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A study on covered lateral ideals of ordered ternary semigroups

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Abstract. We define covered lateral ideals of ordered ternary semigroups and study their properties.

1. Introduction and preliminaries

Kasner's [4] gave the idea of n-ary algebras i.e., the sets with one n-ary operation. Algebras with one 3-ary associative operation are known as ternary semigroups. Ideals in ternary semigroup was studied by Sioson [5]. Fabrici [3] showed some properties and the relation between covered ideals and bases of semigroups. Changphas and Summaprab [1] studied ordered semigroup containing covered ideals. Iampan [3] gave the definition of ordered ternary semigroup and characterized the minimality and maximality concept in ordered ternary semigroups.

For simplicity, a ternary operation [ ] will be identified with a multiplication of three elements, i.e., $[x, y, z]$ will be identified with xyz.

Definition 1.1. A ternary semigroup T is called a *partially ordered ternary semigroup* if there exists a partially ordered relation $\leqslant$ such that for any $a, b, x, y \in T$, $a \leqslant b \Rightarrow axy \leqslant bxy, xay \leqslant xby$, and $xya \leqslant xyb$.

For $H \subseteq T$, we put $(H] = \{s \in T \mid s \leqslant h, \text{ for some } h \in H\}$.

Theorem 1.2. (cf. [3]) *In an ordered ternary semigroup T the following hold:*

1. $A \subseteq (A]$ *and* $((A]] = (A]$*, for all* $A \subseteq T$.
2. *If* $A \subseteq B \subseteq T$*, then* $(A] \subseteq (B]$.
3. $(A](B](C] \subseteq (ABC]$*, for all* A*,* B*,* $C \subseteq T$.

Definition 1.3. A *lateral ideal* M of an ordered ternary semigroup T, i.e., a nonempty subset M of T such that $TMT \subseteq M$, is called an *ordered lateral ideal* of T if for any $b \in T$ and $a \in M$, $b \leqslant a$ implies $b \in M$. If T has no proper lateral ideals, then it is *lateral simple*.

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Lemma 1.4. (cf. [3]) *For any non-empty subset A of an ordered ternary semigroup T, $(TTATT \cup TAT \cup A]$ is the smallest lateral ideal of T containing A. Furthermore, for any $a \in T$, $M(a) = (TTaTT \cup TaT \cup a]$.*

2. Covered lateral deals

Definition 2.1. A proper lateral ideal L of an ordered ternary semigroup T is called a *covered lateral ideal* (CLt-ideal) if $L \subset (T(T-L)T \cup TT(T-L)TT]$.

Lemma 2.2. *If an ordered ternary semigroup T contains two different lateral ideals L_1 and L_2 such that $L_1 \cup L_2 = T$, then L_1, L_2 are not CLt-ideals.*

Proof. We have $L_1 \cup L_2 = T$, it implies T - $L_1 \subset L_2$ and T - $L_2 \subset L_1$. Suppose that L_2 is a covered lateral ideal of T, then $L_2 \subset (T(T-L_2)T \cup TT(T-L_2)TT]$ which implies $L_2 \subset (T(T-L_2)T \cup TT(T-L_2)TT] \subset (TL_1T \cup TTL_1TT] \subset (TL_1T \cup TL_1T] \subseteq L_1$. Similarly $L_1 \subset L_2$. Therefore $L_1 = L_2$. But L_1 and L_2 are different. Thus our assumption is wrong. Hence neither L_1 nor L_2 is a CLt-ideal. □

Corollary 2.3. *If an ordered ternary semigroup T contains more than one maximal lateral ideal, then maximal lateral ideals are not CLt-ideals.*

Proof. Suppose that T contains two maximal lateral ideals M_1 and M_2. We know that union of lateral ideals is a lateral ideal. Then $M_1 \cup M_2$ is a lateral ideal of T and $M_1 \subset M_1 \cup M_2$. As M_1 is a maximal lateral ideal of T. It implies $M_1 \cup M_2 = T$. Hence by Lemma 2.2, neither M_1 nor M_2 is a CLt-ideal of T □

Lemma 2.4. *If L is a lateral ideal of T such that $L \subset (TtT \cup TTtTT]$ and $L \neq (TtT \cup TTtTT]$ for some $t \in T$. Then L will be a CLt-ideal of T.*

Proof. Suppose that L is a lateral ideal of T such that $L \subset (TtT \cup TTtTT]$ and $L \neq (TtT \cup TTtTT]$ for some $t \in T$. Here $t \notin L$, otherwise $(TtT \cup TTtTT] \subseteq (TLT \cup TTLTT] \subseteq L$ and we assume that $L \neq (TtT \cup TTtTT]$. Hence $L \subset (TtT \cup TTtTT] \subset (T(T-L)T \cup TT(T-L)TT]$. Therefore, L is a CLt-ideal of T. □

Corollary 2.5. *An ordered ternary semigroup T in which t does not belongs to $(TtT \cup TTtTT]$ contains CLt-ideal.*

Proof. Let $L = (TtT \cup TTtTT]$. Then L is a lateral ideal of T. If $t \notin L$, we have $L = (TtT \cup TTtTT] \subset (T(T-L)T \cup TT(T-L)TT]$. This implies L is a CLt-ideal of T. □

