \in G - F$. So $x \in F$ and $y \notin F$. Thus by hypothesis $x > y^n$, for some $n \in N$. Since $y \in G$, we get $y^n \in G$, for all $n \in N$. And so $x \in G$, a contradiction. Therefore $F \in nod(F(L))$. □

Definition 3.17. (cf. [20]) By $ASF(L)$ we denote the set of all *associative filters* of L, i.e., subsets F such that

(i) $1 \in F$

(ii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$,

for all $x, y, z \in L$ such that $0 \neq x, z$ and x, y, z are not equal.

Example 3.18. Note that in Example 3.12 we have $F = \{a, b, 1\} \in ASF(L)$. In Example 3.2(i), $F = \{c, d, 1\} \notin ASF(L)$, because $a \to (a \to b) = 1 \in F$ and $a \to a = 1 \in F$ but $b \notin F$.

Theorem 3.19. $ASF(L) \subseteq F(L)$.

Proof. Let $F \in ASF(L)$ and $x, x \to y \in F$. Then $x \to (1 \to y) \in F$ and $x \to 1 = 1 \in F$. Since $F \in ASF(L)$, then $y \in F$. Therefore $F \in F(L)$. □

Theorem 3.20. *Let $F \in F(L)$. Then $F \in ASF(L)$ if and only if $x^n \to z \in F$ implies $z \in F$, for all $n \geqslant 2$.*

Proof. Let $x^n \to z \in F$, for $n \geqslant 2$. Then $x^{n-1} \to (x \to z) = x^n \to z \in F$ and $x^{n-1} \to x = 1 \in F$. Since $F \in ASF(L)$ we get $z \in F$.

Conversely, let $x \to (y \to z), x \to y \in F$. We must show that $z \in F$. By Proposition 2.1, we have $x \to (y \to z) = y \to (x \to z) \leqslant (x \to y) \to (x \to (x \to z))$. Hence $(x \to y) \to (x \to (x \to z)) \in F$ and by $x \to y \in F$, we get $x \to (x \to z) \in F$. Thus $x^2 \to z \in F$. By Proposition 2.1, we have $x^n \leqslant x^2$, for all $n \geqslant 2$. So $x^2 \to z \leqslant x^n \to z$. Hence $x^n \to z \in F$ and consequently $z \in F$. □

Theorem 3.21. *The following statements hold:*

(*i*) $ASF(L) \subseteq PIF(L)$,
(*ii*) $ASF(L) \subseteq IF(L)$,
(*iii*) $ASF(L) \subseteq FF(L)$,
(*iv*) $ASF(L) \subseteq NF(L)$,
(*v*) $ASF(L) \subseteq EF(L)$.

Proof. (*i*). Let $F \in ASF(L)$ and $(x \to y) \to x \in F$. We know $(x \to y) \to (x \to x) = 1 \in F$. Since $(x \to y) \to x \in F$ and $F \in ASF(L)$, then $x \in F$. Therefore $F \in PIF(L)$. So $ASF(L) \subseteq PIF(L)$.

$(ii), (iii), (iv)$. By Theorem 2.2$(ii), (iii), (iv)$, respectively, and part (i).

(v). By Theorem 2.2(v) and (ii). □

Example 3.22. In Example 3.2(i) we have $F = \{c, d, 1\} \notin ASF(L)$, while $F \in \{PIF(L), IF(L), NF(L), FF(L), EF(L)\}$.

Theorem 3.23. *Let $F \in F(L)$. Then $F \in ASF(L)$ if and only if*

$$x \to (y \to z) \in F \iff (x \to y) \to z \in F$$

for all $x, y, z \in L$ where $x, z \neq 0$ and x, y, z are not equal.

Proof. Let $F \in ASF(L)$ and $x \to (y \to z) \in F$. By Proposition 2.1, we get:

$$\begin{aligned}
1 = (y \to z) \to (y \to z) &\leqslant (y \to z) \to ((x \to y) \to (x \to z)) \\
&= (y \to z) \to (x \to ((x \to y) \to z)) \\
&= x \to ((y \to z) \to ((x \to y) \to z)).
\end{aligned}$$

So $x \to ((y \to z) \to ((x \to y) \to z)) = 1 \in F$ and we have $x \to (y \to z) \in F$. Hence by $F \in ASF(L)$ we get $(x \to y) \to z \in F$.

