Quasigroups and Related Systems **26** (2018), 155 − 164

Fuzzy action of fuzzy groups on a set

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Abstract. We describe the notion of fuzzy action of fuzzy groups on a set using method proposed by X. Yuan and E.S. Lee.

1. Introduction

The concept of fuzzy sets was introduced by L. A. Zadeh [8] and applied by A. Rosenfeld [3] to groups. Rosenfeld assumed that subsets of a group G are fuzzy sets and the binary operation on G is nonfuzzy in the classical sense. Another concept of fuzzy subgroups was proposed by X. Yuan and E.S. Lee [7]. They assumed that the binary operation is the fuzzy operation. This approach was developed by some other researchers (see [1, 5, 6]).

On the other hand, Haddadi [2] and Roventa and Spircu [4] studied fuzzy actions of fuzzy submonoids and fuzzy subgroups from an algebraic point of view.

In this paper we investigate a fuzzy action of fuzzy groups on a set using the idea of fuzzy groups proposed by X. Yuan and E. S. Lee [7].

2. Preliminaries

Remember that, a fuzzy subset of a set X is a function from X to the real interval $[0, 1]$. By a fuzzy relation from X to Y we mean a fuzzy subset of $X \times Y$. Throughout this paper $\theta \in [0, 1)$.

Definition 2.1. Let X and Y be nonempty sets. A fuzzy relation f from X to Y is said to be a *fuzzy function* from X to Y if the following are satisfied:

1. for each $x \in X$ there exists $y \in Y$ such that $f(x, y) > \theta$,
2. for each $x \in X$ and $y_1, y_2 \in Y$, $f(x, y_1) > \theta$ and $f(x, y_2) > \theta$ imply $y_1 = y_2$.

A fuzzy function f is one-to-one (or injective) if for each $x_1, x_2 \in X$ and $y \in Y$ $f(x_1, y) > \theta$ and $f(x_2, y) > \theta$ imply $x_1 = x_2$. A fuzzy function f is onto (or surjective) if for each $y \in Y$ there exists $x \in X$ such that $f(x, y) > \theta$.

2010 Mathematics Subject Classification: 03E72, 08A72, 20N25
Keywords: Fuzzy binary operation, fuzzy group based on fuzzy binary operation, fuzzy action

Definition 2.2. (cf. [7]) Let G be a nonempty set. By a *fuzzy binary operation* on G, we mean a fuzzy function R from $G \times G$ into G.

If R is a fuzzy binary operation on G, then we have a mapping:

$$\circ : F(G) \times F(G) \to F(G), \qquad (A, B) \mapsto \circ(A, B),$$

where $F(G) = \{A \mid A\colon G \to [0,1] \text{ is a mapping}\}$ and

$$\circ(A, B)(c) = \bigvee_{x,y \in G} (A(x) \wedge B(y) \wedge R(x, y, c)). \tag{1}$$

For $A = \{a\}$ and $B = \{b\}$, let us denote $\circ(A, B)$ by $a \circ b$. Then

$$(a \circ b)(c) = R(a, b, c) \quad \forall c \in G, \tag{2}$$

$$((a \circ b) \circ c)(z) = \bigvee_{x \in G} (R(a, b, x) \wedge R(x, c, z)), \tag{3}$$

$$(a \circ (b \circ c))(z) = \bigvee_{x \in G} (R(a, x, z) \wedge R(b, c, x)). \tag{4}$$

Using the notations in $(2) - (4)$, we have the following definition:

Definition 2.3. (cf. [7]) Let G be a nonempty set and R be a fuzzy binary operation on G. The pair (G, R) is called a *fuzzy group* if the following conditions are satisfied:

(G1) $\forall a, b, c, z_1, z_2 \in G$, $((a \circ b) \circ c)(z_1) > \theta$ and $(a \circ (b \circ c))(z_2) > \theta$ implies that $z_1 = z_2$,

(G2) $\exists e_\circ \in G$ such that $(e_\circ \circ a)(a) > \theta$ and $(a \circ e_\circ)(a) > \theta$,

(G3) $\forall a \in G$, $\exists b \in G$ such that $(a \circ b)(e_\circ) > \theta$ and $(b \circ a)(e_\circ) > \theta$. In this case b is called an *inverse* of a and is denoted by a^{-1}.

