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Some new approaches on prime and composite order Cayley graphs

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Abstract. Some of the main graph theoretical properties of prime and composite order Cayley graphs of $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$ are studied. For instance, the structure of these two classes of graphs are clarified completely and some of the topological indices of them are obtained.

1. Introduction

The Cayley graph was first introduced for finite groups by Arthur Cayley in 1878. Max Dehn reintroduced Cayley graphs under the name Gruppenbild (group diagram) in his unpublished lectures on group theory, which led to the geometric group theory of today. Let G be a group, and let $S \subseteq G$ be a set of group elements such that the identity element does not belong in S. The *Cayley graph* associated with (G, S) is defined as the directed graph with a vertex associated to each group element and directed edges (g, h) whenever $gh^{-1} \in S$. The Cayley graph may depend on the choice of a generating set, and it is connected if and only if S generates G. If $S = S^{-1}$, then the Cayley graph is undirected which is our favorite graph in this research.

B. Tolue defined the prime and composite order Cayley graphs in [10, 11] and she discussed about some of their properties. For instance, the structure of them for some certain groups were achieved. The planarity of prime order Cayley graph of abelian groups and composite order Cayley graph were presented. Moreover, some graph parameters such as diameter, girth, clique number, independence number, vertex chromatic number and domination number are calculated for the composite order Cayley graph of some certain groups. In this research we present some more new results about them which are complementary to the older ones, so let us recall the following definitions.

Definition 1.1. Let G be a group and S be the set of non-identity prime order elements of G. Consider the Cayley graph $Cay_p(G, S)$ associated to the group G relative to S. We call it a *prime order Cayley graph.*

Now, if we change the subset S to the composite order elements of the group G, then we have the complement of the prime order Cayley graph, which is define as follows.

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Definition 1.2. The *composite order Cayley graph* which is assigned to the group G is a graph with vertex set whole elements of the group G and two distinct vertices x and y are adjacent whenever $xy^{-1} \in S$, where S is a subset of G which contains all elements of composite order. We denote the composite order Cayley graph by $Cay_c(G, S)$.

In the next section, we discussed about the planarity, adjacency matrix, energy, dominating polynomial and independent dominating polynomial of prime order Cayley graph of $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$. We observe $\mathrm{Cay}_p(\mathbb{Z}_{p^{n_1}}, S_1) \times \cdots \times \mathrm{Cay}_p(\mathbb{Z}_{p^{n_t}}, S_t)$ is induced subgraph of $\mathrm{Cay}_p(\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}, S)$. Also in the third section properties of the composite order Cayley graph are studied.

Let Γ be a finite, undirected graph with vertex and edge sets $V(\Gamma)$ and $E(\Gamma)$. For $u, v \in V(\Gamma)$, the minimal path-length distance between u and v is denoted by $d(u, v)$. For the edge $e = \{u, v\}$ of Γ, define

$$N_u(e) = \{x \in V(\Gamma) : d(u, x) < d(v, x)\}$$
$$N_v(e) = \{x \in V(\Gamma) : d(v, x) < d(u, x)\},$$

The size of each sets is denoted by $n_u(e)$ and $n_v(e)$, respectively. If for every edge $e = \{u, v\}$ of the graph Γ, $n_u(e) = n_v(e)$, then Γ is called a *vertex distance-balanced graph*. Moreover, the number of edges which are closer to u than v is denoted by $m_u(e)$. The graph Γ is called *edge distance-balanced graph*, if for every edges of the graph like $e = \{u, v\}$, we have $m_u(e) = m_v(e)$ (see [7] for more details). We have observed that the prime order Cayley graph associated to an elementary abelian p-group and the composite order Cayley graph of $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$ is vertex and edge distance-balanced graph.

For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph Γ, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is a resolving set (or locating set) for Γ if every two vertices of Γ have distinct representations. The metric dimension of Γ is the minimum cardinality of a resolving set for Γ. In the sequel, we have obtained the metric dimension for the prime order Cayley graph of elementary abelian p-group and the composite order Cayley graph of $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$.