Now let $F \in ASF(L)$ and $(x \to y) \to z \in F$. By Proposition 2.1, we get:

$$\begin{aligned}
1 = (x \to y) \to (x \to (y \to 1)) &= (x \to y) \to (x \to (y \to (z \to z))) \\
&= (x \to y) \to (x \to (z \to (y \to z))) \\
&= (x \to y) \to (z \to (x \to (y \to z))).
\end{aligned}$$

So $(x \to y) \to (z \to (x \to (y \to z))) = 1 \in F$ and we have $(x \to y) \to z \in F$. Hence by $F \in ASF(L)$ we get $x \to (y \to z) \in F$.

Conversely, let $x \to (y \to z) \in F$ and $x \to y \in F$. By hypothesis we get $(x \to y) \to z \in F$. Since $x \to y \in F$, we get $z \in F$. Therefore $F \in ASF(L)$. □

Definition 3.24. (cf. [12]) By $STF(L)$ we denote the set of all *strong filters* of L, i.e., filters F such that $(x^{**} \to x)^{**} \in F$, for all $x \in L$.

Note that $F \subseteq G$ and $F \in STF(L)$ imply $G \in STF(L)$. Also, if $F \in STF(L)$ or $G \in STF(L)$ then $\langle F \cup G \rangle \in STF(L)$

Example 3.25. (i). In Example 3.2(i), $STF(L) = F(L)$.

(ii). Let $L = \{0, a, b, 1\}$, where $0 < b < a < 1$. L becomes a residuated lattice relative to the following operations:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	a	1
b	a	1	1	1
1	0	a	b	1

$\{1\} \notin STF(L)$, because $(b^{**} \to b)^{**} = a \notin \{1\}$.

Theorem 3.26. *The following statements hold:*

(i) $IF(L) \subseteq STF(L)$,

(ii) $PIF(L) \subseteq STF(L)$,

(iii) $OF(L) \subseteq STF(L)$,

(iv) $IF(L) \subseteq STF(L) \cap EF(L)$,

(v) $ASF(L) \subseteq STF(L)$,

(vi) $NF(L) \subseteq STF(L)$.

Proof. (i). Let $F \in IF(L)$ and $x, y \in L$. By (R_2), we have $x \leqslant x^{**} \to x$. So, by (R_{11}), $(x^{**} \to x)^* \leqslant x^*$, $(x^{**} \to x)^* \odot x^{**} \leqslant x^* \odot x^{**}$ and by (R_{18}),

$$(x^{**} \to x)^* \odot x^{**} = 0. \qquad \text{(I)}$$

We take $w = (x^{**} \to x)^*$. Then By Proposition 2.1 and (I), we get

$$\begin{aligned} 1 = 0 \to x &= ((x^{**} \to x)^* \odot x^{**}) \to x \\ &= (x^{**} \to x)^* \to (x^{**} \to x) \\ &\leqslant (x^{**} \to x)^* \to (x^{**} \to x)^{**} \\ &= (x^{**} \to x)^* \to ((x^{**} \to x)^* \to 0) \\ &= w \to (w \to 0). \end{aligned}$$

So $w \to (w \to 0) \in F$. We have $w \to w = 1 \in F$, thus by $F \in IF(L)$ we get $w \to 0 \in F$. Therefore $(x^{**} \to x)^{**} \in F$, i.e., $F \in STF(L)$.

$(ii), (iii)$. By Theorem 2.2(ii) and (i).

(iv). By Theorem 2.2(v) and (i).

(v). By Theorem 3.21(i) and (ii).

(vi). By Theorem 2.2(vi) and (i), (ii). $\square$

Example 3.27. In Example 3.2(i), $\{c, d, 1\} \in STF(L)$ but $\{c, d, 1\} \notin ASF(L)$. In Example 3.2($ii$), $F = \{b, 1\} \in STF(L)$ but $F \notin NF(L), PIF(L), OF(L)$. We have $1 = c^{**} \in F$ but $c \notin F$. So $F \notin NF(L)$. Hence by Theorem 2.2(iv), (ii), $F \notin PIF(L), OF(L)$.