3. Fuzzy action of fuzzy groups on a set

Throughout this section G stands for a fuzzy group.

Let X be a nonempty set and R^* be a fuzzy function from $G \times X$ to X. Then we have a function:

$$* : F(G) \times F(X) \to F(X), \qquad (A, Y) \mapsto *(A, Y),$$

here $F(G) = \{A \mid A\colon G \to [0,1] \text{ is a mapping }\}$, $F(X) = \{Y \mid Y\colon X \to [0,1] \text{ is a mapping }\}$ and

$$*(A, Y)(z) = \bigvee_{s \in G, t \in X} (A(s) \wedge Y(t) \wedge R^*(s, t, z)) \quad \forall z \in X. \tag{5}$$

Let $A = \{a\}$ and $Y = \{y\}$, and let $*(A, Y)$ be denoted by $a * y$. Then

$$(a * y)(z) = R^*(a, y, z) \forall a \in G, \forall y, z \in X \tag{6}$$

$$((a \circ b) * y)(z) = \bigvee_{s \in G} (R^\circ(a, b, s) \wedge R^\star(s, y, z)) \forall a, b \in G, \forall y, z \in X \tag{7}$$

$$(a * (b * y))(z) = \bigvee_{t \in X} R^*(a, t, z) \wedge R^*(b, y, t), \forall a \in G, \forall y, z \in X \tag{8}$$

Using the notations in (6) – (8), we have the following definition:

Definition 3.1. A fuzzy function R^* of $G \times X$ into X is called a *fuzzy action* of G on X if the following are satisfied:

1. $(e * x)(x) > \theta$ for all $x \in X$,
2. $((a \circ b) * x)(z_1) > \theta$ and $(a * (b * x))(z_2) > \theta$ implies $z_1 = z_2$.

In this case we say that *G acts on X by a fuzzy action R^**.

Example 3.2. Let (G, R°) be a fuzzy group. Put X to be the set G. Define a fuzzy subset R^* of $G \times X \times X$ by:

$$R^*(a, x, z) = R^\circ(e, x, z) \text{ for all } a \in G,\ x, z \in X.$$

Then R^* is a fuzzy action of G on itself, and it is called the trivial fuzzy action of G onto itself.

Example 3.3. Let (G, R°) be a fuzzy group. Put X to be the set G. Define a fuzzy subset R^* of $G \times X \times X$ by:

$$R^*(a, x, z) = R^\circ(a, x, z) \text{ for all } a \in G,\ x, z \in X.$$

Then R^* is a fuzzy action of G on itself and it is called a fuzzy action of G on itself by left translation.

Example 3.4. Let (G, R°) be a fuzzy group. Put X to be the set G. Define a fuzzy subset R^* of $G \times X \times X$ by:

$$R^*(a, x, z) = ((a \circ x) \circ a^{-1})(z) \text{ for all } a \in G,\ x, z \in X.$$

Then R^* is a fuzzy action of G on itself and it is known as a fuzzy action of G on itself by conjugation.

Example 3.5. Let (G, R) be a fuzzy group and H be a normal fuzzy subgroup of G. Put X to be the set $\frac{G}{H}$. Define a fuzzy subset R^* of $G \times \frac{G}{H} \times \frac{G}{H}$ by:

$$R^*(a, [bH], [cH]) = R(a, b, c) \quad \forall a, b, c \in G.$$

Then R^* is a fuzzy action of G onto its quotient $\frac{G}{H}$.

