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On the extension of continuous homomorphisms of topological n-ary semigroups

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Abstract. We prove that each continuous homomorphism from a topological n-ary semigroup X into a topological binary semigroup Y can be extended to a continuous homomorphism from the free covering semigroup of X into a semigroup Y.

1. Introduction

Topologies of n-ary semigroups are studied by many authors in various directions. One of such directions is investigation of characters of n-ary semigroups, i.e., homomorphisms of topological n-ary semigroups into a multiplicative group of complex numbers with a module equal to 1 and natural topology (see for example [9]). The second direction is investigation of topologies induced by families of some functions such as deviations, for example [8]. The third important direction is investigation of the possibility of embedding of topological n-ary semigroups into binary topological semigroups and groups or finding a way to describe topological n-ary semigroups using other known topological structures (cf. [10]).

In [3, 4, 5, 6] various constructions of topologies for a universal covering semigroup of a topological n-ary semigroup was proposed. In the case on n-ary groups the topology of the covering group and the topology of retracts of n-ary groups, i.e., binary groups obtained from an n-ary group by blocking in an n-ary operation $n-2$ inner elements, are strongly connncted with a topology of initial n-ary group (cf. [2] and [7]). In the case of semigroups, this relationship is not so strong.

In this paper, we consider continuous homomorphisms of topological n-ary semigroups into topological semigroups. It is known that every n-ary semigroup S can be considered as a subset of a binary semigroup S^* that is stable with respect to the multiplication of n elements in S^* (such binary semigroup is called *enveloping* or *covering semigroup* of an n-ary semigroup S).

Questions naturally arise about the possibility of extending a continuous homomorphism of an n-ary semigroup S into binary semigroup G to a continuous homomorphism of its covering semigroup S^* into G. The answers to these questions are devoted to the proposed work.

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2. Preliminaries

The terminology of this work is the same as in [5] and [6].

Recall that a non-empty set X with an associative n-ary operation $[\ ]$ is called an *n-ary semigroup*. The result of application of $[\ ]$ to the sequence $x_1, x_2, \dots, x_n$ is denoted by $[x_1^n]$.

A binary semigroup $(S, \cdot)$ is called a *covering semigroup* for an n-ary semigroup $(X, [\ ])$, if $S \subseteq X$ and $[x_1^n] = x_1 x_2 \dots x_n$ for all $x_1, x_2, \dots, x_n \in X$ and S is generated by X. If, in addition, the sets $X, X^{(2)}, \dots, X^{(n-1)}$, where $X^{(k)} = \{x_1 x_2 \cdots x_k : x_1, \dots, x_k \in X\}$ are disjoint and their union gives S, then $(S, \cdot)$ is called the *universal covering semigroup* of an n-ary semigroup $(X, [\ ])$. Such covering semigroup there exists for each n-ary semigroup [4].

A mapping $f \colon X \to Y$ is called a *homomorphism of an n-ary semigroup* $(X, [\ ])$ *into a binary semigroup* $(Y, *)$ if $f([x_1^n]) = f(x_1) * f(x_2) * \dots * f(x_n)$ is valid for all $x_1, \dots, x_n \in X$. We say that $(X, [\ ], \tau)$ is a topological n-ary semigroup, if $(X, [\ ])$ is an n-ary semigroup and (X, τ) is a topological space such that the n-ary operation $[\ ]$ is continuous in all variables together.

3. Results

Theorem 1. *Let $f \colon X \to Y$ be a homomorphism of an n-ary semigroup $(X, [\ ])$ into a binary semigroup $(Y, *)$ and let $(S, \cdot)$ be the universal covering semigroup of $(X, [\ ])$. If in a subsemigroup of $(Y, *)$ generated by $f(X)$ there is a left or right cancellative element $f(a)$, then f can be uniquely extended to a homomorphism from $(S, \cdot)$ into $(Y, *)$.*

Proof. Let $x \in S$. Then, $x = x_1 x_2 \cdots x_k$ for some $x_1, \dots, x_k \in X$ and $1 \leqslant k < n$. If f can be extended to the homomorphism f_S from $(S, \cdot)$ into $(Y, *)$, then $f(x) = f(x_1) * f(x_2) * \dots * f(x_k)$. Hence this extension is unique, if it exists.