Lemma 3.28. *Let $F \in F(L)$. Then $F \in STF(L)$ if and only if $((x^{**})^n \to x)^{**} \in F$, for all $x \in L$ and for all $n \in N$.*

Proof. By Proposition 2.1, we have $(x^{**})^n \leqslant x^{**}$. Then $x^{**} \to x \leqslant (x^{**})^n \to x$, so by (R_{11}) we get $(x^{**} \to x)^{**} \leqslant ((x^{**})^n \to x)^{**}$. Let $F \in STF(L)$ so $(x^{**} \to x)^{**} \in F$, hence $((x^{**})^n \to x)^{**} \in F$.

Conversely, the proof is easy. □

Definition 3.29. A residuated lattice L is called *strong residuated lattice* if $(x^{**} \to x)^* = 0$, for all $x \in L$.

All of BL-algebras are strong residuated lattice. A residuated lattice from Example 3.25(ii) is not strong, because $(b^{**} \to b)^* = a \neq 0$.

Theorem 3.30. *The following are equivalent on L:*

(i) *L is a strong residuated lattice,*

(ii) *$STF(L) = F(L)$,*

(iii) *$\{1\} \in STF(L)$.*

Proof. (i) $\Rightarrow$ (ii). We have $(x^{**} \to x)^* = 0$, for all $x \in L$. So $(x^{**} \to x)^{**} = 1 \in F$, for all $x \in L$ and for every $F \in F(L)$. Hence $F \in STF(L)$ for every $F \in F(L)$.

(ii) $\Rightarrow$ (iii). The proof is clear.

(iii) $\Rightarrow$ (i). We have $(x^{**} \to x)^{**} = 1 \in \{1\}$, for all $x \in L$. Then $(x^{**} \to x)^* = 0$, for all $x \in L$, i.e., L is a strong residuated lattice. □

Theorem 3.31. *Let $F \in F(L)$. Then $F \in STF(L)$ if and only if L/F is a strong residuated lattice.*

Proof. Let $F \in F(L)$. We have

$$\begin{aligned} F \in STF(L) &\Leftrightarrow (x^{**} \to x)^{**} \in F, \text{ for all } x \in L; \\ &\Leftrightarrow (x^{**} \to x)^{**}/F = 1/F, \text{ for all } x/F \in L/F; \\ &\Leftrightarrow (x^{**}/F \to x/F)^{**} = 1/F, \text{ for all } x/F \in L/F; \\ &\Leftrightarrow (x^{**}/F \to x/F)^{*} = 0/F, \text{ for all } x/F \in L/F; \\ &\Leftrightarrow L/F \text{ is a strong residuated lattice.} \end{aligned}$$

□

Lemma 3.32. *L is a strong residuated lattuce if and only if $((x^{**})^n \to x)^* = 0$, for all $x \in L$ and for all $n \in N$.*

Proof. By Proposition 2.1 we have $(x^{**})^n \leqslant x^{**}$. Then $x^{**} \to x \leqslant (x^{**})^n \to x$, so by (R_{11}) we get $((x^{**})^n \to x)^* \leqslant (x^{**} \to x)^*$. By the fact that L is a strong residuated lattice, $(x^{**} \to x)^* = 0$, so $((x^{**})^n \to x)^* = 0$.

Conversely, the proof is easy. □

Definition 3.33. (cf. [5]) By $INF(L)$ the set of all *integral filters* of L, i.e., proper filters F of L such that for all $x, y \in L$, $(x \odot y)^* \in F$ implies $x^* \in F$ or $y^* \in F$.

Example 3.34. In Example 3.2(ii), $F = \{a, 1\} \in INF(L)$. In $L = \{0, a, b, c, d, 1\}$, with $0 < a < c < 1$, $0 < b < c, d < 1$ and operations $\odot$ and $\rightarrow$ defined by:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$F = \{d, 1\} \notin INF(L)$, since $(b \odot b)^* = 1 \in F$ but $b^* = c \notin F$.