Let us define a relation $\sim$ on X as follows:

For each $x, y \in X$, $x \sim y$ if and only if $R^*(a, x, y) > \theta$ for some $a \in G$.

Theorem 3.6. *The relation $\sim$ is an equivalence relation on X.*

Proof. Let $x, y, z \in X$.

1. Since $R^*(e, x, x) > \theta$ we have $x \sim x$.
2. Let $x \sim y$. Then there exists $a \in G$ such that $R^*(a, x, y) > \theta$. As $a^{-1} \in G$, $y \in X$ and R^* is a fuzzy mapping from $G \times X$ to X, there exists $z \in X$ such that $R^*(a^{-1}, y, z) > \theta$. Now consider the following:

$$(a^{-1}*(a*x))(z) = \bigvee_{t \in X} R^*(a^{-1}, t, z) \wedge R^*(a, x, t) \geqslant R^*(a^{-1}, y, z) \wedge R^*(a, x, y) > \theta.$$

   Also

$$((a^{-1} \circ a)*x)(x) = \bigvee_{s \in G} R^\circ(a^{-1}, a, s) \wedge R^*(s, x, x) \geqslant R^\circ(a^{-1}, a, e) \wedge R^*(e, x, x) > \theta.$$

   This implies that $x = z$, so $R^*(a^{-1}, y, x) > \theta$ and hence $y \sim x$.

3. Let $x \sim y$ and $y \sim z$. There exists $a, b \in G$ such that $R^*(a, x, y) > \theta$ and $R^*(b, y, z) > \theta$. As $a, b \in G$, there exists $c \in G$ such that $R^\circ(b, a, c) > \theta$. Also there exists $w \in X$ such that $R^*(c, x, w) > \theta$. We show that $w = z$. For this, consider:

$$((b \circ a) * x)(w) = \bigvee_{s \in G} R^\circ(b, a, s) \wedge R^*(s, x, w) \geqslant R^\circ(b, a, c) \wedge R^*(c, x, w) > \theta.$$

   Also

$$(b * (a * x))(z) = \bigvee_{t \in X} R^*(b, t, z) \wedge R^*(a, x, t) \geqslant R^\circ(b, y, z) \wedge R^*(a, x, y) > \theta,$$

   which implies $w = z$. So, $x \sim z$.

Therefore $\sim$ is an equivalence relation on X. □

The equivalence class $\overline{x} = \{y \in X \mid x \sim y\}$ is called the *orbit* of x and is denoted by $Or(x)$. The set $St(x) = \{a \in G \mid R^*(a, x, x) > \theta\}$ is called the *stabilizer* of x.

Theorem 3.7. *$St(x)$ is a fuzzy subgroup of G.*

Proof. Since $R^*(e, x, x) > \theta$, then $e \in St(x)$. Now let a, $b \in St(x)$ such that $R^\circ(a, b, c) > \theta$. We show that $c \in St(x)$. Let $z \in X$ such that $R^*(c, x, z) > \theta$. Then

$$((a \circ b) * x)(z) = \bigvee_{s \in G} R^\circ(a, b, s) \wedge R^*(s, x, z) \geqslant R^\circ(a, b, c) \wedge R^*(c, x, z) > \theta$$

and

$$(a*(b*x)(x)=\bigvee_{t\in X} R^*(a,t,x)\wedge R^*(b,x,t) \geqslant R^\circ(a,x,x)\wedge R^*(b,x,x)>\theta,$$

which implies $z=x$. So, $R^*(c,x,x)>\theta$ and hence $c\in St(x)$. Let $a\in St(x)$. Then $R^*(a,x,x)>\theta$. Let $z\in X$ such that $R^*(a^{-1},x,z)>\theta$. Now it can be verified that

$$((a^{-1}\circ a)*x)(x)>\theta \quad \text{and} \quad (a^{-1}*(a*x))(z)>\theta.$$

So, $z=x$. That is, $R^*(a^{-1},x,x)>\theta$ and hence $a^{-1}\in St(x)$. Therefore $St(x)$ is a fuzzy subgroup of G. □

Let G be a fuzzy group acting on a set X by a fuzzy action R^*. For each $a\in G$, let us define $R^*_a : X\times X\to[0,1]$ by:

$$R^*_a(x,y)=R^*(a,x,y) \quad \forall x,y\in X. \tag{9}$$

Then we have the following:

Theorem 3.8. *For each $a\in G$, the fuzzy map R^*_a is a fuzzy bijection of X onto itself.*

Proof. Clearly R^*_a is a fuzzy function.

