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Prime spectra and finitely generated algebras

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Abstract. In this paper we investigate the class of modules whose prime spectrum equipped with the Zariski topology is homeomorphic to maximal spectrum of some finitely generated reduced K-algebra. We investigate the algebraic structure of these modules and provide some methods to construct some examples of such modules.

1. Introduction

Throughout the article, R is a commutative ring with non-zero identity and all modules are unitary. Also, K is an algebraically closed field and $\mathbb{A}^n_K$ denotes *affine n-space over K*. For each set S of polynomials in $K[x_1, \ldots, x_n]$, we define the *zero set* $\mathbb{V}(S)$ to be the set of points in $\mathbb{A}^n_K$ on which the functions in S simultaneously vanish, that is to say

$$\mathbb{V}(S) = \{a \in \mathbb{A}^n_K \mid f(a) = 0, \forall f \in S\}.$$

A subset W of $\mathbb{A}^n_K$ is called an *affine* (*algebraic*) *variety* if $W = \mathbb{V}(S)$ for some $S \subseteq K[x_1, \ldots, x_n]$. Note that here, the affine varieties are not necessarily irreducible. The *ideal* of an affine variety W is denoted by

$$\mathbb{I}(W) = \{f \in K[x_1, \ldots, x_n] \mid \forall a \in W; f(a) = 0\}.$$

We use $K[W]$ to designate the *coordinate ring* $K[x_1, \ldots, x_n]/\mathbb{I}(W)$ of W. The reader can refer to [27] for basic properties of affine varieties. We recall some definitions.

Definition 1.1. Let M be an R-module and let N be a submodule of M.

1. $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity, we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$).

2. N is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then ideal $\mathfrak{p} = (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$-*prime* (see [15, 24]).

3. The set of all prime submodules of an R-module M is called the *prime spectrum* of M and denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of an R-module M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$.

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4. The set of all prime submodules of M containing N is denoted by $V^*(N)$ (see [25]). Following [18], we define $V(N)$ as

$$\{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}.$$

Set $\mathsf{Z}(M) = \{V(N) : N \leqslant M\}$ and $\mathsf{Z}^*(M) = \{V^*(N) : N \leqslant M\}$. Then the elements of the set $\mathsf{Z}(M)$ satisfy the axioms for closed sets in a topological space $\mathrm{Spec}(M)$. The resulting topology due to $\mathsf{Z}(M)$ is called the *Zariski topology relative to* M and denoted by τ (see [18]). There is another topology, τ^* say, on $\mathrm{Spec}(M)$ due to $\mathsf{Z}^*(M)$ as the collection of all closed sets if and only if $\mathsf{Z}^*(M)$ is closed under finite union. When this is the case, we call the topology τ^* the *quasi-Zariski topology on* $\mathrm{Spec}(M)$ and M is called a *top* module (see [25]).

The concept of prime submodule has led to the development of topologies on the spectrum of modules. Topologies are considered by Duraivel, McCasland, Moore, Smith, and Lu in [11, 18, 25]. It is well-known that Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in algebraic geometry. In the literature, there are many papers devoted to the Zariski topology on the spectrum of modules [1, 2, 8, 13, 21, 26, 28]. Finding relationship between topological properties of prime spectra of modules and algebraic properties of those modules is one of interesting subject in many articles. In this paper, we consider the class of modules whose prime spectrum equipped with the Zariski topology is homeomorphic to the maximal spectrum of some finitely generated reduced K-algebra, namely QC modules. We are interesting to study the algebraic structure of these modules and provide some methods to construct some examples of such modules. For this aim, we use some notions that come from algebraic geometry, such as affine variety.

2. Preliminaries

In the present section, we recall briefly definitions and basic properties of certain topological spaces that we shall use.

Remark 2.1. Let M be an R-module and N be a submodule of M.

1. Note that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some non-zero R-module M. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module has no prime submodule for any prime integer p (see [17]). Such a module is said to be *primeless*.

2. M is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the *natural map* $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective (see [20]).

