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Purity of ideals and generalized ideals on ordered semigroups

Thawhat Changphas and Pisit Kummoon

Abstract. The purpose of this paper is to study purity of ideals and of generalized ideals of an ordered semigroup, a semigroup together with a partial order that is compatible with the semigroup operation.

1. Preliminaries

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leqslant y \text{ implies } zx \leqslant zy \text{ and } xz \leqslant yz,$$

is called a *partially ordered semigroup*, or simply an *ordered semigroup* ([7]). A non-empty subset T of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *subsemigroup* of S if, for any x, y in T, $xy \in T$. Ordered semigroups have been widely studied ([1], [2], [3], [4]).

For A, B non-empty subsets of an ordered semigroup $(S, \cdot, \leqslant)$, the set product AB is defined to be the set of all elements ab in S with a in A and b in B. In particular, Ax will be written for $A\{x\}$, and similarly for xA. What is more, the subset $(A]$ of S is defined to be the set of all elements x in S such that $x \leqslant a$ for some a in A. It is observed that the following hold: (1) $A \subseteq (A]$ (hence, $S = (S]$; (2) $A \subseteq B \Longrightarrow (A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $(A] = ((A]]$; (5) $(A \cup B] = (A] \cup (B]$; (6) $\big((A](B]\big] = (AB]$.

As in [7] (Theorem 4.2, p. 234), a non-empty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *left ideal* (resp. *right ideal*) of S if the following two conditions hold:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

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(ii) if $x \in A$ and $y \in S$ such that $y \leqslant x$, then $y \in A$, i.e., $A = (A]$.

And A is called a *two-sided ideal* (or simply *ideal*) of S if A is both a left and a right ideal of S. It is observed that the following hold: If A is a left ideal of S and B is a right ideal of S, then $(AB]$ is an ideal of S. And the intersection of all ideals containing a non-empty subset A of S, denote by $I(A)$, is an ideal of S containing A, and it is of the form

$$I(A) = (A \cup AS \cup SA \cup SAS].$$

In particular, we write $I(\{a\})$ by $I(a) = (a \cup Sa \cup aS \cup SaS]$ which is called the *principal ideal of S generated by a.*

Let $(S, \cdot, \leqslant)$ be an ordered semigroup, and let $\mathfrak{J}(S)$ be the set of all ideals of S. For A, B in $\mathfrak{J}(S)$, define

$$A \circ B := (AB].$$

Then $(\mathfrak{J}(S), \circ)$ is a semigroup. Indeed, for A, B, C in $(\mathfrak{J}(S)$, we have

$$(A \circ B) \circ C = (AB] \circ C = ((AB]C] = (ABC] = (A(BC]] = A \circ (BC] = A \circ (B \circ C).$$

Moreover, based on the inclusion, we have $(\mathfrak{J}(S), \circ, \subseteq)$ is an ordered semigroup which is called *ideals ordered semigroup* of S. Hereafter, the operation $\circ$ will be skipped, however, it will be clearly seen from the context.

Finally, a subsemigroup B of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *bi-ideal* of S if

(i) $BSB \subseteq B$;

(ii) if $x \in B$ and $y \in S$ such that $y \leqslant x$, then $y \in B$, i.e., $(B] = B$.

It is well-known that the intersection of bi-ideals of S, if it is non-empty, is a bi-ideal of S. The bi-ideal of S generated by a non-empty set A of S is of the form

$$B(A) = (A \cup A^2 \cup ASA].$$

And, for a in S, we write $B(\{a\})$ as $B(a)$, and $B(a) = (a \cup a^2 \cup aSa]$. As in [10], for x, y in S, it is asserted that B is a bi-ideal of S implies $(xBy]$ is a bi-ideal of S. To see this:

$$(xBy](xBy] \subseteq (xByxBy] \subseteq (xBSBy] \subseteq (xBy]$$

and

$$(xBy]S(xBy] = (xBy](S](xBy] \subseteq (xBySxBy] \subseteq (xBSBy] \subseteq (xBy].$$

2. Pure (two-sided) ideals

We begin this section with recalling the following definitions ([5], [6]):

Definition 2.1. An ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *globally idempotent* if $A = (A^2]$.

Definition 2.2. An ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *complete* if $A = (AS] = (SA]$.