(1) We show that R^*_a is one-to-one. Let $x_1,x_2,y\in X$ be such that $R^*_a(x_1,y)>\theta$ and $R^*_a(x_2,y)>\theta$. Then $R^*(a,x_1,y)>\theta$ and $R^*(a,x_2,y)>\theta$. Also let $z\in X$ be such that $R^*(a^{-1},y,z)>\theta$. Now we have:

$$(a^{-1}*(a*x_1))(z)>\theta \quad \text{and} \quad (a^{-1}\circ a)*x_1)(x_1)>\theta,$$

which gives$z=x_1$. In a similar way we get $z=x_2$. Thus R^*_a is one-to-one.

(2) To show that it is onto, let $y\in X$. As $a^{-1}\in G$ and R^* is a fuzzy action, there exists $x\in X$ such that $R^*(a^{-1},y,x)>\theta$. Also as $a\in G$ and $x\in X$ there exists $z\in X$ such that $R^*(a,x,z)>\theta$. Then:

$$(a*(a^{-1}*y)(z)=\bigvee_{t\in X} R^*(a,t,z)\wedge R^*(a^{-1},y,t) \geqslant R^*(a,x,z)\wedge R^*(a^{-1},y,x)>\theta$$

and

$$((a\circ a^{-1})*y)(y)=\bigvee_{s\in G} R^\circ(a,a^{-1},s)\wedge R^*(s,y,y) \geqslant R^\circ(a,a^{-1},e)\wedge R^*(e,y,y)>\theta,$$

which implies $z=y$. That $R^*_a(x,y)=R^*(a,x,y)>\theta$. So, R^*_a is onto. Therefore it is a bijection. □

Put $\Im^*(X) = \{R^*_a \,|\, a \in G\}$ and define a fuzzy subset H of $\Im^*(X) \times \Im^*(X) \times \Im^*(X)$ by:

$$H(R^*_a, R^*_b, R^*_c) = R(a, b, c) \quad \forall a, b, c \in G.$$

Then we have the following results:

Theorem 3.9. *$(\Im^*(X), H)$ is a fuzzy group.*

Corollary 3.10. *The function $f : G \to \Im^*(X)$ defined by $f(a) = R^*_a$ for all $a \in G$ is an epimorphism.*

The epimorphism f defined above is called the *homomorphism associated with* R^*. If f is one-to-one then R^* is called an *effective fuzzy action* of G on X. By the *kernel* of R^*, we mean the kernel of the homomorphism f.

Lemma 3.11. *The fuzzy action R^* of a fuzzy group G on a set X is effective if and only if $kerR^* = \{e\}$.*

Corollary 3.12. *If R^* is the fuzzy action of G on itself by left translation, then R^* is effective.*

Corollary 3.13. *Let R^* be the fuzzy action of G on itself by conjugation. Then, R^* is effective if and only if the center $Z(G)$ of G is $\{e\}$.*

Proof. This can be verified by showing that $kerf = Z(G)$. □

Theorem 3.14. *There is a fuzzy one-to-one correspondence between $Or(x)$ and the collection of all left cosets of $St(x)$.*

Proof. Let $H = St(x)$ and $\Sigma = \{[aH] \,|\, a \in G\}$. Define a fuzzy subset α of $Or(x) \times \Sigma$ by:

$$\alpha(y, [aH]) = R^*(a, x, y)$$

for all $a \in G, y \in Or(x)$.
First we show that α is a fuzzy function.

