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Retractions of cyclic finitely supported Cb-sets

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Abstract. The monoid Cb of name substitutions originated by Pitts in name abstraction, and the notion of a finitely supported Cb-set appeared in the study of models of homotopy type theory in the works of Gabbay and Pitts. On the other hand, retracts and retractions play a crucial role in most branches of mathematics as well as in computer science where partial morphisms need to be completed. Retracts are the subobjects whose related inclusion morphism have a left inverse, called retraction.

In this paper, we study the retracts and retractions of cyclic finitely supported Cb-sets. We find the general definition of retractions from a cyclic Cb-set, and give necessary conditions under which retractions exist. Also, fix-simple retracts of a cyclic Cb-set are characterized. Further, the cyclic finitely supported Cb-sets all whose subobjects are retract, are studied. In particular, we give a necessary condition for a cyclic finitely supported Cb-set to be retractable.

1. Introduction and preliminaries

The notion of a nominal set was originated by Fraenkel in 1922 and developed by Mostowski in the 1930s under the name of legal sets. The legal sets were applied to prove the independence of the axiom of choice with the other axioms (in the classical Zermelo-Fraenkel (ZF) set theory).

In 2001, Gabbay and Pitts rediscovered those sets in the context of name abstraction. They called them nominal sets, and applied this notion to properly model the syntax of formal systems involving variable binding operations (see [5]).

In [10], Pitts generalized the notion of nominal sets, by first adding two elements 0, 1 to $\mathbb{D}$, then generalizing the notion of a finitary permutation to *finite substitution*, and considering the monoid Cb instead of the group G. Then he defined the notion of a support for Cb-sets, sets with an action of Cb on them, and invented the notion of *finitely supported* Cb-sets, a generalization of nominal sets. He has shown that the category of finitely supported Cb-sets is in fact isomorphic to the category of nominal sets equipped with two families of unary operations which substitute names (elements of $\mathbb{D}$) by the constants 0 or 1; and the category of finitely supported Cb-sets is a coreflective subcategory of the category of Cb-sets.

The notion of retractions appears when one can find a left inverse (reflection) for a morphism. This notion plays a crucial role in many areas of mathematics, such as homological algebra, topology, ordered algebraic structures, etc. The retracts

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are also known as complete or partial objects in recursion theory by computer scientists (see [7]).

On the other hand, we recall from [6] that every Cb-set is a disjoint union of all its indecomposable sub Cb-sets, where an indecomposable Cb-set is a Cb-set which can not be written as a disjoint union of non-empty sub Cb-sets. Therefore, to find retractions of a Cb-set, it is sufficient to obtain retractions of its indecomposable sub Cb-sets. Also, it is known that cyclic finitely supported Cb-sets are indecomposable (see Proposition I.5.8 of [6]). These facts provided our motivation to study the retracts of cyclic finitely supported Cb-sets in this paper. First, applying the characterization of cyclic finitely supported Cb-sets from [3], and assuming the existence of retractions from a cyclic Cb-set to its proper sub Cb-sets, we find the possible definition of them.

Moreover, using the characterization of cyclic fix-simple finitely supported Cb-sets given in [3], we find a characterization of retracts of cyclic finitely supported Cb-sets. Also, we prove that simple finitely supported Cb-sets which are fix-simple with one zero element are retracts of cyclic finitely supported Cb-sets.

Finally, retractable (the ones all whose subobjects are retract) finitely supported Cb-sets are studied; and a necessary condition for cyclic finitely supported Cb-sets to be retractable is obtained.

1.1. M-sets

In the following, we recall some notions and facts about M-sets, for a general monoid M. For more information, see ([2, 6]).

A (left) *M-set* for a monoid M with identity e is a set X equipped with a map $M \times X \to X, (m,x) \rightsquigarrow mx$, called an *action* of M on X, such that $ex = x$ and $m(m'x) = (mm')x$, for all $x \in X$ and $m, m' \in M$. An *equivariant map* from an M-set X to an M-set Y is a map $f : X \to Y$ with $f(mx) = mf(x)$, for all $x \in X, m \in M$.