3. The *radical* of N, denoted by $\mathrm{rad}_M(N)$ or briefly $\mathrm{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. In the case where there are no such prime submodules, $\mathrm{rad}(N)$ is defined as M. If $\mathrm{rad}(N) = N$, we say that N is a *radical submodule* (see [16, 23]).

4. Let $\mathfrak{p}$ be a prime ideal of R. By the *saturation of N with respect to $\mathfrak{p}$*, we mean the contraction of $N_{\mathfrak{p}}$ in M and designate it by $S_{\mathfrak{p}}(N)$ (see [19]).

5. M is said to be *multiplication* (see [7] and [12]) if every submodule N of M is of the form IM for some ideal I of R and M is called *weak multiplication* if every prime submodule P of M is of the form IM for some ideal I of R (see [3] and [5]).

6. (See [25, Theorem 3.5].) Consider the following statements. (1) M is a multiplication module; (2) M is a top module; (3) $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leqslant 1$ for every prime ideal $\mathfrak{p}$ of R; (4) $M/\mathfrak{p}M$ is cyclic for every maximal ideal $\mathfrak{p}$ of R. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, if M is finitely generated then (4) $\Rightarrow$ (1).

7. (See [18, Theorem 6.1].) The following statements are equivalent: (1) $(\mathrm{Spec}(M), \tau)$ is a T_0-space; (2) $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leqslant 1$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

8. If M is a top R-module, then $(\mathrm{Spec}(M), \tau)$ is a T_0-space ([18, Corollary 6.2]).

Remark 2.2. Let X be a topological space.

1. X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition. X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect ([9]). For more examples of modules with Noetherian spectrum we refer the reader to [21] and [1].

2. Let M be an R-module and Y be a subset of $\mathrm{Spec}(M)$.
   - (a) We will denote the intersection of all elements in Y by $\Im(Y)$ and the *closure* of Y in $\mathrm{Spec}(M)$ with respect to the (quasi-)Zariski topology by $Cl(Y)$. By [18, Proposition 5.1], $V(\Im(Y)) = Cl(Y)$.
   - (b) An element $y \in Y$ is called a *generic point* of Y if $Y = Cl(\{y\})$.

3. Following M. Hochster [14], we say that a topological space Y is a *spectral space* in the case where Y is homeomorphic to $\mathrm{Spec}(S)$, with the Zariski topology, for some ring S. For some examples of modules such that their prime spectrum are spectral see [1, 18].

4. A Noetherian space is spectral if and only if it is T_0 and every non-empty irreducible closed subspace has a generic point ([14, pp. 57-58]). We recall that if M is a top R-module, then $(\mathrm{Spec}(M), \tau^*)$ is a T_0-space and every irreducible closed subset of $\mathrm{Spec}(M)$ has a generic point (see [4, Theorem 3.3]).

3. Main results

Definition 3.1. An R-module M is called *quasi-coordinate* (or briefly QC) if $(\mathrm{Spec}(M), \tau)$ is homeomorphic to $\mathrm{Max}(B)$, for some finitely generated reduced K-algebra B.

Recall that any finitely generated reduced K-algebra B is the coordinate ring of some affine variety (see [27]). Hilbert's Nullstellensatz allows us to identify the points of any affine variety with the maximal ideals of its coordinate ring. Therefore, we can consider the prime spectra of a QC module as a geometric object, in a sense. Recall that, affine varieties and their morphisms are essentially equivalent to finitely generated reduced K-algebras and their homomorphisms, only with the arrows reversed.

Let N be a QC R-module. Then there exists a finitely generated reduced K-algebra D such that $(\mathrm{Spec}(N), \tau)$ is homeomorphic to $\mathrm{Max}(D)$. We use notation $[N]$ to designate the class of all affine variety W such that $K[W]$, coordinate ring of W, is isomorphic to D.

In the next theorem, we present some properties of the QC modules. Recall that if R is an integral domain with the *quotient field* $Q(R)$, the *rank* of an R-module M which is written as $rank_R M$, is the dimension of the vector space $Q(R)M = M \otimes_R Q(R)$ over the field $Q(R)$; i.e., $rank_R M = \dim_{Q(R)} Q(R)M$ (see, [22, p. 84]).