Theorem 3.35. *Let $F, G \in F(L)$ be proper filters of L, such that $F \subseteq G$. If $F \in INF(L)$, then $G \in INF(L)$.*

Proof. Let $(x \odot y)^* \in G$, for $x, y \in L$. We know that $((x \odot y) \odot (x \odot y)^*)^* = 1 \in F$. Since $F \in INF(L)$, then $(x \odot y)^* \in F$ or $(x \odot y)^{**} \in F$.

If $(x \odot y)^* \in F$ then $x^* \in F$ or $y^* \in F$. Hence $x^* \in G$ or $y^* \in G$ i.e., $G \in INF(L)$. If $(x \odot y)^{**} \in F$, then $(x \odot y)^{**} \in G$. So by $(x \odot y)^* \in G$, we have $0 = (x \odot y)^* \odot (x \odot y)^{**} \in G$. And so $G = L$, that is a contradiction. □

Theorem 3.36. *The following statements hold.*

(i) $INF(L) \subseteq PF(L)$,

(ii) $OF(L) \subseteq INF(L)$,

(iii) $INF(L) \cap NF(L) = OF(L)$.

Proof. (i). It is clear.

(ii). Let $F \in OF(L)$ and $(x \odot y)^* \in F$, but $x^*, y^* \notin F$, for $x, y \in L$. Since $F \in OF(L)$, then $x^{**}, y^{**} \in F$. By Theorem 2.2$(ii), (iv)$, we know that $OF(L) \subseteq NF(L)$, so $F \in NF(L)$. Hence $x, y \in F$. Then $x \odot y \in F$, and so $0 = (x \odot y) \odot (x \odot y)^* \in F$, which is a contradiction because $F \in OF(L)$. Therefore $x^* \in F$ or $y^* \in F$.

(iii). Let $F \in INF(L) \cap NF(L)$ and $x \notin F$. It is enough to show that $x^* \in F$. We have $(x \odot x^*)^* = 1 \in F$. Since $F \in INF(L)$, then $x^* \in F$ or $x^{**} \in F$. Let $x^{**} \in F$. By $F \in NF(L)$, we get $x \in F$, which is a contradiction. And so $x^* \in F$, i.e., $F \in OF(L)$. Therefore $INF(L) \cap NF(L) \subseteq OF(L)$.

Conversely, by Theorem 2.2$(ii), (iv)$, we have $OF(L) \subseteq NF(L)$. Hence by (ii), the proof is complete. □

Example 3.37. In the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example 3.34 $F = \{d, 1\} \in PF(L)$, but $F \notin INF(L)$. In Example 3.2(ii). $F = \{a, 1\} \in INF(L)$, but $F \notin OF(L)$. Because $b, b^* \notin F$.

Definition 3.38. L is called an *integral residuated lattice*, if $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$.

The residuated lattice in Example 3.2(ii) is an integral residuated lattice, but $L = \{0, a, b, c, d, 1\}$ from Example 3.34 is not an integral residuated lattice, because $a \odot b = 0$, for $a, b \neq 0$.

Theorem 3.39. *Let F be a proper filter of L. Then L/F is an integral residuated lattice if and only if $F \in INF(L)$.*

Proof. Let L/F be an integral residuated lattice and $(x \odot y)^* \in F$, for $x, y \in L$. Then $(x \odot y)/F = 0/F$. Since L/F is an integral residuated lattice then $x/F = 0/F$ or $y/F = 0/F$. Hence $x^* \in F$ or $y^* \in F$, i.e., $F \in INF(L)$.