An element x of an M-set X is called a *zero* (or a *fixed*) element if $mx = x$, for all $m \in M$. We denote the set of all zero elements of an M-set X by $\mathrm{Fix}\, X$. The M-set X all of whose elements are zero is called a *discrete* M-set, or an M-set with *identity action*.

An equivalence relation ρ on an M-set X is called a *congruence* on X if $x\rho x'$ implies $mx\,\rho\, mx'$, for $x, x' \in X$, $m \in M$. We denote the set of all congruences on X by $\mathrm{Con}(X)$. Also, for $x, x' \in X$, the smallest congruence on X containing (x, x') is denoted by $\rho(x, x')$. It is in fact, the equivalence relation generated by $\{(mx, mx') \mid m \in M\}$.

A subset Y of an M-set X is a *sub M-set* (or *M-subset*) of Y if for all $m \in M$ and $y \in Y$ we have $my \in Y$. The subset $\mathrm{Fix}\, X$ of X is in fact a sub M-set.

1.2. Cb-sets

Now, we give some basic notions about the monoid Cb, and Cb-sets. For more information one can see [9, 10].

Let $\mathbb{D}$ be an infinite countable set, whose elements are sometimes called *directions* (*atomic names* or *data values*) and Perm$\mathbb{D}$ be the group of all permutations (bijection maps) on $\mathbb{D}$. A permutation $\pi \in$ Perm$\mathbb{D}$ is said to be *finite* if $\{d \in \mathbb{D} \mid \pi(d) \neq d\}$ is finite. Clearly the set $\mathrm{Perm_f}\mathbb{D}$ of all finitary permutations is a subgroup of Perm$\mathbb{D}$.

Also, we take $2 = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

Definition 1.1. (a) A finite *substitution* is a function $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$ for which $\mathrm{Dom_f}\sigma = \{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite.

(b) If $d \in \mathbb{D}$ and $b \in 2$, we write (b/d) for the finite substitution which maps d to b, and maps identically on other elements of $\mathbb{D}$. Each (b/d) is called a *basic substitution*.

(c) If $d, d' \in \mathbb{D}$ then we write $(d\,d')$ for the finite substitution that transposes d and d', and keeps fixed all other elements. Each $(d\,d')$ is called a *transposition substitution*.

Definition 1.2. (a) Let Sb be the monoid whose elements are finite substitutions, with the monoid operation given by $\sigma \cdot \sigma' = \hat{\sigma}\sigma'$, where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined the same as σ. The identity element of Sb is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup 2$.

(b) Let Cb be the submonoid of Sb satisfying the following *injectivity condition*:

$$(\forall d, d' \in \mathbb{D}),\ \sigma(d) = \sigma(d') \notin 2 \Rightarrow d = d'.$$

(c) Take S to be the subsemigroup of Cb generated by basic substitutions. The members of S are of the form $\delta = (b_1/d_1) \cdots (b_k/d_k) \in S$ for some $d_i \in \mathbb{D}$ and $b_i \in 2$, and we denote the set $\{d_1, \ldots, d_k\}$ by $\mathbb{D}_\delta$.

Remark 1.3. (1) Notice that each finite permutation π on $\mathbb{D}$, can be considered as a finite substitution $\iota \circ \pi : \mathbb{D} \to \mathbb{D} \cup 2$. Doing so, throughout this paper, we consider the group $\mathrm{Perm_f}\mathbb{D}$ as a submonoid of Cb, and denote $\iota \circ \pi$ with the same notation π.

(2) Let $d \in \mathbb{D}$ and $b \in 2$. Then, for a finite permutation π and a basic substitution (b/d), one can compute that in Cb, $\pi(b/d) = (b/\pi(d))\pi$ and $(b/d)\pi = \pi(b/\pi^{-1}(d))$. Then, by induction, we also have:

$$\pi(b_1/d_1) \cdots (b_k/d_k) = (b_1/\pi d_1) \cdots (b_k/\pi d_k)\pi,$$

and

$$(b_1/d_1) \cdots (b_k/d_k)\pi = \pi(b_1/\pi^{-1}d_1) \cdots (b_k/\pi^{-1}d_k),$$

for $\pi \in \mathrm{Perm_f}(\mathbb{D})$, $d_1, \cdots, d_k \in \mathbb{D}$, and $b_i \in 2$, for $i = 1, \ldots, k$.