Theorem 3.2. *Let M be a non-primeless QC R-module.*

(1) *Every prime submodule of M is a maximal element of $\mathrm{Spec}(M)$ and moreover, is of the form $S_{\mathfrak{p}}(\mathfrak{p}M)$ for some prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$.*

(2) $\mathrm{rad}(\mathbf{0})$ *is a prime submodule of M if and only if $\mathrm{Spec}(M)$ is a singleton set.*

(3) *If R is an integral domain and $\mathrm{Spec}(M)$ is not singleton, then M is not torsion-free.*

(4) *If $\{(P : M) \mid P \in \mathrm{Spec}(M)\} \subseteq \mathrm{Max}(R)$, then $\mathrm{Spec}(M) = \mathrm{Max}(M)$ and M is weak multiplication.*

(5) *If R is an integral domain, then $rank_R M \leqslant 1$.*

(6) *$M/\mathfrak{m}M$ is cyclic for every maximal ideal $\mathfrak{m}$ of R.*

(7) *If M is finitely generated, then M is multiplication.*

(8) *If M is free, then M is cyclic.*

(9) *Let M be a flat R-module. Then, M is weak multiplication and $(\mathrm{Spec}(M), \tau)$ is a spectral space.*

Proof. Suppose that $W \in [M]$.

(1). By [14, Proposition 11], $(\mathrm{Spec}(M), \tau)$ is a T_1-space. Therefore, by Remark 2.2(2),

$$Cl(\{P\}) = V(P) = \{P\},$$

for any prime submodule P of M. If there exists a prime submodule Q of M such that $P \subseteq Q$, then $(P : M) \subseteq (Q : M)$. Hence, $Q \in V(P) = \{P\}$. This implies that $P = Q$. Thus, every prime submodule of M is a maximal element of $\mathrm{Spec}(M)$. Now, let P be a $\mathfrak{p}$-prime submodule of M. Then, by [19, Corollary 3.7], $S_{\mathfrak{p}}(\mathfrak{p}M)$ is a $\mathfrak{p}$-prime submodule of M. So, in the light of Remark 2.1(2.1), $P = S_{\mathfrak{p}}(\mathfrak{p}M)$ since $(\mathrm{Spec}(M), \tau)$ is a T_1-space.

(2). If $\mathrm{Spec}(M) := \{Q\}$ is a singleton set, clearly $\mathrm{rad}(\mathbf{0}) = Q$ is a prime submodule of M. Conversely, if $\mathrm{rad}(\mathbf{0})$ is a prime submodule of M, then it is a maximal element of $\mathrm{Spec}(M)$ which is contained in every prime submodule of M, by (1). Hence, $\mathrm{Spec}(M) = \{\mathrm{rad}(\mathbf{0})\}$ is a singleton set.

(3). Suppose that M is torsion-free. Then by [19, Lemma 4.5], $(\mathbf{0})$ is a prime submodule of M. By (2), M has only one prime submodule, a contradiction.

(4). Let P be a prime submodule of M and N be a proper submodule of M such that $P \subseteq N$. Then

$$(P : M) = (N : M) \in \mathrm{Max}(R).$$

This implies that N is a prime submodule by [15, Proposition 2] and so $P = N$ by (1). Hence, $\mathrm{Max}(M) = \mathrm{Spec}(M)$. Also, $(P : M)M \subseteq P$ is a $(P : M)$-prime submodule. By Remark 2.1(2.1), $P = (P : M)M$. This shows that M is weak multiplication.

(5). Suppose that $rank_R M = \dim Q(R)M > 1$ and x_1, x_2 are linearly independent element of $Q(R)M$. Then, $(x_1) \subseteq (x_1, x_2)$ is a chain of subspaces of $Q(R)M$. But every proper subspace of a vector space is a (0)-prime submodule. Then

$$(x_1) \cap M \subseteq (x_1, x_2) \cap M$$

is a chain of (0)-prime submodules of M, by [17, Proposition 1]. By Remark 2.1(2.1),

$$(x_1) \cap M = (x_1, x_2) \cap M.$$

This implies that $(x_1) = (x_1, x_2)$, a contradiction.