In chemical graph theory, the Wiener index is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the molecule. More precisely, for the connected graph Γ, the Wiener index $W(\Gamma) = 1/2 \sum_{(u,v) \in V(\Gamma) \times V(\Gamma)} d(u, v)$. The Wiener index is named after Harry Wiener, who introduced it in 1947. Based on its success, many other topological indices were introduced. The first and second Zagreb index are important topological indices which are defined by Gutman and Trinajestic [6]. The graph invariants $M_1(\Gamma)$ and $M_2(\Gamma)$, known as first and second Zagrebs equal to $\sum_{v \in V(\Gamma)} (\deg(v))^2$ and $\sum_{\{u,v\} \in E(\Gamma)} \deg(u)\deg(v)$, respectively. In the following, we compute the first and second Zagreb indices for the prime and composite order Cayley graph of $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$.

Another topological index is eccentric connectivity which is defined by $\xi^c(\Gamma) = \sum_{u\in V(\Gamma)} \deg(u)\varepsilon_\Gamma(u)$, where $\varepsilon_\Gamma(u)$ is the eccentric of the vertex u in the graph Γ. The eccentric connectivity of the prime and composite order graph of elementary abelian p-group and $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$ is achieved, respectively.

The vertex and edge Padmakar-Ivan indices was defined by Padmakar V. Khadikar, which has the usage in chemical sciences. The vertex Padmakar-Ivan and edge Padmakar-Ivan indices of the graph Γ are $PI_v(\Gamma) = \Sigma_{e=\{u,v\}\in E(\Gamma)}(n_u(e)+n_v(e))$ and $PI_e(\Gamma) = \Sigma_{e=\{u,v\}\in E(\Gamma)}(m_u(e)+m_v(e))$, respectively [8]. We compute the vertex and edge Padmakar-Ivan indices for some certain prime and composite order graphs.

Throughout the paper, all the notations and terminologies about the graphs are found in [2, 4].

2. The prime order Cayley graph

Let p be a prime number. It is clear that, if G is an elementary abelian p-group of order p^t, then the prime order Cayley graph, $\mathrm{Cay}_p(G,S)$ is a complete graph. Therefore, it is (p^t-1) regular and $\omega(\mathrm{Cay}_p(G,S)) = \chi(\mathrm{Cay}_p(G,S)) = p^t$. It is vertex and edge distance balanced, also vertex and edge transitive. Moreover, its metric dimension is p^t-1. The prime order Cayley graph of elementary abelian p-groups is planar if and only if $G \cong \mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_2\times\mathbb{Z}_2$.

The Wiener index, first and second Zagreb index, the eccentric connectivity, vertex and edge Padmakar-Ivan index were obtained for the complete graphs so the proof of the following result is straightforward and we omit it.

Proposition 2.1. *Let G be an elementary abelian group of order p^t. Then*

(*i*) *The Wiener index is $W(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)/2$*

(*ii*) *The first Zagreb index is $M_1(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)^2$.*

(*iii*) *The second Zagreb index is $M_2(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)^3/2$.*

(*iv*) *The eccentric connectivity is $\xi^c(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)$.*

(*v*) *The vertex and edge Padmakar-Ivan indices are $PI_v(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)^2$ and $PI_e(\mathrm{Cay}_p(G,S)) = p^t(p^t-1)(p^t-2)$, respectively.*

Assume $G = \mathbb{Z}_{p^n}\times\mathbb{Z}_{p^m}$. Then the set of all prime order elements is $S = \{(x,y)\in G : x = kp^{n-1},\ y = k'p^{m-1},\ 0 \leqslant k,k' \leqslant p-1\}$. Therefore $\mathrm{Cay}_p(G,S)$ is (p^2-1) regular. If (x_1,y_1) and (x_2,y_2) are two adjacent vertices, then $x_2 = x_1 - kp^{n-1}$ and $y_2 = y_1 - k'p^{m-1}$, where $0 \leqslant k,k' \leqslant p-1$. It is obvious that all vertices which join (x_2,y_2) has the form $(x_2 - lp^{n-1}, y_2 - qp^{m-1})$, where $0 \leqslant l,q \leqslant p-1$. Hence $\mathrm{Cay}_p(G,S)$ is disjoint union of $(n+m-2)$ complete components of size p^2.