(3) Let $d \neq d' \in \mathbb{D}$ and $b, b' \in 2$. Then $(b/d)(b'/d') = (b'/d')(b/d)$. But, we see that $(1/d)(0/d) = (0/d)$ and $(0/d)(1/d) = (1/d)$, and hence $(1/d)(0/d) \neq (0/d)(1/d)$.

Theorem 1.4. [3] *For the monoid Cb, we have:*

$$Cb = \mathrm{Perm_f}(\mathbb{D}) \cup \mathrm{Perm_f}(\mathbb{D})S, \quad \mathrm{Perm_f}(\mathbb{D}) \cap \mathrm{Perm_f}(\mathbb{D})S = \emptyset.$$

1.3. Finitely supported Cb-sets

In this subsection, we give some basic notions of finitely supported Cb-sets needed in the sequel, some of which are given in [10].

The following definition introduces the notion of a, so called, *support*, which is the central notion to define finitely supported Cb-sets.

Definition 1.5. (a) Suppose X is a Cb-set. A subset $C_x \subseteq \mathbb{D}$ supports an element x of X if, for every $\sigma, \sigma' \in Cb$,

$$(\sigma(c) = \sigma'(c), (\forall c \in C_x)) \Rightarrow \sigma x = \sigma' x$$

If there is a finite (possibly empty) support C_x then we say that x is *finitely supported*.

(b) A Cb-set X all of whose elements has a finite support, is called a *finitely supported Cb-set*.

We denote the category of all Cb-sets with equivariant maps between them by $\mathbf{Set}^{Cb}$, and its full subcategory of all finitely supported Cb-sets by $\mathbf{Set}^{Cb}_{\mathrm{fs}}$.

Remark 1.6. [3] Suppose that X is a non-empty finitely supported Cb-set and $x \in X \setminus \mathrm{Fix}\, X$.

(1) By Remark 1.3(3), it is clear that

$$\{d \in \mathbb{D} \mid (0/d)\, x \neq x\} = \{d \in \mathbb{D} \mid (1/d)\, x \neq x\}.$$

This set is in fact the least finite support of x. First notice that, by Lemma 2.4 of [10], this set is a finite support for x. Now, let C be a finite support for x. Then for any $d \in \mathbb{D}$ with $(0/d)\, x \neq x$, by taking $\sigma = (0/d)$ and $\sigma' = \iota$ in the Definition 1.5(a), we get $(0/d)d' \neq d'$, for some $d' \in C$. So, by the definition of $(0/d)$, we have $d = d'$, and therefore $d \in C$.

From now on, we call the least finite support for x *the support* for x, and denote it by $\mathrm{supp}\, x$.

(2) Let $\delta \in S$. Then, by (1),

$$\delta\, x = x \text{ if and only if } \mathbb{D}_\delta \cap \mathrm{supp}\, x = \emptyset.$$

(3) Let $\{d_1, \ldots, d_k\} \subseteq \mathrm{supp}\, x$. Then,

$$\mathrm{supp}\, (b_1/d_1) \cdots (b_k/d_k)\, x \subseteq \mathrm{supp}\, x \setminus \{d_1, \ldots, d_k\},$$

for any $b_i \in 2$ and $i = 1, \cdots, k$.

(4) Let $\delta \in S$. Then,

$$\delta\, x = x \quad \text{if and only if} \quad |\mathrm{supp}\, \delta\, x| = |\mathrm{supp}\, x|.$$

(5) By (3) and (4), we have

$$\delta\, x \neq x \quad \text{if and only if} \quad |\text{supp}\, \delta\, x| < |\text{supp}\, x|.$$

(6) Let $\pi \in \text{Perm}_{\text{f}}(\mathbb{D})$. Then, $\text{supp}\, \pi x = \pi\, \text{supp}\, x$, and so $|\text{supp}\, \pi x| = |\pi \text{supp}\, x| = |\text{supp}\, x|$.