(6). Use Remark 2.1(2.1) and (1).

(7). Use Remark 2.1(2.1) and (1).

(8). Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a basis for M and $\mathfrak{m}$ be a maximal ideal of R. Then $M/\mathfrak{m}M$ is a free $R/\mathfrak{m}$-module with basis $\{f_\lambda + \mathfrak{m}M\}_{\lambda\in\Lambda}$. But $M/\mathfrak{m}M$ is cyclic by (6). Thus, M is cyclic.

(9). Let P be a $\mathfrak{p}$-prime submodule of M. Then $\mathfrak{p}M \subseteq P$. By [15, Theorem 3], $\mathfrak{p}M$ is a prime submodule of M. Now, according to (1), $P = \mathfrak{p}M$. Therefore, M is weak multiplication. By definition $(\mathrm{Spec}(M), \tau)$ is a Noetherian space. Also, by (1) and Remark 2.2(4), it is enough to show that every irreducible closed subset of $(\mathrm{Spec}(M), \tau)$ has a generic point. Let $Y = V(N)$ be an irreducible closed subset of $(\mathrm{Spec}(M), \tau)$, where N is an arbitrary submodule of M. By [18, Proposition 5.4], $\mathfrak{p} := (\Im(Y) : M)$ is a prime ideal of R. It follows [18, Result 3 and Proposition 5.1] that

$$V(\mathfrak{p}M) = V((\Im(V(N)) : M)M) = V(\Im(V(N))) = V(N) = Y.$$

Since M is flat and $\mathfrak{p}M \subseteq \Im(Y) \neq M$, we deduce from [15, Theorem 3] that $\mathfrak{p}M$ is a prime submodule of M. Hence, Y has a generic point. □

We will show that if M is a QC R-module, then the elements of $[M]$ determine some properties of the ring R.

Corollary 3.3. *Let M be a QC R-module and $W \in [M]$ be a variety with infinitely many points. Then, R has infinitely many prime ideals.*

Proof. Let $\mathrm{Spec}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Then for any prime submodule P of M there exists $\mathfrak{p}_i \in \mathrm{Spec}(R)$ such that $(P : M) = \mathfrak{p}_i$. As we mentioned in the proof Theorem 3.2(1), $\mathrm{Spec}(M)$ is a T_1-space. Hence, by Remark 2.1(2.1), $\mathrm{Spec}(M)$ is a finite set, a contradiction. □

Recall that the *dimension*, $\dim V$, of a variety V is defined to be the length d of the longest possible chain of distinct non-empty irreducible subvarieties of V, $V_d \supsetneq \cdots \supsetneq V_1 \supsetneq V_0$.

Proposition 3.4. *Let M be a QC R-module and $W \in [M]$ be a variety of dimension n. Then* $\dim R \geqslant n$.

Proof. Let

$$V_n \supsetneq \cdots \supsetneq V_1 \supsetneq V_0$$

be a descending chain of irreducible closed subsets of $\mathrm{Spec}(M)$. Then

$$\Im(V_n) \subsetneq \cdots \subsetneq \Im(V_1) \subsetneq \Im(V_0)$$

is a strictly ascending chain of submodules of M, by Remark 2.2(2). Thus, we obtain an acsending chain

$$(\Im(V_n) : M) \subsetneq \cdots \subsetneq (\Im(V_1) : M) \subsetneq (\Im(V_0) : M)$$

of prime ideals of R, by [18, Proposition 5.4]. Therefore, $\dim R \geqslant n$. □

Corollary 3.5. *Any non-primeless QC module M over a zero-dimensional ring R has only finitely many prime submodules.*

Proof. By Proposition 3.4, every irreducible closed subset of $\mathrm{Spec}(M)$ is an irreducible component of $\mathrm{Spec}(M)$. As we mentioned in the proof of Theorem 3.2, $\mathrm{Spec}(M)$ is a Noetherian topological space, and so has finitely many irreducible components. Since $Cl\{P\} = V(P) = \{P\}$, we infer that $\mathrm{Spec}(M)$ is a finite set. □

In the sequel, we provide some examples of QC modules. First, we show that every affine variety is homeomorphic to the prime spectrum of a certain module.