1. Let $y \in Or(x)$. There exists $a \in G$ such that $R^*(a, x, y) > \theta$. That is, $\alpha(y, [aH]) > \theta$.

2. Let $y \in Or(x)$ and $a, b \in G$ be such that $\alpha(y, [aH]) > \theta$ and $\alpha(y, [bH]) > \theta$. Let $z \in G$ be such that $R^*(a^{-1}, y, z) > \theta$. Then

$$(a^{-1} * (a * x))(z) = \bigvee_{t \in X} R^*(a^{-1}, t, z) \wedge R^*(a, x, t) \geqslant R^*(a^{-1}, y, z) \wedge R^*(a, x, y) > \theta$$

and

$$((a^{-1} \circ a) * x)(x) = \bigvee_{s \in G} R^\circ(a^{-1}, a, s) \wedge R^*(s, x, x) \geqslant R^\circ(a^{-1}, a, e) \wedge R^*(e, x, x) > \theta,$$

which implies $z = x$. So, $R^*(a^{-1}, y, x) > \theta$. Similarly $R^*(b^{-1}, y, x) > \theta$.

Let $c \in G$ be such that $R^\circ(a^{-1}, b, c) > \theta$. We need to show that $c \in H$. Let $z \in X$ be such that $R^*(c, x, z) > \theta$. Then

$$((a^{-1} \circ b) * x)(z) = \bigvee_{s \in G} R^\circ(a^{-1}, b, s) \wedge R^*(s, x, z) \geqslant R^\circ(a^{-1}, b, c) \wedge R^*(c, x, z) > \theta$$

and

$$(a^{-1} * (b * x))(x) = \bigvee_{t \in X} R^*(a^{-1}, t, x) \wedge R^*(b, x, t) \geqslant R^*(a^{-1}, y, x) \wedge R^*(b, x, y) > \theta,$$

which gives $z = x$. So, $R^*(c, x, x) > \theta$. Hence $c \in H$. Thus $[aH] = [bH]$.

Therefore α is a fuzzy function. Since R^* is a fuzzy action of G on X, it follows that α is one-to-one and onto. Hence α is a fuzzy one-to-one correspondence. □

Definition 3.15. A fuzzy action of a fuzzy group G on a set X is said to be *transitive* if there is only one orbit in X, i.e, $Or(x) = X$ for all $x \in X$. In this case, we say that G acts transitively on X.

Lemma 3.16. *A fuzzy action R^* is transitive if and only if, for any pairs of elements $x, y \in X$, there is an element $a \in G$ such that $R^*(a, x, y) > \theta$.*

Corollary 3.17. *Let R^* be a fuzzy action of a fuzzy group (G, R°) on itself by left translation. Then R^* is transitive.*

Definition 3.18. Let G be acting on two sets X_1 and X_2 by fuzzy actions R^* and $R^\triangle$ respectively. These two fuzzy actions are said to be *equivalent* if there exists a bijection $\alpha : X_1 \to X_2$ such that

$$R^*(a, x, y) > \theta \Rightarrow R^\triangle(a, \alpha(x), \alpha(y)) > \theta$$

for all $a \in G$, $x, y \in X_1$.

Lemma 3.19. *Let R^* be the fuzzy action of G on itself by left translation and let $R^\triangle$ be the fuzzy action of G on itself defined by:*

$$R^\triangle(a, x, y) = R^\circ(x, a^{-1}, y)$$

for all $a, x, y \in G$. Then the fuzzy actions R^* and $R^\triangle$ are equivalent.