(7) X has a zero element.

Remark 1.7. [3] Let X be a finitely supported Cb-set and $x \in X$. Then,

(1) $S_x \doteq \{\delta \in S \mid \delta\, x = x\}$ is a subsemigroup of S;

(2) $S'_x \doteq S \setminus S_x = \{\delta \in S \mid \delta\, x \neq x\}$ is also a subsemigroup of S;

(3) If $\delta \in S'_x$ then $\delta\, x = \delta_1\, x$, for some $\delta_1 \in S'_x$ with $\mathbb{D}_{\delta_1} \subseteq \text{supp}\, x$;

(4) If $\delta \in S'_x$ then $\pi x \neq \pi' \delta x$, for some $\pi, \pi' \in \text{Perm}_{\text{f}}(\mathbb{D})$. Since otherwise, if $\pi x = \pi' \delta x$ then by Remark 1.6(5,6),

$$|\text{supp}\, x| = |\text{supp}\, \pi x| = |\text{supp}\, \pi' \delta x| = |\text{supp}\, \delta x| < |\text{supp}\, x|$$

which is impossible.

Definition 1.8. A cyclic finitely supported Cb-set X is said to be *cyclic*, if it is generated by only one element, that is $X = Cb\, x$, for some $x \in X$.

Lemma 1.9. [3] *Let $Cb\, x$ be a cyclic finitely supported Cb-set. Then,*

$$Cb\, x = \text{Perm}_{\text{f}}(\mathbb{D}) S'_x\, x \cup \text{Perm}_{\text{f}}(\mathbb{D})\, x, \quad \text{Perm}_{\text{f}}(\mathbb{D}) S'_x\, x \cap \text{Perm}_{\text{f}}(\mathbb{D})\, x = \emptyset.$$

2. Retractions of cyclic finitely supported Cb-sets

In this section, we study retracts and retractions of cyclic finitely supported Cb-sets. We find the general definition of a retraction, and give some necessary and sufficient conditions for a sub Cb-set of a cyclic finitely supported Cb-set to be a retract.

First, we give the definition of a *retraction*.

Let X be an object of a category $\mathfrak{C}$. A subobject J of X is called a *retract* of X if there exists a morphism $g : X \to J$, called *a retraction*, such that $g|_J = id_J$.

Notice that, for a proper sub Cb-set Cbx' of Cbx, Cbx' is a retract of Cbx if and only if $Cb\delta_0 x$ is a retract of Cbx, where $x' = \pi \delta_0 x$, for some $\pi \in \text{Perm}_{\text{f}}(\mathbb{D})$ and $\delta_0 \in S'_x$.

Lemma 2.1. *Suppose $\pi\delta_0 x$ is a non-zero element in Cbx, where $\pi \in \text{Perm}_{\text{f}}(\mathbb{D})$ and $\delta_0 \in S'_x$. If there exists a retraction φ from Cbx to $Cb\delta_0 x$ then*

(i) $\varphi(x) \in \text{Perm}_{\text{f}}(\mathbb{D})\delta_0 x$;

(ii) $\varphi(x) = \delta_0 x$.

Proof. (i) We have $\varphi(x) \in Cb\delta_0 x$. So by Lemma 1.9, $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})S'_{\delta_0 x}\delta_0 x$ or $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})\delta_0 x$. We show $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})\delta_0 x$. On the contrary, let $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})S'_{\delta_0 x}\delta_0 x$. Then, $\varphi(x) = \pi'\delta'\delta_0 x$ where $\delta' \in S'_{\delta_0 x}$ and $\pi' \in \mathrm{Perm_f}(\mathbb{D})$. Since φ is a retraction and $\delta_0 x \in Cb\delta_0 x$, we get

$$\delta_0 x = \varphi(\delta_0 x) = \delta_0\varphi(x) = \delta_0\pi'\delta'\delta_0 x.$$

Now, by Remark 1.6(5,6),

$$|\mathrm{supp}\,\delta_0 x| = |\mathrm{supp}\,\delta_0\pi'\delta'\delta_0 x| < |\mathrm{supp}\,\delta_0 x|,$$

which is impossible.