Assume that A is a globally idempotent ideal of an ordered semigroup $(S, \cdot, \leqslant)$. Then

$$A = (A^2] \subseteq (AS] \subseteq (A] = A.$$

Similarly,

$$A = (A^2] \subseteq (SA] \subseteq (A] = A.$$

We can deduce that

$$A = (AS] = (SA].$$

This proves the following theorem:

Theorem 2.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If every ideal of S is globally idempotent, then every ideal of S is complete.*

Theorem 2.3 leads to the following question: *What kinds of ordered semigroups in order that every ideal is complete implies every ideal is globally idempotent?* To answer this question, as in [8], we define the main concepts of this section as follows:

Definition 2.4. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal A of S is said to be $\mathfrak{J}$-*pure* if

$$A \cap (XS] = (XA] \text{ and } A \cap (SX] = (AX]$$

for all X in $\mathfrak{J}(S)$.

Definition 2.5. An ordered semigroup $(S, \cdot, \leqslant)$ is said to be $\mathfrak{J}^*$-*pure* if every ideal of S is $\mathfrak{J}$-pure.

Example 2.6. Let $S = \{a, b, c\}$ be an ordered semigroup with the operation and the order relation are defined by:

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	a	c

$$\leqslant := \{(a,a),(b,b),(c,c),(a,b)\}.$$

It is easy to see that $\{a\}$ and S are the ideals of S, and S is $\mathfrak{J}^*$-pure.

Example 2.7. Let $S = \{a,b,c,d,e\}$ be an ordered semigroup with the operation and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,e),(c,b),(d,b),(d,e),(c,e)\}.$$

The ideals of S are $\{a,b,c,d\}$ and S, and we have S is $\mathfrak{J}^*$-pure.

Theorem 2.8. *The following conditions are equivalent for a $\mathfrak{J}^*$-pure ordered semigroup $(S,\cdot,\leqslant)$:*

(1) *every ideal of S is globally idempotent;*

(2) *every ideal of S is complete.*

Proof. Assume that A is a complete ideal of S; then $A = (AS]$. Since S is $\mathfrak{J}^*$-pure, we have

$$A \cap (AS] = (AA].$$

This gives

$$A = A \cap A = A \cap (AS] = (AA].$$

This means that A is a globally idempotent ideal of S. Hence (2) $\Longrightarrow$ (1) holds. The opposite direction follows from Theorem 2.3. □

Example 2.9. Let $S = \{a,b,c\}$ be an ordered semigroup with the operation and the order relation are defined by:

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

$$\leqslant := \{(a,a),(b,b),(c,c),(a,b)\}.$$

It can be shown that $\{a,b\}$ is a complete ideal of S. But $\{a,b\}$ is not a globally idempotent ideal of S. Therefore, this example shows that the converse of Theorem 2.8 does not hold in general.

Next, we characterize when an ideal of an ordered semigroup is $\mathfrak{J}$-pure.

Theorem 2.10. *For an ideal A of an ordered semigroup $(S, \cdot, \leqslant)$, the following statements are equivalent:*

(1) *A is $\mathfrak{J}$-pure;*

(2) *$A \cap (XS] \subseteq (XA]$ and $A \cap (SX] \subseteq (AX]$ for all X in $\mathfrak{J}(S)$;*

(3) *$A \cap (I(x)S] = (I(x)A]$ and $A \cap (SI(x)] = (AI(x)]$ for all x in S;*

(4) *$A \cap (I(x)S] \subseteq (I(x)A]$ and $A \cap (SI(x)] \subseteq (AI(x)]$ for all x in S.*

Proof. (1) $\Longleftrightarrow$ (2). Assume that (2) holds. By

$$(XA] \subseteq (XS] \quad \text{and} \quad (XA] \subseteq (A] = A$$

it follows that

$$(XA] \subseteq (XS] \cap A.$$

Similarly,

$$(AX] \subseteq (AX] \cap A.$$

Hence (1) follows. The converse statement, (1) $\Longrightarrow$ (2), follows by Definition 2.4.