The subset of vertices T is called a *dominating set* for the graph Γ, if every vertices outside of T join to at least one vertex of T. The size of the smallest dominating set is called the domination number and is denoted by $\gamma(\Gamma)$. A set of vertices is an independent set if no two vertices in it are adjacent. The maximum size of the independent set is called an *independence number* of the graph. If T is an independent set, then it is called an *independent dominating set* and the size of the smallest is denoted by $\gamma_i(\Gamma)$.

Saeid Alikhani and Peng, Y.H. [1], have introduced the domination polynomial of a graph. Then domination polynomial of Γ is $D(\Gamma,x)=\Sigma_{i=\gamma(\Gamma)}^{|V(\Gamma)|}d(\Gamma,i)x^i$, where $d(\Gamma,i)$ is the number of the dominating sets of size i. After that P.M. Shivaswamy et al. defined independent polynomial by $D_i(\Gamma,x)=\Sigma_{j=\gamma_i(\Gamma)}^{|V(\Gamma)|}d_i(\Gamma,j)x^j$, where $d_i(\Gamma,j)$ is the number of the independent dominating sets of size j [9]. There are other graph polynomials too which are powerful and well-developed tools to express graph parameters.

Proposition 2.2. *Let $G=\mathbb{Z}_{p^{n_1}}\times\cdots\times\mathbb{Z}_{p^{n_t}}$, where n_i are positive integers and p is prime number.*

(*i*) $\mathrm{Cay}_p(G,S)$ *is* (p^t-1) *regular.*

(*ii*) $\mathrm{Cay}_p(G,S)$ *is disjoint union of* $p^{(\Sigma_{i=1}^t n_i)-t}$ *complete components of size* p^t.

(*iii*) $\mathrm{Cay}_p(G,S)$ *is planar if and only if* $G\cong\mathbb{Z}_{2^{n_1}},\ \mathbb{Z}_{2^{n_1}}\times\mathbb{Z}_{2^{n_2}},\ \mathbb{Z}_3$, *where* $1\leqslant n_1,n_2\leqslant 2$.

(*iv*) $\omega(\mathrm{Cay}_p(G,S))=\chi(\mathrm{Cay}_p(G,S))=p^t$ *and* $\alpha(\mathrm{Cay}_p(G,S))=p^{(\Sigma_{i=1}^t n_i)-t}$.

(*v*) *The adjacency matrix of* $\mathrm{Cay}_p(G,S)$ *is* $l\times l$ *matrix where* $l=\prod_{i=1}^t p^{n_i}$, *it contains* $p^t\times p^t$ *sub-matrix on the main diagonal such that all their components are one except the diagonal which is zero, and the components out of this sub-matrices are zero.*

(*vi*) *The energy of the graph with respect to the adjacency matrix is equal to* $2p^{(\Sigma_{i=1}^t n_i)-t}(p^t-1)$.

(*vii*) $\gamma(\mathrm{Cay}_p(G,S))=\gamma_i(\mathrm{Cay}_p(G,S))=p^{(\Sigma_{i=1}^t n_i)-t}$.

(*viii*) *The dominating polynomial of the graph* $\mathrm{Cay}_p(G,S)$ *is,*

$$D(\mathrm{Cay}_p(G,S),x)=\sum_{s=\gamma(\mathrm{Cay}_p(G,S))}^{|V(\mathrm{Cay}_p(G,S))|}\binom{P^t}{k_1}\binom{P^t}{k_2}\cdots\binom{P^t}{k_{p^{(\Sigma_{i=1}^t n_i)-t}}}x^s,$$

where $\sum_{1\leqslant j\leqslant p^{(\Sigma_{i=1}^t n_i)-t}}k_j=s$ *and* $k_j>0$.