(ii) By (i), we get $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})\delta_0 x$. So there exists $\pi' \in \mathrm{Perm_f}(\mathbb{D})$ such that $\varphi(x) = \pi'\delta_0 x$. Since φ is a retraction and $\delta_0 x \in Cb\delta_0 x$, we get

$$\delta_0 x = \varphi(\delta_0 x) = \delta_0\varphi(x) = \delta_0\pi'\delta_0 x = \pi'\delta'_0\delta_0 x,$$

where the last equality is true by Remark 1.3(2). Now, $\delta'_0 \in S_{\delta_0 x}$, since otherwise, if $\delta'_0 \in S'_{\delta_0 x}$ then by Remark 1.6(5,6),

$$|\mathrm{supp}\,\delta_0 x| = |\mathrm{supp}\,\pi'\delta'_0\delta_0 x| = |\mathrm{supp}\,\delta'_0\delta_0 x| < |\mathrm{supp}\,\delta_0 x|,$$

which is impossible. Thus, $\delta'_0 \in S_{\delta_0 x}$ and so $\delta_0 x = \pi'\delta'_0\delta_0 x = \pi'\delta_0 x$. Therefore, $\varphi(x) = \delta_0 x$. □

Corollary 2.2. *Suppose $\pi\delta_0 x$ is a non-zero element of Cbx, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta_0 \in S'_x$. Let $\varphi : Cbx \to Cb\delta_0 x$ be a retraction. Then,*

(i) *If $\delta x \in Cb\delta_0 x$ then $\delta\delta_0 x = \delta x$;*

(ii) *If $\pi\delta_0 x = \pi'\delta'_0 x$ then $\delta_0 x = \delta'_0 x$.*

Proof. (i) Since φ is a retraction, by Lemma 2.1, we get

$$\delta x = \varphi(\delta x) = \delta\varphi(x) = \delta\delta_0 x.$$

(ii) Let $\pi\delta_0 x = \pi'\delta'_0 x$. Then $\delta'_0 x \in Cb\delta_0 x$ and $|\mathrm{supp}\,\delta'_0 x| = |\mathrm{supp}\,\delta_0 x|$. Since $\delta'_0 x \in Cb\delta_0 x$, by (i) $\delta'_0 x = \delta'_0\delta_0 x$. So $\mathrm{supp}\,\delta'_0 x \subseteq \mathrm{supp}\,\delta_0 x$. Now, since $|\mathrm{supp}\,\delta'_0 x| = |\mathrm{supp}\,\delta_0 x|$ and $\mathrm{supp}\,\delta_0 x$ is finite, we get $\mathrm{supp}\,\delta'_0 x = \mathrm{supp}\,\delta_0 x$. Thus for all $d \in \mathbb{D}_{\delta'_0}$, we have $d \notin \mathrm{supp}\,\delta_0 x$ and so $\delta'_0 x = \delta'_0\delta_0 x = \delta_0 x$. □

Remark 2.3. Let $Cb\delta_0 x$ be a proper sub Cb-set of Cbx. Then,

(1) $B = \{\pi\delta x \in \mathrm{Perm_f}(\mathbb{D})S'_x x \mid \mathbb{D}_\delta \cap \mathrm{supp}\,\delta_0 x \neq \emptyset\}$ is a proper sub Cb-set of Cbx.

(2) If $a \in B$ then $a = \pi\delta x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S'_{\delta_0 x}$. This is because, since $\delta x \in B$, we get $\mathbb{D}_\delta \cap \mathrm{supp}\,\delta_0 x \neq \emptyset$. So by Remark 1.6(2), $\delta\delta_0 x \neq \delta_0 x$. Therefore, $\delta \in S'_{\delta_0 x}$.