(3) $\Longleftrightarrow$ (4). It is clear that (3) $\Longrightarrow$ (4). Conversely, assume that (4) holds. Let $x \in S$. By

$$(I(x)A] \subseteq (I(x)S] \text{ and } (I(x)A] \subseteq (A] = A,$$

it follows that

$$(I(x)A] \subseteq (I(x)S] \cap A.$$

Similarly,

$$(AI(x)] \subseteq (AI(x)] \cap A.$$

We conclude that (3) holds.

(2) $\Longleftrightarrow$ (4). Clearly, (2) $\Longrightarrow$ (4). Assume that (4) holds. Let $X \in \mathfrak{J}(S)$. To show that $A \cap (XS] \subseteq (XA]$, let $a \in A \cap (XS]$. Then $a \in A$ and $a \leqslant xs$ for some $x \in X$ and $s \in S$. Because

$$xs \in I(x)S \subseteq (I(x)S],$$

we have $a \in (I(x)S]$. And by assumption,

$$a \in A \cap (I(x)S] \subseteq (I(x)A] \subseteq (XA].$$

Thus

$$A \cap (XS] \subseteq (XA].$$

Similarly,

$$A \cap (SX] \subseteq (AX].$$

Then (2) holds. This completes the proof of Theorem 2.10. □

Corollary 2.11. *For an element a of an ordered semigroup $(S, \cdot, \leqslant)$, the following statements are equivalent:*

(1) *$I(a)$ is $\mathfrak{J}$-pure;*

(2) *$I(a) \cap (XS] \subseteq (XI(a)]$ and $I(a) \cap (SX] \subseteq (I(a)X]$ for all X in $\mathfrak{J}(S)$;*

(3) *$I(a) \cap (I(x)S] = I(x)I(a)$ and $I(a) \cap (SI(x)] = I(a)I(x)$ for all x in S;*

(4) *$I(a) \cap (I(x)S] \subseteq (I(x)I(a)]$ and $I(a) \cap (SI(x)] \subseteq (I(a)I(x)]$ for all x in S.*

Using the last two results, we have the following:

Theorem 2.12. *For an ordered semigroup $(S, \cdot, \leqslant)$, the following statements are equivalent:*

(1) *S is $\mathfrak{J}^*$-pure;*

(2) *for every ideal A of S, $A \cap (XS] \subseteq (XA]$ and $A \cap (SX] \subseteq (AX]$ for all X in $\mathfrak{J}(S)$;*

(3) *for every ideal A of S, $A \cap (I(x)S] = (I(x)A]$ and $A \cap (SI(x)] = (AI(x)]$ for all x in S;*

(4) *for every ideal A of S, $A \cap (I(x)S] \subseteq (I(x)A]$ and $A \cap (SI(x)] \subseteq (AI(x)]$ for all x in S;*

(5) *every principal ideal of S is $\mathfrak{J}$-pure;*

(6) *for every element a of S, $I(a) \cap (XS] \subseteq (XI(a)]$ and $I(a) \cap (SX] \subseteq (I(a)X]$ for all X in $\mathfrak{J}(S)$;*

(7) *for every element a of S, $I(a) \cap (I(x)S] = (I(x)I(a)]$ and $I(a) \cap (SI(x)] = (I(a)I(x)]$ for all x in S;*

(8) *for every element a of S, $I(a) \cap (I(x)S] \subseteq (I(x)I(a)]$ and $I(a) \cap (SI(x)] \subseteq (I(a)I(x)]$ for all x in S.*

Proof. By Corollary 2.11, (1)-(4) are equivalent, and by Theorem 2.11, (5) – (8) are equivalent. It remains to show that (2) $\Longleftrightarrow$ (6). It is obvious that (2) $\Longrightarrow$ (6). Then we assume that (6) holds. Let A and X be ideals of S, and let $a \in A \cap (XS]$. By assumption,

$$a \in I(a) \cap (XS] \subseteq (XI(a)] \subseteq (XA].$$

Thus

$$A \cap (XS] \subseteq (XA].$$

Similarly,

$$A \cap (SX] \subseteq (AX].$$

Therefore, (2) holds. This completes the proof of Theorem 2.12. □

The following can be found in [5] (Theorem 2.5).