We show that the above equality defines a mapping from S to Y. To do this, it suffices to show that if $x_1 x_2 \cdots x_k = y_1 * y_2 * \dots * y_m$, where $1 \leqslant k, m < n$ and all x_i, y_j are from X, then

$$f(x_1) * f(x_2) * \dots * f(x_k) = f(x_1) * f(x_2) * \dots * f(x_m).$$

Let an element $f(a)$, where $a \in X$, be left cancellative in a subsemigroup of a semigroup $(Y, *)$ generated by the set $f(X)$. Then $k = m$ and, consequently, $a^{n-k} x_1 x_2 \cdots x_k = a^{n-k} y_1 y_2 \cdots y_k$. Thus $f([a^{n-k} x_1^k]) = f([a^{n-k} y_1^k])$. Therefore,

$$f(a)^{n-k} * f(x_1) * \dots * f(x_k) = f(a)^{n-k} * f(y_1) * \dots * f(y_k).$$

Since $f(a)$ is left cancellative, the last implies $f(x_1) * \dots * f(x_k) = f(y_1) * \dots * f(y_k)$. This proves that f_S is correctly defined. Obviously, f_S is an extension of the homomorphism f to the mapping from $(S, \cdot)$ to $(Y, *)$.

Let $x, y \in S$. Then $x = x_1x_2\cdots x_k$ and $y = y_1y_2\cdots y_m$ for some $x_i, y_j \in X$ and $1 \leqslant k, m < n$. If $k+m < n$, then

$$f_S(xy) = f(x_1) * \ldots * f(x_k) * f(y_1) * \ldots * f(y_m) = f_S(x) * f_S(y).$$

In the case $k+m = n$, we have

$$f_S(xy) = f_S([x_1^k y_1^m]) = f(x_1) * \ldots * f(x_k) * f(y_1) * \ldots * f(y_m) = f_S(x) * f_S(y).$$

Now, if $k+m > n$, then

$$\begin{aligned} f_S(xy) &= f_S([x_1^k y_1^{n-k}]y_{n-k+1}\cdots y_m) = f_S([x_1^k y_1^{n-k}]) * f_S(y_{n-k+1}\cdots y_m) \\ &= f(x_1) * \ldots * f(x_k) * f(y_1) * \ldots * f(y_m) = f_S(x) * f_S(y.) \end{aligned}$$

So, f_S is a homomorphism from $(S, \cdot)$ into $(Y, *)$.

For a right cancellative elenent the proof is analogous. □

In [6], the topology τ_S on $(S, \cdot)$ was constructed using the following construction. On a free semigroup $F = \bigcup_{k=1}^{\infty} X^k$ we consider the relation Ω defined by

$$(x_1, x_2, \ldots, x_p)\Omega(y_1, y_2, \ldots, y_m) \Longleftrightarrow x_1x_2\cdots x_p = y_1y_2\cdots y_m.$$

This relation is a congruence. The mapping φ from a quotient semigroup F/Ω into S which the equivalence class of an element $(x_1, x_2, \ldots . x_p) \in F$ transforms into an element $x_1x_2\cdots x_p \in S$, is an isomorphism of semigroups F/Ω and S.

Let τ be a topology on X, τ_k – a topology on the Cartesian product X^k that is the product of topologies on factors, τ_F – a topology on F that is the sum of the topologies τ_k, $k = 1, 2, \ldots$

Using the mapping φ we transfer the factor topology on F/Ω onto $(S, \cdot)$ and denote it by τ_S. If π is a canonical mapping from F to F/Ω, then the topology τ_S is characterized as the strongest of the topologies on S for which the mapping $\varphi \circ \pi$ is continuous.