(3) If $a \in Cbx \setminus B$ then $a = \pi x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ or $a = \pi\delta x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S_{\delta_0 x}$. Notice that, if $\delta x \notin B$ then $\mathbb{D}_\delta \cap \mathrm{supp}\,x = \emptyset$, and so by Remark 1.6(2), $\delta\delta_0 x = \delta_0 x$. Thus $\delta \in S_{\delta_0 x}$.

Theorem 2.4. *Let $Cb\delta_0 x$ be a proper sub Cb-set of Cbx. Then, $Cb\delta_0 x$ is a retract of Cbx if and only if the assignment $\varphi : Cbx \to Cb\delta_0 x$ defined by*

$$\varphi(a) = \begin{cases} \pi\delta\delta_0 x, & \text{if } a = \pi\delta x \in B \\ \pi\delta_0 x, & \text{if } a = \pi x \text{ or } \pi\delta x \notin B \end{cases}$$

is a map retraction, where B is considered as in Remark 2.3.

Proof. To prove the non-trivial part, let $Cb\delta_0 x$ be a retract of Cbx and $\psi : Cbx \to Cb\delta_0 x$ be a retraction. Then, by Lemma 2.1, $\psi(x) = \delta_0 x$. Let $a = a'$. Then, we show $\varphi(a) = \varphi(a')$.

Case (1): Suppose $a = a' \in B$. By Remark 2.3(2), $\pi\delta x = a = a' = \pi'\delta' x$, for some $\pi, \pi' \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta, \delta' \in S'_{\delta_0 x}$. Now, $\psi(a) = \psi(a')$ and so

$$\begin{aligned}\varphi(a) &= \pi\delta\delta_0 x = \pi\delta\psi(x) = \psi(\pi\delta x) = \psi(a) = \psi(a') = \psi(\pi'\delta' x)\\ &= \pi'\delta'\psi(x) = \pi'\delta'\delta_0 x = \varphi(a').\end{aligned}$$

Case (2): Suppose $a = a' \notin B$. By Remark 2.3(3), $a = \pi x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ or $a = \pi\delta x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S_{\delta_0 x}$. Then, by Remark 1.7(4), we have the following subcases;

Subcase (2a): If $\pi x = a = a' = \pi' x$ then

$$\begin{aligned}\varphi(a) &= \pi\delta_0 x = \pi\psi(x) = \psi(\pi x) = \psi(a) = \psi(a') = \psi(\pi' x) = \pi'\psi(x)\\ &= \pi'\delta_0 x = \varphi(a').\end{aligned}$$

Subcase (2b): If $\pi\delta x = a = a' = \pi'\delta' x$ then $\delta\delta_0 x = \delta_0 x$ and so

$$\begin{aligned}\varphi(a) &= \pi\delta_0 x = \pi\delta\delta_0 x = \pi\delta\psi(x) = \psi(\pi\delta x) = \psi(a) = \psi(a') = \psi(\pi'\delta' x)\\ &= \pi'\delta'\psi(x) = \pi'\delta'\delta_0 x = \pi'\delta_0 x = \varphi(a').\end{aligned}$$

Now, we show φ is equivariant and $\varphi \mid_{Cb\delta_0 x} = id \mid_{Cb\delta_0 x}$. Suppose $a \in Cb\,x$ and $\sigma_1 \in Cb$. We have the following cases:

Case (a): Let $a \in B$. Then, $\sigma_1 a \in B$ and by Remark 2.3(2), $a = \pi\delta x$, where $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S'_{\delta_0 x}$. Now,

$$\sigma_1\varphi(a) = \sigma_1\pi\delta\delta_0 x = \varphi(\sigma_1\pi\delta x) = \varphi(\sigma_1 a).$$

Case (b): Let $a \notin B$. Then, by Remark 2.3(3), $a = \pi x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ or $a = \pi\delta x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S_{\delta_0 x}$ and by Remark 1.7(4), we have the following subcases;