Lemma 2.13. *For an ordered semigroup $(S,\cdot,\leqslant)$, the following statements are equivalent:*

(1) $\mathfrak{J}(S)$ *is idempotent;*

(2) $A\cap B=(AB]$ *for all* $A,B\in\mathfrak{J}(S)$*;*

(3) *every principal ideal of* S *is globally idempotent.*

Corollary 2.14. *An idempotent ideal ordered semigroup* $\mathfrak{J}(S)$ *of an ordered semigroup* $(S,\cdot,\leqslant)$ *is commutative.*

Finally, we have the following:

Theorem 2.15. *If* $(S,\cdot,\leqslant)$ *is an ordered semigroup such that the ideal ordered semigroup* $\mathfrak{J}(S)$ *is idempotent, then* S *is* $\mathfrak{J}^*$*-pure.*

Proof. Let A be any ideal of S. To show that A is $\mathfrak{J}$-pure, let $X\in\mathfrak{J}(S)$. Since $\mathfrak{J}(S)$ is idempotent, it follows from Theorem 2.3 that

$$X=(XS]=(SX].$$

Since A, $(SX]$ and $(XS]$ are ideals of S, it follows from Lemma 2.13 that

$$A\cap(SX]=(A(SX]]=(AX] \text{ and } A\cap(XS]=(A(XS]]=(AX]$$

By Theorem 2.14,

$$A\cap(XS]=(AX]=(XA].$$

Hence, A is $\mathfrak{J}$-pure. □

3. Pure bi-ideals

Let $\mathfrak{B}(S)$ denote the set of all bi-ideals of an ordered semigroup $(S,\cdot,\leqslant)$. For B_1,B_2 in $\mathfrak{B}(S)$, define

$$B_1*B_2:=(B_1B_2].$$

It is observed that

$$(B_1*B_2)(B_1*B_2)=(B_1B_2](B_1B_2]\subseteq(B_1B_2B_1B_2]\subseteq(B_1B_2].$$

Then $(\mathfrak{B}(S),*)$ is a semigroup. Moreover, $(\mathfrak{B}(S),*,\subseteq)$ is an ordered semigroup.

An ordered semigroup $(S,\cdot,\leqslant)$ is said to be *regular* ([2]) if for any a in S there exists x in S such that $a\leqslant axa$.

Theorem 3.1. *For an ordered semigroup $(S,\cdot,\leqslant)$, the following conditions are equivalent:*

(1) *S is regular;*

(2) *$\mathfrak{B}(S)$ is regular.*

Proof. Assume that S is regular. Let $B \in \mathfrak{B}(S)$. For each $b \in B$, by assumption then there exists x_b in S such that $b \leqslant bx_bb$. Setting $B' := \{x_b \mid b \in B\}$; hence

$$B \subseteq BB(B')B.$$

Conversely, assume that $\mathfrak{B}(S)$ is regular. Let $a \in S$. Consider $B(a)$ the principal bi-ideal generated by a, there exists B in $\mathfrak{B}(S)$ such that

$$B(a) \subseteq B(a)BB(a).$$

But,

$$(a \cup a^2 \cup aSa]B(a \cup a^2 \cup aSa]$$

is a subset of

$$(aBa \cup aBa^2 \cup aBaSa \cup a^2Ba \cup a^2Ba^2 \cup a^2BaSa \cup aSaBa \cup aSaBa^2 \cup aSaBaSa],$$

it then follows that a is regular. □

Consequently,

Corollary 3.2. *For an ordered semigroup $(S, \cdot, \leqslant)$, if $\mathfrak{B}(S)$ is idempotent, then $\mathfrak{B}(S)$ is regular.*

Analogous to [9], we now introduce the main concept of this section:

Definition 3.3. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A bi-ideal B of S is said to be *$\mathfrak{T}$-pure* if

$$B \cap (xSy] = (xBy]$$

for all x, y in S.

Definition 3.4. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then S is said to be *$\mathfrak{T}^*$-pure* if every bi-ideal of S is $\mathfrak{T}$-pure.

Example 3.5. Let $S = \{a, b, c\}$ be an ordered semigroup with the operation and the order relation are defined by:

$\cdot$	a	b	c
a	a	b	a
b	b	b	b
c	c	b	c

$$\leqslant := \{(a, a), (b, b), (c, c), (a, b), (b, c)\}.$$

The only $\mathfrak{T}$-pure of S is itself.