Lemma 2.6. *If L_1 and L_2 are two covered lateral ideals of an ordered ternary semigroup T. Then $L_1 \cup L_2$ is a CLt-ideal of T.*

Proof. To prove $L_1 \cup L_2$ is a CLt-ideal of T. We have to show that $L_1 \cup L_2 \subset (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT]$. As L_1 is a CLt-ideal i.e. $L_1 \subset (T(T-L_1)T \cup TT(T-L_1)TT]$, which implies for any $m \in L_1$, there exists $m_1, m_2 \in T - L_1$ such that $m \in (Tm_1T \cup TTm_2TT]$. Now we have following four cases:

1. If $m_1, m_2 \in (T-L_1)-L_2$. Then $m \in (T((T-L_1)-L_2)T \cup TT((T-L_1)-L_2)TT] \subseteq (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT]$.

2. If $m_1, m_2 \in (T-L_1) \cap L_2$, then $m_1, m_2 \in L_2 \subset (T(T-L_2)T \cup TT(T-L_2)TT]$. Then there exists $m_3, m_4, m_5, m_6 \in T-L_2$ s.t. $m_1 \in (Tm_3T \cup TTm_4TT]$ and $m_2 \in (Tm_5T \cup TTm_6TT]$. Here $m_3, m_4 \notin L_1$, otherwise $m_1 \in (Tm_3T \cup TTm_4TT] \subseteq (TL_1T \cup TTL_1TT] \subseteq L_1$. Hence $m_1 \in L_1$, which is contradiction as $m_1 \in T - L_1$. Thus we have $m_3, m_4 \in T - L_1$. Therefore $m_3, m_4 \in T - L_1 \cap T - L_2 = T - (L_1 \cup L_2)$. Similarly $m_5, m_6 \in T - (L_1 \cup L_2)$. Now

$$\begin{aligned}
m \in\ & (Tm_1T \cup TTm_2TT] \\
& \subset (T(Tm_3T \cup TTm_4TT]T \cup TT(Tm_5T \cup TTm_6TT]TT] \\
& \subseteq ((T](Tm_3T \cup TTm_4TT](T] \cup (T](T](Tm_5T \cup TTm_6TT](T](T]] \\
& \subseteq ((T(Tm_3T \cup TTm_4TT)T \cup TT(Tm_5T \cup TTm_6TT)TT]] \\
& = (T(Tm_3T \cup TTm_4TT)T \cup TT(Tm_5T \cup TTm_6TT)TT] \\
& \subseteq (TTm_3TT \cup Tm_4T \cup Tm_5T \cup TTm_6TT] \\
& \subset (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT].
\end{aligned}$$

3. If $m_1 \in (T-L_1)-L_2$ and $m_2 \in (T-L_1) \cap L_2$. As $m_1 \in (T-L_1)-L_2$, it implies $m_1 \in T-(L_1 \cup L_2)$. From case 2, $m_2 \in T-(L_1 \cup L_2)$. Therefore we have $m \in (Tm_1T \cup TTm_2TT] \subset (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT]$.

4. If $m_2 \in T - L_1 - L_2$ and $m_1 \in (T-L_1) \cap L_2$. Then this is similar to case 3.

Therefore in all these cases $m \in (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT]$. Similarly we can prove this for $m \in L_2$. Thus $L_1 \cup L_2 \subset (T[T-(L_1 \cup L_2)]T \cup TT[T-(L_1 \cup L_2)]TT]$ and hence $L_1 \cup L_2$ is a CLt-ideal of T. □

Lemma 2.7. *If L_1 is a covered and L_2 is lateral ideal of an ordered ternary semigroup T. Then $L_1 \cap L_2$ is a CLt-ideal of T, provided $L_1 \cap L_2 \neq \emptyset$.*

Proof. Assume that L_1 is a covered lateral ideal and L_2 is a lateral ideal of T such that $L_1 \cap L_2 \neq \emptyset$. Then $L_1 \subset (T(T-L_1)T \cup TT(T-L_1)TT]$. It implies $L_1 \cap L_2 \subset (T(T-L_1)T \cup TT(T-L_1)TT] \subset (T[T-(L_1 \cap L_2)]T \cup TT[T-(L_1 \cap L_2)]TT]$. Therefore $L_1 \cap L_2$ is a CLt-ideal of T. □

Theorem 2.8. *If T is not a simple ordered ternary semigroup such that there is no any two proper lateral ideals in which there intersection is empty. Then T contains at least one CLt-ideal.*

Proof. Let M be a proper lateral ideal of T. Then $M_1 = (T(T-M)T \cup TT(T-M)TT]$ is also a lateral ideal of T. By assumption $M \cap M_1 \neq \emptyset$. Thus $M_c = M \cap M_1$ is a lateral ideal of T and $M_c \subset M$, it implies $T - M_c \supset T - M$. Now we have, $M_c \subset M_1 = (T(T-M)T \cup TT(T-M)TT] \subset (T(T-M_c)T \cup TT(T-M_c)TT]$. This shows that M_c is a CLt-ideal of T. □

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