For an ideal I of R we recall that the *I-torsion submodule of M* is $\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \in \mathbb{N}\}$ and M is said to be *I-torsion* if $M = \Gamma_I(M)$ (see [10]). We recall that a family $\{M_i\}_{i\in I}$ of R-modules is said to be *prime-compatible* if, for all $i \neq j$ in I, there does not exist a prime ideal $\mathfrak{p}$ in R with $\mathrm{Spec}_{\mathfrak{p}}(M_i)$ and $\mathrm{Spec}_{\mathfrak{p}}(M_j)$ both non-empty (see [25]).

Theorem 3.6. *For any affine variety $W \subseteq \mathbb{A}^n_K$ there exists a QC R-module M such that $W \in [M]$.*

Proof. Let $R = K[x_1, \ldots, x_n]$. If $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{A}^n_K$, then

$$\mathfrak{m}_\lambda := (x_1 - \lambda_1, \ldots, x_n - \lambda_n)$$

is a maximal ideal of R. Let $\{M_\lambda\}_{\lambda \in W}$ be a family of R-modules such that M_λ is $\mathfrak{q}_\lambda$-torsion with a unique prime submodule, where $\mathfrak{q}_\lambda$ is an $\mathfrak{m}_\lambda$-primary ideal of R. Put $M = \bigoplus_{\lambda \in W} M_\lambda$. Let $\alpha, \beta \in W$ such that $P_\alpha \in \mathrm{Spec}_\mathfrak{p}(M_\alpha)$ and $P_\beta \in \mathrm{Spec}_\mathfrak{p}(M_\beta)$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$. Then it is easy to see that $\mathfrak{q}_\alpha \subseteq \mathfrak{p}$ and $\mathfrak{q}_\beta \subseteq \mathfrak{p}$. So, $\mathfrak{q}_\alpha + \mathfrak{q}_\beta \subseteq \mathfrak{p}$, a contradiction. Hence, the family $\{M_\lambda\}_{\lambda \in W}$ is prime-compatible. Therefore, M is a top R-module by [25, Theorem 5.1]. Hence, by Remark 2.1(2.1), $(\mathrm{Spec}(M), \tau)$ is a T_0-space. According to Remark 2.1(2.1),

$$\mathrm{Spec}(M) = \{\mathfrak{m}_\lambda M \mid \lambda \in W\}.$$

We define

$$\begin{aligned} \beta : \mathrm{Spec}(M) &\longrightarrow \mathrm{Max}(K[W]) \\ \mathfrak{m}M &\mapsto \overline{\mathfrak{m}} := \mathfrak{m}/\mathbb{I}(W) \end{aligned}$$

Obviously, β is bijective. Suppose that $V(\overline{J})$ is a closed subset of $\mathrm{Max}(K[W])$, for some ideal $\overline{J} = J/\mathbb{I}(W)$ of $K[W]$. Then

$$\begin{aligned} \beta^{-1}(V(\overline{J})) &= \{\mathfrak{m}M \mid \beta(\mathfrak{m}M) \in V(\overline{J})\} \\ &= \{\mathfrak{m}M \mid \overline{\mathfrak{m}} \in V(\overline{J})\} \\ &= \{\mathfrak{m}M \mid J \subseteq \mathfrak{m}\} = V(JM). \end{aligned}$$