Proof. This can be verified by taking $\alpha : G \to G$ defined by $\alpha(x) = x^{-1}$ for all $x \in G$. □

Below we obtain an internal characterization of transitive fuzzy actions

Theorem 3.20. *Let R^* be a transitive fuzzy action of G on a set X, $x \in X$ and $H = St(x)$. Then, R^* is equivalent to the fuzzy action of G on the set of left cosets of H in G by left translation.*

Proof. Put $H = St(x)$, $X_1 = X$ and $X_2 = \{[aH] \mid a \in G\}$. Since R^* is transitive we have $Or(x) = X_1$. So for each $y \in X_1$ there exists $b \in G$ such that $R^*(b, x, y) > \theta$. Now define $f : X_1 \to X_2$ by:

$$f(y) = [aH]$$

for all $y \in X_1$. This f is a bijection. Remember that a fuzzy subset $R^\diamond$ of $G \times X_2 \times X_2$ defined by

$$R^\diamond(a, [bH], [cH]) = R^\circ(a, b, c) \;\; \forall a, b, c \in G$$

is a fuzzy action of G on X_2 by left translation. Let $a \in G$ and $y_1, y_2 \in X_1$ such that $R^*(a, y_1, y_2) > \theta$. Now we show that

$$R^\diamond(a, f(y_1), f(y_2)) > \theta.$$

As $y_1 \in X_1 = Or(x)$, there exists $b \in G$ such that $R^*(b, x, y_1) > \theta$. As $a, b \in G$, $\exists c \in G$ such that $R^\circ(a, b, c) > \theta$. Let $z \in X_1$ be such that $R^*(c, x, z) > \theta$. Then:

$$((a \circ b) * x)(z) = \bigvee_{s \in G} R^\circ(a, b, s) \wedge R^*(s, x, z) \geqslant R^\circ(a, b, c) \wedge R^*(c, x, z) > \theta$$

and

$$(a * (b * x))(y_2) = \bigvee_{t \in X} R^*(a, t, y_2) \wedge R^*(b, x, t) \geqslant R^*(a, y_1, y_2) \wedge R^*(b, x, y_1) > \theta,$$

which implies $y_2 = z$. So $R^*(c, x, y_2) > \theta$. That is $f(y_2) = [cH]$. Also we have $f(y_1) = [bH]$. Thus

$$R^\diamond(a, f(y_1), f(y_2)) = R^\diamond(a, [bH], [cH]) = R^\circ(a, b, c) > \theta.$$

Hence R^* and $R^\diamond$ are equivalent. □

Among the transitive fuzzy actions, there is a special class, namely primitive fuzzy actions, which deserves emphasis.

Definition 3.21. Let R^* be a fuzzy action of a fuzzy group G on a set X. An equivalence relation ψ on X is said to be *compatible* with the fuzzy action R^* if, for any $x_1, x_2, y_1, y_2 \in X$, and $a \in G$ with $R^*(a, x_1, y_1) > \theta$ and $R^*(a, x_2, y_2) > \theta$

$$(x_1, x_2) \in \psi \Rightarrow (y_1, y_2) \in \psi.$$

Clearly the whole of $X \times X$ and the diagonal Δ_X are equivalence relation X which are compatible with every fuzzy action of G on X.

Definition 3.22. A fuzzy action of a fuzzy group G on a set X is called *primitive* if $X \times X$ and the diagonal Δ_X are the only equivalence relations on X which are compatible with it. A fuzzy action which is not primitive is called *imprimitive*.

Theorem 3.23. *Let R^* and $R^{\triangle}$ be equivalent fuzzy actions of G on X_1 and X_2, respectively. Then, R^* is primitive if and only if $R^{\triangle}$ too is so.*

Proof. Let R^* and $R^{\triangle}$ be equivalent fuzzy actions. Then there exists a bijection $f : X_1 \to X_2$ such that for any $a \in G$ and $x, y \in X_1$:

$$R^*(a, x, y) > \theta \Rightarrow R^{\triangle}(a, f(x), f(y)) > \theta.$$

Suppose that R^* is primitive. We show that $R^{\triangle}$ is also primitive. Let ψ be an equivalence relation on X_2 which is compatible with $R^{\triangle}$. Then it can be seen that $f^{-1}(\psi) = \{(x, y) \in X_1 \times X_2 \,|\, (f(x), f(y)) \in \psi\}$ is an equivalence relation on X_1 which is compatible with R^*. That is $f^{-1}(\psi) = \Delta_{X_1}$ or $f^{-1}(\psi) = X_1 \times X_1$ which is implies that $\psi = \Delta_{X_2}$ or $\psi = X_2 \times X_2$. □