Proof. (i). It is enough to compute the number of all prime order elements of the group G. Similar to the above argument an element of order p has the form $(k_1 p^{n_1-1}, \cdots, k_t p^{n_t-1})$, where $0 \leqslant k_i \leqslant p-1$, $1 \leqslant i \leqslant t$. Note that we omit the identity element and so the assertion is clear.

$(ii)-(v)$. It is clear by the discussion before the proposition and the structure of the graph.

(vi). If the graph is not connected, then the energy of the graph is the sum of the energy of connected components [5] . Since every component in this graph is K_{p^t}, the eigenvalues of each components are p^t-1 and -1 with multiplicity 1 and p^t-1, respectively.

(vii). It is clear by the structure of the graph.

$(viii)$. The smallest dominating sets have $\gamma(\mathrm{Cay}_p(G,S)) = \gamma$ elements. The number of dominating sets of size s is obtained by choosing enough vertices from each components in such a way the summation of them is equal to s. Thus the number of dominating set of size s is

$$\binom{P^t}{k_1}\binom{P^t}{k_2}\cdots\binom{P^t}{k_{p^{(\Sigma_{i=1}^t n_i)-t}}},$$

where $\sum_{1\leqslant j\leqslant p^{(\Sigma_{i=1}^t n_i)-t}} k_j = s$ and $k_j > 1$. □

By the above theorem the prime order Cayley graph of the group $\mathbb{Z}_{2^3} \times \mathbb{Z}_2$ is union of 4 complete components isomorphic to K_4, where $S = \{(0,1), (4,0), (4,1)\}$.

Theorem 2.3. (cf. [9]) *Let Γ be a graph with components Γ_i, $1 \leqslant i \leqslant n$. Then the independent dominating polynomial of the graph Γ is $\prod_{i=1}^n D_i(\Gamma_i, x)$, where $D_i(\Gamma_i, x)$ is independent dominating polynomial of the graph Γ_i.*

It is clear that for a complete graph K_n independent dominating polynomial of the graph K_n is nx. By the above argument we have the following result.

Proposition 2.4. *The independent dominating polynomial of the graph* $\mathrm{Cay}_p(G,S)$ *is $D_i(\mathrm{Cay}_p(G,S), x) = p^{tp^{(\Sigma_{i=1}^t n_i)-t}} x^{p^{(\Sigma_{i=1}^t n_i)-t}}$, where $G = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$.*

Since the first and second Zagreb of a complete graph is clear, by considering the structure of $\mathrm{Cay}_p(G,S)$ of the group $G = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$ we conclude the following result.

Proposition 2.5. *With the same notations in Proposition* 2.2 *we have,*

(i) *The first Zagreb is* $M_1(\mathrm{Cay}_p(G,S)) = p^{(\Sigma_{i=1}^t n_i)-t} p^t (p^t-1)^2$.

(ii) *The second Zagreb is* $M_2(\mathrm{Cay}_p(G,S)) = p^{(\Sigma_{i=1}^t n_i)-t} p^t (p^t-1)^3/2$.

Theorem 2.6. (cf. [12]) *The graph mK_n and its complement $T_{m,mn}$ are circulant and edge transitive, where $T_{m,mn}$ is a complete m-partite graph with n vertices in each part.*

Let $G = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$. By Theorem 2.6, $\mathrm{Cay}_p(G, S)$ is edge transitive and circulant. Since every circulant graph is vertex transitive so $\mathrm{Cay}_p(G, S)$ is edge transitive too.

The Cartesian product $\Gamma \times \Delta$ of graphs Γ and Δ is a graph such that the vertex set of $\Gamma \times \Delta$ is the Cartesian product $V(\Gamma) \times V(\Delta)$ and any two vertices (u, u') and (v, v') are adjacent in $\Gamma \times \Delta$ if and only if either $u = v$ and u' is adjacent with v' in Δ, or $u' = v'$ and u is adjacent with v in Γ.