Hence, β is a continuous map. We claim that β is a closed map. Let $V(N)$ be a closed set of $\mathrm{Spec}(M)$, for some submodule N of M. Then

$$\begin{aligned} \beta(V(N)) &= \{\beta(\mathfrak{m}M) \mid \mathfrak{m}M \in V(N)\} \\ &= \{\overline{\mathfrak{m}} \mid (N : M) \subseteq (\mathfrak{m}M : M) = \mathfrak{m}\} \\ &= \{\overline{\mathfrak{m}} \mid \overline{(N : M) + \mathbb{I}(W)} \subseteq \overline{\mathfrak{m}}\} = V(\overline{(N : M) + \mathbb{I}(W)}). \end{aligned}$$

Therefore, $(\mathrm{Spec}(M), \tau)$ is homeomorphic to $\mathrm{Max}(K[W])$. Thus, M is a QC R-module such that $W \in [M]$. □

Example 3.7. Let $R := \mathbb{C}[x, y]$ and $W := \mathbb{V}(y - x^2)$ be a variety of $\mathbb{A}^2_\mathbb{C}$. Then in the light of Theorem 3.6,

$$M = \bigoplus_{(a,b) \in W} R/(x - a, y - b)^3$$

is a QC R-module such that $W \in [M]$. Note that by the construction of M, this R-module is not unique. For example,

$$M' = \bigoplus_{(a,b) \in W} R/((x - a)^3, y - b)^5$$

is a QC R-module such that $W \in [M']$.

Example 3.8. In Example 3.7, we showed that for a variety W of $\mathbb{A}^{\mathbf{s}}_K$, there exists a QC module over the ring $R = K[x_1, \ldots, x_{\mathbf{s}}]$ such that $W \in [M]$. Here, we show that for a variety $W \subseteq \mathbb{A}^s_K$ it may be found a QC module M over $K[x_1, \ldots, x_m]$, where $s > m \geqslant \dim W$ and $W \in [M]$ (see Proposition 3.4).

Let $R = \mathbb{C}[x, y]$ and $W = \mathbb{V}(y - x^2)$ be a variety of $\mathbb{A}^2_{\mathbb{C}}$. Then by the proof of Theorem 3.6, we have

$$M = \bigoplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t - a)$$

is a QC $\mathbb{C}[t]$-module such that $W \in [M]$.

It follows from Theorem 3.6 and Example 3.7 that there exists a QC R-module M such that $(\mathrm{Spec}(M), \tau)$ is homeomorphic to the affine n-space $\mathbb{A}^n_K$ and every variety W of $\mathbb{A}^n_K$ is homeomorphic to the prime spectrum of a certain submodule N of M. More precisely:

Proposition 3.9. *Let $R := K[x_1, \ldots, x_n]$ and $\mathfrak{m}_\lambda := (x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ be a maximal ideal of R. Let $\{M_\lambda\}_{\lambda \in K^n}$ be a family of R-modules such that for any $\lambda \in K^n$, M_λ is $\mathfrak{q}_\lambda$-torsion with a unique prime submodule, where $\mathfrak{q}_\lambda$ is an $\mathfrak{m}_\lambda$-primary ideal of R. Put $M = \bigoplus_{\lambda \in K^n} M_\lambda$. Then M is a QC R-module such that $\mathbb{A}^n_K \in [M]$. Moreover, if W is an affine variety of $\mathbb{A}^n_K$, then $N = \bigoplus_{\lambda \in W} M_\lambda$ is a QC submodule of M such that $W \in [N]$.*

Suppose that $\{M_i\}_{i=1}^n$ is a family of the QC R-modules. As we mentioned, the prime spectra of each M_i presents an affine variety, in a sense. The union of these varieties is a variety, namely W. In the next proposition, we state a method to obtain a new QC module M from $\{M_i\}_{i=1}^n$ such that $W \in [M]$.

Proposition 3.10. *Let $\{M_i\}_{i=1}^t$ be a family of prime-compatible QC R-modules. Then $M = \bigoplus_{i=1}^t M_i$ is a QC R-module.*

Proof. By Theorem 3.2, $(\mathrm{Spec}(M_i), \tau)$ is a T_1-space, for each $i \in \{1, \ldots, t\}$. We claim that $(\mathrm{Spec}(M), \tau)$ is a T_0-space.