Conversely, let $F \in INF(L)$ and $x/F \odot y/F = 0/F$, for $x/F, y/F \in L/F$. Then $(x \odot y)/F = 0/F$, i.e $(x \odot y)^* \in F$. Since $F \in INF(L)$, then $x^* \in F$ or $y^* \in F$. And so $x/F = 0/F$ or $y/F = 0/F$. Therefore L/F is an integral residuated lattice. □

Corollary 3.40. *Let F be a proper filter of L.*

(i) *If $F \in IF(L) \cap Max(L)$. then L/F is an integral residuated lattice.*

(ii) *If $F \in PIF(L) \cap Max(L)$, then L/F is an integral residuated lattice.*

(iii) *If $F \in OF(L)$, then L/F is an integral residuated lattice.*

Proof. By Theorem 3.13 in [5] and our Theorem 3.39. □

Theorem 3.41. *The following conditions are equivalent:*

(i) *L is an integral residuated lattice,*

(ii) *$\{1\} \in INF(L)$,*

(iii) *$F(L) = INF(L)$.*

Proof. (i) $\Rightarrow$ (ii). Let L be an integral residuated lattice. Then $L/\{1\}$ is an integral residuated lattice. So, by Theorem 3.39, $\{1\} \in INF(L)$.

(ii) $\Rightarrow$ (iii). Let $\{1\} \in INF(L)$. Then by Theorem 3.35, the proof is clear.

(iii) $\Rightarrow$ (i). Let $F(L) = INF(L)$. Then $\{1\} \in INF(L)$. Hence by Theorem 3.39, $L/\{1\}$ is an integral residuated lattice. Therefore L is an integral residuated lattice. □

Theorem 3.42. *Let $F \in F(L)$ be a proper and for each $a \in L$, $a^2 = a$. Then*

(i) *$F \in PF(L)$ if and only if L/F is an integral residuated lattice,*

(ii) *L is an integral residuated lattice if and only if L is a local residuated lattice,*

(iii) *$INF(L) = PF(L)$.*

Proof. (i). Let $F \in PF(L)$ and $x/F \odot y/F = 0/F$, for $x/F, y/F \in L/F$. Then $(x \odot y)/F = 0/F$, i.e., $(x \odot y)^* \in F$. So there exist $m, n \in N$ such that $(x^n)^* \in F$ or $(y^m)^* \in F$. By hypothesis we get $x^* \in F$ or $y^* \in F$. Hence $x/F = 0/F$ or $y/F = 0/F$, i.e., L/F is an integral residuated lattice.

Conversely, by Theorem 3.39 and Theorem 3.36(i) the proof is clear.

(ii). Let L be an integral residuated lattice. By Theorem 3.41, $\{1\} \in INF(L)$ hence by Theorem 3.36(i), $\{1\} \in PF(L)$. So $L/\{1\}$ is a local residuated lattice, i.e., L is a local residuated lattice.

Conversely, let L be a local residuated lattice. Then $L/\{1\}$ is a local residuated lattice. Hence $\{1\} \in PF(L)$, and by (i), $L/\{1\}$ is an integral residuated lattice. So L is an integral residuated lattice.

(iii). By (i) and Theorem 3.39, respectively, we have $F \in PF(L)$ if and only if L/F is an integral residuated lattice if and only if $F \in INF(L)$. □

Example 3.43. Consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ in Example 3.34. $F = \{d, 1\} \in PF(L)$, hence L/F is a local residuated lattice. But L/F is not an integral residuated lattice, since $c/F \odot c/F = a/F = 0/F$, but $c/F \neq 0/F$ ($c/F = b/F$).

4. Positive implicative filters in residuated lattices

We start with

Theorem 4.1. *The following conditions are equivalent:*

(i) $F(L) = PIF(L)$,

(ii) $\{1\} \in PIF(L)$,

(iii) $(x \to y) \to x = x$, *for all* $x, y \in L$,

(iv) *L is a Boolean algebra,*

(v) $x^* \to x = x$, *for all* $x \in L$,

(vi) $((x \to y) \to x) \to x = 1$, *for all* $x, y \in L$.

Proof. By Proposition 25 in [8], we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (v). Let $(x \to y) \to x = x$, for all $x, y \in L$. Take $y = 0$, so $(x \to 0) \to x = x$, for all $x \in L$. Thus $x^* \to x = x$, for all $x \in L$.

(v) $\Rightarrow$ (iii). Let $x^* \to x = x$, for all $x \in L$. By Proposition 2.1 we have $0 \leqslant y$ then $x \to 0 \leqslant x \to y$ so $(x^* \to x) \to x \leqslant ((x \to y) \to x) \to x$. By hypothesis we get $(x^* \to x) \to x = 1$ and so $((x \to y) \to x) \to x = 1$, for all $x, y \in L$. Hence by Proposition 2.1 we get $(x \to y) \to x = x$, for all $x, y \in L$.