We claim that $\mathrm{Cay}_p(\mathbb{Z}_{p^n}, S_1) \times \mathrm{Cay}_p(\mathbb{Z}_{p^m}, S_2)$ is induced subgraph of $\mathrm{Cay}_p(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}, S)$. Obviously, $S = \{(k_1 p^{n-1}, k_2 p^{m-1}) : 0 \leqslant k_1, k_2 \leqslant p-1\} - \{(0,0)\}$. By the argument after Proposition 2.1, $\{(x_1, y_1), (x_2, y_2)\}$ is an edge of $\mathrm{Cay}_p(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}, S)$ whenever $x_2 = x_1 - kp^{n-1}$ and $y_2 = y_1 - k'p^{n-1}$. If $\{(x_1, y_1), (x_2, y_2)\}$ is an edge of $\mathrm{Cay}_p(\mathbb{Z}_{p^n}, S_1) \times \mathrm{Cay}_p(\mathbb{Z}_{p^m}, S_2)$ with out loss of generality we can suppose $x_1 = x_2$ and $\{y_1, y_2\}$ is an edge of $\mathrm{Cay}_p(\mathbb{Z}_{p^m}, S_2)$, so $y_1 - y_2 \in S_2$ and clearly $\{(x_1, y_1), (x_2, y_2)\}$ is an edge of $\mathrm{Cay}_p(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}, S)$. It is not hard to deduce that $\mathrm{Cay}_p(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}, S)$ have more edges than $\mathrm{Cay}(\mathbb{Z}_{p^n}, S_1) \times \mathrm{Cay}_p(\mathbb{Z}_{p^m}, S_2)$. By induction we can prove the following result.

Proposition 2.7. $\mathrm{Cay}_p(\mathbb{Z}_{p^{n_1}}, S_1) \times \cdots \times \mathrm{Cay}_p(\mathbb{Z}_{p^{n_t}}, S_t)$ *is induced subgraph of* $\mathrm{Cay}_p(\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}, S)$.

3. The composite order Cayley graph

It is clear that $\mathrm{Cay}_c(G, S)$ is an empty graph for elementary abelian p-group G. In this section we discuss about the composite order Cayley graph of the group $G = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$. By Theorem 2.6, it is vertex and edge transitive. In the following proposition some more results about its structure is presented.

Proposition 3.1. *Let* $G = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}$, *where* n_i *are positive integers. Then*

(*i*) $\mathrm{Cay}_c(G, S)$ *is* $(p^{(\sum_{i=1}^t n_i)} - p^t)$ *regular.*

(*ii*) $\mathrm{Cay}_c(G, S)$ *is* $p^{(\sum_{i=1}^t n_i)-t}$*-partite graph.*

(*iii*) $\mathrm{Cay}_c(G, S)$ *is not a planar graph, for* $t > 2$.

(*iv*) $\omega(\mathrm{Cay}_c(G, S)) = \chi(\mathrm{Cay}_c(G, S)) = p^{(\sum_{i=1}^t n_i)-t}$ *and* $\alpha(\mathrm{Cay}_c(G, S)) = p^t$.

(*v*) *The adjacency matrix of* $\mathrm{Cay}_c(G, S)$ *is* $l \times l$ *matrix where* $l = \prod_{i=1}^t p^{n_i}$, *it contains* $p^t \times p^t$ *sub-matrix on the main diagonal such that all their components are zero as the diagonal which is zero, and the components out of this sub-matrices are one.*

(*vi*) $\gamma(\mathrm{Cay}_c(G, S)) = \gamma_i(\mathrm{Cay}_c(G, S)) = p^t$.

(*vii*) $\gamma(\mathrm{Cay}_p(G,S)) = 2$ *and the dominating polynomial of the graph* $\mathrm{Cay}_c(G,S)$ *is*

$$D(\mathrm{Cay}_c(G,S),x) = \sum_{k=2}^{|V(\mathrm{Cay}_p(G,S))|} \binom{l_1}{i_1}\binom{l_2}{i_2}\cdots\binom{l_s}{i_s}x^k,$$

where $s = p^{\Sigma_{i=1}^t n_i - t}$, $2 \leqslant i_1 + i_2 + \cdots + i_s = k$ *and at least two of* i_j*'s are not equal to zero,* $1 \leqslant j \leqslant s$.