In [6], it was shown that if $(X, [\,], \tau)$ is a topological n-ary semigroup, then the topological space (X, τ) is a topological subspace of (S, τ_S).

Theorem 2. *Let f be a continuous homomorphism from a topological n-ary semigroup $(X, [\,], \tau)$ into a topological binary semigroup $(Y, *, \tau_Y)$ and let $(S, \cdot, \tau_S)$ be the universal covering semigroup of the n-ary semigroup $(X, [\,])$ endowed with the topology τ_S described above. If f can be extended to a homomorphism from $(S, \cdot)$ to $(Y, *)$, then this extension is a continuous mapping.*

Proof. Let a homomorphism f be extend to a homomorphism from $(S, \cdot)$ to $(Y, *)$. From the proof of Theorem 1 it follows that this extension is representable in the form f_S. If $x = (x_1, x_2, \ldots, x_p) \in F$, then $g(x) = f(x_1) * f(x_2) * \ldots * f(x_p)$ is a continuous mapping from F to Y. Moreover, we also have $f_S(\varphi(\pi(x))) = f(x_1) * f(x_2) * \ldots * f(x_p) = g(x)$.

From the properties of the final topology it follows that the homomorphism f_S is a continuous mapping (cf. [1]). This completes the proof. □

Let $(X,[\ ])$ be an n-ary semigroup and let $\alpha=(a_1,\ldots,a_p)$, $\beta=(b_1,\ldots,b_q)$ be elements of $F=\bigcup_{k=1}^{\infty}X^k$. We put $\alpha\#\beta$ if and only if there are $(d_1,d_1,\ldots,d_t)\in F$ and two sequences of natural numbers $k_1<k_2<\ldots<k_p=t$, $m_1<m_2<\ldots<m_q=t$ such that

$$a_1=[d_1\ldots d_{k_1}],\quad a_2=[d_{k_1+1}\ldots d_{k_2}],\quad \ldots\quad a_p=[d_{k_{p-1}+1}\ldots d_{k_p}]$$

and

$$b_1=[d_1\ldots d_{m_1}],\quad b_2=[d_{m_1+1}\ldots d_{m_2}],\quad \ldots\quad q_p=[d_{m_{q-1}+1}\ldots d_{m_q}].$$

The relation $\#$ is reflexive and symmetric on F. Its transitive closure $\approx$ is a congruence on F. A factor semigroup $F/\#$ is called a *free covering semigroup* of $(X,[\ ])$ and is denoted by $\widehat{F}$. By $\star$ is denoted a binary operation on $\widehat{F}$. The equivalence class of $\alpha\in F$ is denoted by $\widehat{\alpha}$. The topology on $\widehat{F}$ is constructed in the same way as in [5].

Let φ be the canonical mapping F onto $\widehat{F}$. The subset $\varphi(X)$ of $\widehat{F}$ is stable with respect to the n-th iteration of $\star$ and it is isomorphic to $(X,[\ ])$. So, $\varphi(X)$ can be identified with X. Thus $\widehat{F}$ can be represented as a union of pairwise disjoint sets $\widehat{F}=X_1\cup X_2\cup\ldots\cup X_{n-1}$, where $X_1=\varphi(X)=X$, $X_i=\varphi(X^i)=X\star X\star\ldots\star X$ (i-times), $i=2,3,\ldots,n-1$.

Theorem 3. *Every homomorphism of an n-ary semigroup $(X,[\ ])$ into a binary semigroup $(Y,*)$ can be extended to a homomorphism from a free covering semigroup of $(X,[\ ])$ into $(Y,*)$.*