Let Q_1 and Q_2 be two $\mathfrak{q}$-prime submodules of M. Then there are $a, b \in \{1, \ldots, t\}$ such that $M_a \nsubseteq Q_1$ and $M_b \nsubseteq Q_2$. By [25, Lemma 1.6],

$$Q_1 \cap M_a \in \mathrm{Spec}_{\mathfrak{q}}(M_a) \qquad \text{and} \qquad Q_2 \cap M_b \in \mathrm{Spec}_{\mathfrak{q}}(M_b).$$

Since M_a and M_b are prime-compatible, we conclude that $a = b$. Thus, it follows from Remark 2.1(2.1) that

$$Q_1 \cap M_a = Q_2 \cap M_a$$

For all $i \in \{1, \ldots, t\}$ with $i \neq a$, $\mathrm{Spec}_{\mathfrak{q}}(M_i)$ is an empty set by hypothesis and hence, again using [25, Lemma 1.6], $Q_1 \cap M_i = M_i$. Therefore,

$$Q_1 = (Q_1 \cap M_a) \oplus (\bigoplus_{i=1, i \neq a}^{t} M_i) = Q_2.$$

By Remark 2.1(2.1), we deduce that $(\mathrm{Spec}(M), \tau)$ is a T_0-space.

Let $j \in \{1, \ldots, t\}$ and put

$$X_j = \{P_j \oplus (\bigoplus_{i=1, i\neq j}^{t} M_i) \mid P_j \in \mathrm{Spec}(M_j)\}.$$

Then $X_j \subseteq \mathrm{Spec}(M)$, by [20, Lemma 4.6]. On the other hand, if P is a $\mathfrak{p}$-prime submodule of M, then there exists $c \in \{1, \ldots, t\}$ such that $M_c \nsubseteq P$. Thus,

$$P \cap M_c \in \mathrm{Spec}_{\mathfrak{p}}(M_c) \quad \text{and} \quad (P \cap M_c) \oplus (\bigoplus_{i=1, i\neq c}^{t} M_c)$$

is a $\mathfrak{p}$-prime submodule of M. Since $(\mathrm{Spec}(M), \tau)$ is a T_0-space, we infer that

$$P = (P \cap M_c) \oplus (\bigoplus_{i=1, i\neq c}^{t} M_c),$$

by Remark 2.1(2.1). This implies that $P \in X_c$. Therefore, $\mathrm{Spec}(M) = \bigcup_{i=1}^{t} X_i$, as disjoint union. It is easy to see that X_j is homeomorphic to $\mathrm{Spec}(M_j)$. By assumption for each $j \in \{1, \ldots, t\}$, there exists an affine variety $W_j \in [M_j]$ such that $W_j \cong X_j$. Hence, $(\mathrm{Spec}(M), \tau)$ is homeomorphic with $\bigcup_{i=1}^{t} W_i$. Since a finite union of the affine varieties is also an affine variety, M is a QC R-module and $\bigcup_{i=1}^{t} W_i \in [M]$. □

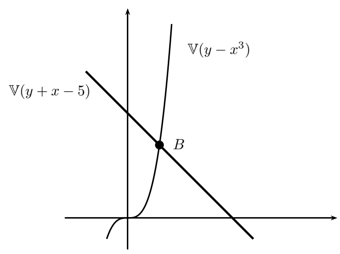
Remark 3.11. The "prime-compatible" condition in Proposition 3.10 is necessary. For example, consider varieties $W_1 = \mathbb{V}(y - x^3)$ and $W_2 = \mathbb{V}(y + x - 5)$ in $\mathbb{A}^2_{\mathbb{C}}$. If

$$M_1 = \bigoplus_{(a,b)\in W_1} \mathbb{C}[x, y]/(x - a, y - b)^3$$

and

$$M_2 = \bigoplus_{(a,b)\in W_2} \mathbb{C}[x, y]/((x - a)^3, y - b)^5,$$

then $W_i \in [M_i]$ for $i = 1, 2$. But, $M = M_1 \oplus M_2$ is not a QC R-module. Since, as shown in the figure below, there are prime submodules $P_i \in \mathrm{Spec}(M_i)$ for $i = 1, 2$ such that $P_1 \oplus M_2$ and $M_1 \oplus P_2$ are correspond to the point $B \in \mathbb{A}^2_{\mathbb{C}}$. Thus, $\mathrm{Spec}(M)$ is not a T_0-space. This shows that M is not a QC R-module.