(*viii*) $\gamma_i(\mathrm{Cay}_p(G,S)) = p^t$ *and the independent dominating polynomial of the graph* $\mathrm{Cay}_c(G,S)$ *is,*

$$D_i(\mathrm{Cay}_c(G,S),x) = p^{(\sum_{i=1}^t n_i)-t}x^{p^t}$$

(*ix*) *It is vertex and edge distance balanced.*

Proof. The proof is clear by the fact that $\mathrm{Cay}_c(G,S)$ is complement of $\mathrm{Cay}_p(G,S)$ and Proposition 2.2. □

A strongly regular graph with parameters (v,k,λ,μ) is a graph on v vertices, regular of valency k, such that any two adjacent (nonadjacent) vertices have precisely λ (resp. μ) common neighbors. The complete a-partite graph $K_{a\times m}$, has parameters $(v,k,\lambda,\mu) = (am,(a-1)m,(a-2)m,(a-1)m)$ and spectrum $k, 0, -m$ with multiplicity $1, a(m-1), a-1$, respectively ([3, Remark 11.8]).

Proposition 3.2. *Let* $G = \mathbb{Z}_{p^{n_1}} \times\cdots\times \mathbb{Z}_{p^{n_t}}$. *The eigenvalues of adjacency matrix of* $\mathrm{Cay}_c(G,S)$ *are* $p^{\sum_{i=1}^t n_i} - p^t$, 0, *and* $-p^t$ *with multiplicity* 1, $p^{\sum_{i=1}^t n_i} - p^t$ *and* $p^{(\sum_{i=1}^t n_i)-t} - 1$, *respectively. Moreover, the energy of the graph is* $2(p^{(\sum_{i=1}^t n_i} - p^t)$.

Proof. $\mathrm{Cay}_c(G,S)$ is strongly regular graph with parameters $(p^{\sum_{i=1}^t n_i},\ p^{\sum_{i=1}^t n_i} - p^t,\ p^{\sum_{i=1}^t n_i} - 2p^t,\ p^{\sum_{i=1}^t n_i} - p^t)$ so by the argument before the theorem the assertion is clear. □

Proposition 3.3. *The metric dimension of* $\mathrm{Cay}_c(G,S)$ *is* $p^{(\sum_{i=1}^t n_i)-t}(p^t-1)$.

Proof. In order to construct the smallest resolving set, it is enough to choose $p^t - 1$ vertices from each parts. By the structure of the graph the assertion is clear. □

Proposition 3.4. *With the same notations in Proposition 3.1 we have,*

(*i*) *The Wiener index is* $W(\mathrm{Cay}_c(G,S)) = p^{\sum_{i=1}^t n_i} + \frac{1}{4}(p^{\sum_{i=1}^t n_i}(p^{\sum_{i=1}^t n_i} - p^t))$.

(*ii*) *The first Zagreb is* $M_1(\mathrm{Cay}_c(G,S)) = p^{\Sigma_{i=1}^t n_i}(p^{\Sigma_{i=1}^t n_i} - p^t)^2$.

(*iii*) *The second Zagreb is* $M_2(\mathrm{Cay}_c(G,S)) = p^{\Sigma_{i=1}^t n_i}(p^{\Sigma_{i=1}^t n_i} - p^t)^3/2$.

(*iv*) *The eccentric connectivity* $\xi^c(\mathrm{Cay}_c(G,S)) = 2p^{\Sigma_{i=1}^t n_i}(p^{\Sigma_{i=1}^t n_i} - p^t)$.

Proof. (i). According to the definition of the Wiener index, it is enough to break the summation to two cases. In the first case, we add the distances for the vertices in the same part, while the rest is the summation of the distances of the vertices in the two distinct parts.