Proof. Let $f\colon X\to Y$ be a homomorphism of an n-ary semigroup $(X,[\ ])$ into a binary semigroup $(Y,*)$. Let $\widehat{\gamma}\in\widehat{F}$, where $\gamma=(x_1,x_2,\ldots,x_p)\in F$. Define $f_{\widehat{F}}\colon\widehat{F}\to Y$ by putting $f_{\widehat{F}}(\widehat{\gamma})=f(x_1)*f(x_2)*\ldots*f(x_p)$. To show that $f_{\widehat{F}}$ is a homomorphism consider an arbitrary $\widehat{\delta}\in\widehat{F}$ such that $\delta=(y_1,y_2,\ldots,y_q)\in F$ and $\gamma\#\delta$. Then there are $(d_1,d_2,\ldots,d_t)\in F$ and two sequences of natural numbers $k_1,k_2<\ldots<k_p=t$, $m_1<m_2<\ldots m_q=t$ such that

$$x_1=[d_1\ldots d_{k_1}],\quad x_2=[d_{k_1+1}\ldots d_{k_2}],\quad \ldots\quad x_p=[d_{k_{p-1}+1}\ldots d_{k_p}]$$

and

$$y_1=[d_1\ldots d_{m_1}],\quad y_2=[d_{m_1+1}\ldots d_{m_2}],\quad \ldots\quad y_q=[d_{m_{q-1}+1}\ldots d_{m_q}].$$

Since $f\colon X\to Y$ is a homomorphism, we have

$$\begin{aligned}f_{\widehat{F}}(\widehat{\gamma})&=f(x_1)*f(x_2)*\ldots*f(x_p)=f([d_1^{k_1}])*f([d_{k_1+1}^{k_2}])*\ldots*f([d_{k_{p-1}+1}^{k_p}])\\&=f(d_1)*f(d_2)*\ldots*f(d_{k_p})=f([d_1^{m_1}])*f([d_{m_1+1}^{m_2}])*\ldots*f([d_{m_{q-1}+1}^{m_q}])\\&=f(y_1)*f(y_2)*\ldots*f(y_q)=f_{\widehat{F}}(\widehat{\delta}).\end{aligned}$$

But the relation $\approx$ is a transitive closure of the relation $\#$, so we have $f_{\widehat{F}}(\widehat{\alpha})=f_{\widehat{F}}(\widehat{\beta})$ for all $\widehat{\alpha},\widehat{\beta}\in\widehat{F}$ such that $\widehat{\alpha}=\widehat{\beta}$. This means that the mapping $f_{\widehat{F}}$ is correctly defined. Obviously, $f_{\widehat{F}}$ is a homomorphism and it is an extension of f. □

Let $(X, [\,], \tau)$ be a topological n-ary semigroup, τ_k – a topology on the Cartesian product X^k that is a product of topologies on factors, τ_F – a topology on F that is the sum of the topologies τ_k, $k = 1, 2, \ldots$, τ_S – the factor topology on $\widehat{F}$. If π is a canonical mapping from F to $\widehat{F}$, then the topology τ_S is characterized as the strongest of the topologies on $\widehat{F}$ for which the mapping π is continuous.

In [5], it was shown that if $(X, [\,], \tau)$ is a topological n-ary semigroup, then (X, τ) is a topological subspace of $(\widehat{F}, \tau_s)$.

Theorem 4. *Let f be a continuous homomorphism from a topological n-ary semigroup $(X, [\,], \tau)$ into a topological binary semigroup $(Y, *, \tau_y)$. Then f can be extended to a continuous homomorphism from the free covering semigroup $(\widehat{F}, *, \tau_S)$ of $(X, [\,], \tau)$ into $(Y, *, \tau_Y)$.*

Proof. From the proof of Theorem 3 it follows that the extension of f to a homomorphism from $(\widehat{F}, *, \tau_s)$ to $(Y, *, \tau_y)$ exists and has the form $f_{\widehat{F}}$.

If $x = (x_1, x_2, \ldots, x_p) \in F$, then $g(x) = f(x_1) * f(x_2) * \ldots * f(x_p)$ is a continuous mapping from F into Y and $f_{\widehat{F}}(\pi(x)) = f(x_1) * f(x_2) * \ldots * f(x_p) = g(x)$.

From the properties of the final topology it follows that the homomorphism $f_{\widehat{F}}$ is a continuous mapping (cf. [1]). The theorem is proved. □

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