(iii) $\Rightarrow$ (vi). Let $(x \to y) \to x = x$, for all $x, y \in L$. Then $((x \to y) \to x) \to x = 1$, for all $x, y \in L$.

(vi) $\Rightarrow$ (iii). Let $((x \to y) \to x) \to x = 1$, for all $x, y \in L$. Then we have $(x \to y) \to x \leqslant x$. Also $x \leqslant (x \to y) \to x$, by Proposition 2.1. Therefore $(x \to y) \to x = x$, for all $x, y \in L$. □

Theorem 4.2. *Let $F \in F(L)$. The following conditions are equivalent.*

(i) $F \in PIF(L)$,

(ii) $((x \to y) \to x) \to x \in F$, *for all* $x, y \in L$,

(iii) $(x^* \to x) \to x \in F$, *for all* $x \in L$,

(iv) $((x^n \to y) \to x) \to x \in F$, *for all* $x, y \in L$ *and for all* $n \in N$.

Proof. $(i) \Rightarrow (ii)$. Let $F \in PIF(L)$. Then by Proposition 27[8], L/F is a Boolean algebra. Then by Theorem 4.1 $(iv) \Leftrightarrow (vi)$, we get $(((x \to y) \to x) \to x)/F = 1/F$, for all $x/F, y/F \in L/F$. Hence $((x \to y) \to x) \to x \in F$, for all $x, y \in L$.

$(ii) \Rightarrow (i)$. Let $(x \to y) \to x \in F$. By hypothesis $((x \to y) \to x) \to x \in F$, for all $x, y \in L$, so we get $x \in F$. Therefore $F \in PIF(L)$.

$(ii) \Rightarrow (iii)$. In (ii), take $y = 0$, so $(x^* \to x) \to x \in F$.

$(iii) \Rightarrow (ii)$. Let $(x^* \to x) \to x \in F$, for each $x \in L$. By Proposition 2.1 we have $0 \leqslant y$ then $x \to 0 \leqslant x \to y$ so $(x^* \to x) \to x \leqslant ((x \to y) \to x) \to x$. Hence $((x \to y) \to x) \to x \in F$.

$(ii) \Rightarrow (iv)$. Let $((x \to y) \to x) \to x \in F$, for all $x, y \in L$. By Proposition 2.1 we have $x \to y \leqslant x \to (x \to y)$ then

$$\begin{aligned} ((x \to y) \to x) \to x &\leqslant ((x \to (x \to y)) \to x) \to x \\ &\leqslant (x \to (x \to (x \to y)) \to x) \to x \\ &= ((x^3 \to y) \to x) \to x = \dots \\ &= ((x^n \to y) \to x) \to x. \end{aligned}$$

Therefore $((x^n \to y) \to x) \to x \in F$, for all $x, y \in L$.

$(iv) \Rightarrow (ii)$ In (iv), take $n = 1$. □

By Theorem 4.2, it is clear that extension property hold for positive implicative filters.

Corollary 4.3. *L is a boolean algebra if and only if $(x^n \to y) \to x = x$, for all $x, y \in L$ and for all $n \in N$.*

Proof. Let L be a boolean algebra. By Theorem 4.1, $\{1\} \in PIF(L)$. So by Theorem 4.2 we have $((x^n \to y) \to x) \to x \in \{1\}$, for all $x, y \in L$ and for all $n \in N$. By Proposition 2.1, we get $(x^n \to y) \to x = x$, for all $x, y \in L$ and for all $n \in N$.

Conversely, by Theorem 4.1 the proof is easy. □

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S. Motamed
Department of Mathematics, Bandar Abbas Branch, Islamic Azad University Bandar Abbas, Iran
E-mails: s.motamed63@yahoo.com , somayeh.motamed@iauba.ac.ir

J. Moghaderi
Department of Mathematics, University of Hormozgan, Bandar Abbas, Iran
E-mails: j.moghaderi@hormozgan.ac.ir, j.moghaderi@gmail.com