It is clear that $\deg(v) = (p^{\Sigma_{i=1}^{t} n_i} - p^t)$, the number of edges of the graph is $p^{\Sigma_{i=1}^{t} n_i}(p^{\Sigma_{i=1}^{t} n_i} - p^t)/2$ and by the definitions of these indices the assertion of (ii) and (iii) follows. The forth part is clear by the fact that the greatest distance for any vertex is 2. □

Theorem 3.5. *If $G \cong D_{2n}$, v is a vertex of the graph $\mathrm{Cay}_c(G, S)$ and p is a prime number, then*

(i) $\mathrm{Cay}_c(G, S)$ *is an empty graph, for prime n.*

(ii) $\deg(v) = p^\alpha - p$, $\mathrm{diam}(\mathrm{Cay}_c(G, S)) = 2$ *and* $\gamma(\mathrm{Cay}_c(G, S)) = 2p$ *for* $n = p^\alpha$, *where $\alpha > 1$. Moreover, if $p^\alpha \neq 4$, then we have* $\mathrm{girth}(\mathrm{Cay}_c(G, S)) = 3$ *and* $\mathrm{girth}(\mathrm{Cay}_c(D_8, S)) = 4$.

(iii) $\deg(v) = \varphi(n)$, *for $n = p_1p_2$, where φ is Eulerian function, p_1 and p_2 are distinct prime numbers. Also* $\mathrm{girth}(\mathrm{Cay}_c(G, S)) = 6,\ 4$ *or* 3, *for* $n = 6,\ 2p$ *and the rest cases, respectively.*

(iv) *For $n = \prod_{i=1}^{l} p_i$,*

$$deg(v) = \varphi(n) + \sum_{j=2}^{l-1} \binom{M}{j},$$

where M is the set of all prime numbers which divides n and the notation $\binom{M}{j}$ stands for the sum of the Eulerian function of multiplication of j prime numbers belongs to M which are chosen randomly. Furthermore, $\mathrm{diam}(\mathrm{Cay}_c(G, S)) = 3$ *and* $\mathrm{girth}(\mathrm{Cay}_c(G, S)) = 3$.

(v) *If $n = \prod_{i=1}^{l} p_i^{\alpha_i}$, where p_i, p_j are distinct prime numbers for $i \neq j$, $1 < \alpha_i$ for some i and $l > 2$, then*

$$deg(v) = \sum_{j=2}^{l} \binom{M}{j},$$

where M is the set of all power of prime numbers which divides n and the notation $\binom{M}{j}$ stands for the sum of the Eulerian function of multiplication of j power of prime numbers belongs to M which are chosen randomly. It is obvious that $\mathrm{diam}(\mathrm{Cay}_c(G, S)) = 2$, $\mathrm{girth}(\mathrm{Cay}_c(G, S)) = 3$ *and* $\gamma(\mathrm{Cay}_c(G, S)) = 2p_i$, *where p_i is the smallest prime number dividing n.*

(vi) *The* $\mathrm{Cay}_c(G, S)$ *the union of two isomorphic components with n vertices* $\{a^ib;\ 0 \leqslant i \leqslant n-1\}$ *and* $\{a^i;\ 0 \leqslant i \leqslant n-1\}$.

(vii) $\mathrm{Cay}_c(G,S)$ *is planar if and only if* $G \cong D_8,\ D_{12}$ *and* D_{2p}, *where* $p > 4$ *is prime number.*

Proof. (vi). Suppose $\mathrm{Cay}_c(G,S)$ is planar. By considering the degree of vertices in the different cases, we conclude that n is divisible just by two distinct prime numbers. As the degree of vertices are less or equal than 4, only two pairs of primes $2,3$ and $2,5$ are acceptable. The rest of the proof is clear by computation and [10, Theorem 2.2] left for the readers. □

Finally, we deduce the following result similar to the Proposition 2.7.

Proposition 3.6. $\mathrm{Cay}_c(\mathbb{Z}_{p^{n_1}}, S_1) \times \cdots \times \mathrm{Cay}_c(\mathbb{Z}_{p^{n_t}}, S_t)$ *is induced subgraph of* $\mathrm{Cay}_c(\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_t}}, S)$.

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