Example 3.6. Let $S = \{a, b, c\}$ be an ordered semigroup with the operation and the order relation are defined by:

$\cdot$	a	b	c
a	b	c	c
b	c	c	c
c	c	c	c

$$\leqslant := \{(a, a), (b, b), (c, c), (c, a), (c, b)\}.$$

The $\mathfrak{T}$-pures of S are $\{c\}, \{a, c\}, \{b, c\}$ and S.

In Example 3.5 we have S is $\mathfrak{T}^*$-pure, and in Example 3.6 we have S is also $\mathfrak{T}^*$-pure.

Lemma 3.7. *Let B be a bi-ideal of an ordered semigroup $(S, \cdot, \leqslant)$. The following are equivalent:*

(1) $B \cap (B_1SB_2] = (B_1BB_2]$ *for all* B_1, B_2 *in* $\mathfrak{B}(S)$*;*

(2) $B \cap (B(x)SB(y)] = (B(x)BB(y)]$ *for all* x, y *in* S*.*

Proof. That (1) $\Longrightarrow$ (2) is clear. Assume that (2) holds. Let $B_1, B_2 \in \mathfrak{B}(S)$. Let $x \in B \cap (B_1SB_2]$. Since $x \in (B_1SB_2]$, we have $x \leqslant b_1sb_2$ for some $b_1 \in B_1, s \in S, b_2 \in B_2$. By $b_1sb_2 \in B(b_1)SB(b_2)$, it follows that

$$x \in B \cap (B(b_1)SB(b_2)] = (B(b_1)BB(b_2)] \subseteq (B_1BB_2].$$

Thus

$$B \cap (B_1SB_2] \subseteq (B_1BB_2].$$

Now, we prove the reverse inclusion. Since $(B_1BB_2] \subseteq (B_1SB_2]$, it suffices to show that $(B_1BB_2] \subseteq B$. Let $x \in (B_1BB_2]$; then $x \leqslant b_1'b'b_2'$ for some $b_1' \in B_1, b' \in B, b_2' \in B_2$. By assumption,

$$b_1'b'b_2' \in B(b_1')BB(b_2') \subseteq (B(b_1')BB(b_2')] = B \cap (B(b_1')SB(b_2')] \subseteq B.$$

And, $x \in B$. Hence the proof is complete. □

Lemma 3.8. *If B is a $\mathfrak{T}$-pure bi-ideal of an ordered semigroup $(S, \cdot, \leqslant)$, then one of the equivalent conditions in Lemma* 3.7 *holds.*

Proof. Assume that B is a $\mathfrak{T}$-pure bi-ideal of an ordered semigroup S. We will show that the condition (1) of Lemma 3.7 holds. Let $B_1, B_2 \in \mathfrak{B}(S)$. Chose $x \in B \cap (B_1SB_2]$; then $x \leqslant b_1sb_2$ for some $b_1 \in B_1, s \in S, b_2 \in B_2$. By assumption,

$$x \in B \cap (b_1Sb_2] = (b_1Bb_2] \subseteq (B_1BB_2].$$

Then

$$B \cap (B_1 S B_2] \subseteq (B_1 B B_2].$$

Let $x \in (B_1 B B_2]$; then $x \leqslant b_1' b' b_2'$ for some $b_1' \in B_1, b' \in B, b_2' \in B_2$. Since

$$b_1' b' b_2' \in b_1 B b_2 \subseteq (b_1 B b_2] = B \cap (b_1 S b_2] \subseteq B \cap (B_1 S B_2],$$

we have $x \in B \cap (B_1 S B_2]$. □

Theorem 3.9. *For an ordered semigroup $(S, \cdot, \leqslant)$, the following conditions are equivalent:*