Theorem 3.24. *Let R^* be a fuzzy action of G on a set X. Then, R^* is imprimitive if and only if there exists a proper subset Y of X with $|Y| > 1$ such that, for any $a \in G$, either $R^*(a, Y) = Y$ or $R^*(a, Y) \cap Y = \emptyset$, where $R^*(a, Y) = \{z \in X |\, R^*(a, y, z) > \theta$ for some $y \in Y\}$.*

Proof. Suppose that R^* is imprimitive. Then, there exists an equivalence relation ψ on X which is compatible with R^* such that $\psi \neq X \times X$ and $\psi \neq \Delta_X$. Choose $x \neq y \in X$ such that $(x, y) \in \psi$. Put $Y =$ the equivalence class of ψ containing x. That is $Y = \psi(x) = \{z \in X \,|\, (x, z) \in \psi\}$. Since $x \neq y \in Y$, $|Y| > 1$. Moreover, since $\psi \neq X \times X$, Y is a proper subset of X. Now let $a \in G$ such that $R^*(a, Y) \cap Y \neq \emptyset$. Then choose an element $z_1 \in Y$ such that $R^*(a, z_1, z_2) > \theta$ for some $z_2 \in Y$ so, $(x, z_2) \in \psi$. Let $R^*(a, x, u) > \theta$ for some $u \in X$. Since $(x, z_1) \in \psi$ and ψ is compatible with R^*, we get $(u, z_2) \in \psi$ which implies that $(x, u) \in \psi$. Now it can be easily verified that $R^*(a, Y) = Y$.

Conversely suppose that there is a proper subset Y of X with $|Y| > 1$ such that either $R^*(a, Y) \cap Y = \emptyset$ or $R^*(a, Y) = Y$. Then for any a and $b \in G$,

$$\text{either} \quad R^*(a, Y) = R^*(b, Y) \quad \text{or} \quad R^*(a, Y) \cap R^*(b, Y) = \emptyset.$$

Put $Z = X - (\bigcup_{a \in G} R^*(a, Y))$. Then $\mathrm{P} = \{R^*(a, Y) \,|\, a \in G\} \cup \{Z\}$ is a partition of X and the corresponding equivalence relation ψ on X is compatible with the fuzzy action R^*. Since $Y = R^*(e, Y)$ is an equivalence class and $Y \neq X$, it follows that $\psi \neq X \times X$. Also since $|Y| > 1$, $\psi \neq \Delta_X$. Thus R^* is imprimitive. □

Lemma 3.25. *Let R^* be a fuzzy action of G on a set X. Define*

$$\psi^* = \{(x, y) \in X \times X \,|\, R^*(a, x, y) > \theta \text{ for some } a \in G\}.$$

Then, ψ^* is an equivalence relation on X, which is compatible with the fuzzy action action R^*.

Corollary 3.26. *If a fuzzy action R^* of G on X is primitive, then either R^* is transitive or*

$$R^*(a,x,y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in G$ and $x, y \in X$.

Proof. If R^* is fuzzy primitive, then $\psi^* = \Delta_X$ or $X \times X$ and hence all orbits are singleton sets or there is only one orbit. □

In particular, a nontrivial primitive fuzzy action must be necessarily transitive and hence the class of nontrivial primitive fuzzy actions of a fuzzy group G on a set X is a subclass of the transitive fuzzy actions of G on X. But in general a transitive fuzzy action need not be primitive.

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Received April 21, 2018

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