As an important consequence of Proposition 3.10, a QC R-module induces different QC R-modules which all of them geometrically are the same.

Corollary 3.12. *Let M_1 be a QC R-module and M_2 be a primeless R-module, then $M = M_1 \oplus M_2$ is a QC R-module.*

Proof. Use Proposition 3.10. □

Proposition 3.13. *Let M be an R-module such that*

$$\{(P : M) \mid P \in \operatorname{Spec}(M)\} \subseteq \operatorname{Max}(R).$$

Let Y be a subspace of $(\operatorname{Spec}(M), \tau)$ such that with induced topology is homeomorphic with $\operatorname{Max}(B)$ for some finitely generated reduced K-algebra B. Then M induces a top R-module L such that $\operatorname{Max}(B) \in [L]$.

Proof. We define $L = \bigoplus_{P \in Y} M/P$. Similar to the proof of Theorem 3.2, it is easy to see that every prime submodule in Y is a maximal submodule of M, since $\operatorname{Max}(B)$ is a T_1-space and

$$\{(P : M) \mid P \in \operatorname{Spec}(M)\} \subseteq \operatorname{Max}(R).$$

Therefore, the R-module M/P has only one prime submodule, for every $P \in Y$, and hence is a top module. Since Y is a T_1-space, $\{M/P\}_{P \in Y}$ is a prime-compatible family of the top modules, by Remark 2.1(2.1). Hence, L is a top R-module (see [25, Theorem 5.1]).

Let $Q \in \operatorname{Spec}_{\mathfrak{q}}(L)$, where $\mathfrak{q} \in \operatorname{Spec}(R)$. Then there is an element $P \in Y$ such that $M/P \nsubseteq Q$. Therefore, $Q \cap (M/P)$ is a $\mathfrak{q}$-prime submodule of M/P and

$$\mathfrak{q} = (Q : L) = (P : M) \in \operatorname{Max}(R).$$

Hence,

$$\operatorname{Spec}(L) = \{(P : L)L \mid P \in Y\}.$$

Now, it is easy to see that $\operatorname{Max}(B) \in [L]$. □

In Proposition 3.10, we considered the relationship between the union of varieties and the QC modules. It is natural to ask about the relationship between the intersection of varieties and the QC modules. As an immediate consequence of Proposition 3.13, we obtain the following:

Corollary 3.14. *Let $\{M_j\}_{j\in J}$ be a family of the QC R-modules such that $W_j \in [M_j]$, for each $j \in J$. Let*

$$\{(P : M_\alpha) \mid P \in \mathrm{Spec}(M_\alpha)\} \subseteq \mathrm{Max}(R)$$

for some $\alpha \in J$. Then there exists an R-module M such that $\bigcap_{j\in J} W_j \in [M]$.

Proof. Let $W = \bigcap_{j\in J} W_j$ and Y be a subspace of $(\mathrm{Spec}(M_\alpha), \tau)$ homeomorphic to W. By Proposition 3.13, $M = \bigoplus_{P\in Y}(M_\alpha/P)$ is the desired module. □

Corollary 3.15. *Let M be an Artinian R-module and let Y be a subspace of* $(\mathrm{Spec}(M), \tau)$ *such that with induced topology is homeomorphic to* $\mathrm{Max}(B)$ *for some finitely generated reduced K-algebra B. Then, M induces a top R-module L such that* $\mathrm{Max}(B) \in [L]$.

Proof. By [6, Corollary 2.4], we have $\{(P : M) \mid P \in \mathrm{Spec}(M)\} \subseteq \mathrm{Max}(R)$. Thus, the result follows from Proposition 3.13. □

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