(1) *S is $\mathfrak{T}^*$-pure;*

(2) *every principal bi-ideal of S is $\mathfrak{T}$-pure.*

Proof. It is clear that (1) $\Longrightarrow$ (2). Assume that every principal bi-ideal of S is $\mathfrak{T}$-pure. Let B be a bi-ideal of S, and let $x, y \in S$. If $a \in B \cap (xSy]$, then

$$a \in B(a) \cap (xSy] = (xB(a)y] \subseteq (xBy].$$

Thus

$$B \cap (xSy] \subseteq (xBy].$$

For the reverse inclusion, since $(xBy] \subseteq (xSy]$, it suffices to show that $(xBy] \subseteq B$. If $a \in (xBy]$, then $a \leqslant xb'y$ for some $b' \in B$. By assumption we have

$$xb'y \in xB(b')y \subseteq (xB(b')y] = B(b') \cap (xSy] \subseteq B,$$

it follows that $a \in B$. □

Theorem 3.10. *For a minimal bi-ideal B of an ordered semigroup $(S, \cdot, \leqslant)$, B is $\mathfrak{T}$-pure if and only if $B = (xBy]$ for all x, y in S.*

Proof. Assume that B is $\mathfrak{T}$-pure. If $x, y \in S$, then

$$(xBy] = B \cap (xSy] \subseteq B.$$

Since $(xBy]$ is a bi-ideal of S, it follows by the minimality of B that $B = (xBy]$.

Conversely, assume that $B = (xBy]$ for all x, y in S. Then, for any $x, y \in S$,

$$B \cap (xSy] = (xBy] \cap (xSy] = (xBy].$$

This means B is $\mathfrak{T}$-pure. □

Corollary 3.11. *Every minimal $\mathfrak{T}$-pure bi-ideal of an ordered semigroup $(S, \cdot, \leqslant)$ is regular.*

Proof. If B is a $\mathfrak{T}$-pure bi-ideal of an ordered semigroup S, then, for any $a \in B$, $B = (aBa]$; hence a is regular. □

Lemma 3.12. *For a bi-ideal B of an ordered semigroup $(S, \cdot, \leqslant)$, the following are equivalent:*

(1) $B = (XBY]$ *for all* X, Y *in* $\mathfrak{B}(S)$;

(2) $B = (B(x)BB(y)]$ *for all* $x, y \in S$.

Proof. Assume that $B = (B(x)BB(y)]$ for all $x, y \in S$. Let $X, Y \in \mathfrak{B}(S)$. Choose $x \in X$ and $y \in Y$; then because of assumption we have

$$B = (B(x)BB(y)] \subseteq (XBY].$$

Let $s \in (XBY]$; then $s \leqslant xby$ for some $x \in X, b \in B, y \in Y$. By

$$xby \in B(x)BB(y) \subseteq (B(x)BB(y)] = B$$

it follows that $s \in B$. This proves (2) $\Rightarrow$ (1). That (1) $\Rightarrow$ (2) is clear. □

Theorem 3.13. *For a minimal $\mathfrak{T}$-pure bi-ideal B of an ordered semigroup $(S, \cdot, \leqslant)$, one of the equivalent conditions in Lemma* 3.12 *holds.*

Proof. If $X, Y \in \mathfrak{B}(S)$, then by B is $\mathfrak{T}$-pure we have

$$(XBY] = B \cap (XSY] \subseteq B.$$

Using the minimality of B, $B = (XBY]$. □

Theorem 3.14. *For an ordered semigroup $(S, \cdot, \leqslant)$, the following conditions are equivalent:*

(1) S *is regular;*

(2) $\mathfrak{B}(S)$ *is regular;*

(3) $\mathfrak{B}(S)$ *is idempotent.*

Proof. We have (1) $\Longleftrightarrow$ (2) by Theorem 3.1. And (3) $\Longrightarrow$ (2) is clear, so we will show that (2) $\Longrightarrow$ (3). Assume that (2) holds. Let B be a bi-ideal of S. Then there exists $B_1 \in \mathfrak{B}(S)$ such that $B = BB_1B$. Since

$$B = BB_1B \subseteq BSB \subseteq (BSB] \subseteq (B] = B$$

we have $B = (BSB]$. Since B is $\mathfrak{T}$-pure, it follows by Lemma 3.8 that

$$(BBB] = B \cap (BSB] = B \cap B = B.$$

Thus

$$B = (BBB] \subseteq (BB] \subseteq (B] = B.$$

That is, $\mathfrak{B}(S)$ is idempotent. □

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T. Changphas
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayuttaya Rd., Bangkok 10400, Thailand
E-mail: thacha@kku.ac.th

P. Kummoon
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
E-mail: pisit_2534@hotmail.com