Subcase (b1): Let $a = \pi x$ and $\sigma_1 = \pi_1$. Then,

$$\sigma_1\varphi(a) = \pi_1\varphi(\pi x) = \pi_1\pi\delta_0 x = \varphi(\pi_1\pi x) = \varphi(\pi_1 a) = \varphi(\sigma_1 a).$$

Subcase (b2): Let $a = \pi\delta x$ and $\sigma_1 = \pi_1$. Then,

$$\sigma_1\varphi(a) = \pi_1\varphi(\pi\delta x) = \pi_1\pi\delta_0 x = \varphi(\pi_1\pi\delta x) = \varphi(\pi_1 a) = \varphi(\sigma_1 a).$$

SUBCASE (B3): If $a = \pi x$ and $\sigma_1 = \pi_1\delta_1$ or $a = \pi\delta x$ and $\sigma_1 = \pi_1\delta_1$ then

$$\sigma_1\varphi(a) = \pi_1\delta_1\varphi(a) = \pi_1\delta_1\pi\delta_0 x = \pi_1\pi\delta_1'\delta_0 x.$$

Now, if $\sigma_1 a \in B$ then

$$\sigma_1 a = \pi_1\delta_1\pi x = \pi_1\pi\delta_1' x \text{ or } \sigma_1 a = \pi_1\delta_1\pi\delta x = \pi_1\pi\delta_1'\delta x,$$

and so $\mathbb{D}_{\delta_1'} \cap \operatorname{supp}\delta_0 x \neq \emptyset$. Thus $\varphi(\sigma_1 a) = \pi_1\pi\delta_1'\delta_0 x = \sigma_1\varphi(a)$.

Also, if $\sigma_1 a \notin B$ then

$$\pi_1\delta_1 a = \pi_1\delta_1\pi x = \pi_1\pi\delta_1' x \text{ or } \pi_1\delta_1 a = \pi_1\delta_1\pi\delta x = \pi_1\pi\delta_1'\delta x,$$

and so $\mathbb{D}_{\delta_1'} \cap \operatorname{supp}\delta_0 x = \emptyset$. Thus $\delta_1'\delta_0 x = \delta_0 x$, and so

$$\varphi(\sigma_1 a) = \pi_1\pi\delta_0 x = \pi_1\pi\delta_1'\delta_0 x = \sigma_1\varphi(a).$$

It remains to show $\varphi\,|_{Cb\delta_0 x} = id_{Cb\delta_0 x}$. Let $a \in Cb\delta_0 x$. Then, $a = \pi'\delta'\delta_0 x$, for some $\pi' \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta' \in S'_{\delta_0 x}$ or $a = \pi'\delta_0 x$, for some $\pi' \in \mathrm{Perm_f}(\mathbb{D})$. Now, if $a = \pi'\delta'\delta_0 x$ then $a \in B$ and so $\varphi(a) = \pi'\delta'\delta_0 x = a$. Also, if $a = \pi'\delta_0 x$ then $a \notin B$, and so $\varphi(a) = \pi'\delta_0 x = a$. □

Now, we recall the following definition and theorem from [3].

Definition 2.5. We call a finitely supported Cb-set X, *fix-simple* if its only non-trivial sub Cb-sets are of the form $\bigcup_{i\in I}\{\theta_i\}$, for a set I, and $\theta_i \in \operatorname{Fix} X$.

If X is a fix-simple Cb-set and $\operatorname{Fix} X = \{\theta_1, \ldots, \theta_k\}$, then we simply call X, $\{\theta_1, \ldots, \theta_k\}$-simple. A $\{\theta\}$-simple Cb-set is said to be θ-simple or 0-simple.

Theorem 2.6. *If X is a non-discrete fix-simple finitely supported Cb-set, then X is cyclic and of one of the forms*

$$\mathrm{Perm_f}(\mathbb{D})\, x \cup \{\theta\} \quad \text{or} \quad \mathrm{Perm_f}(\mathbb{D})\, x \cup \{\theta_1, \theta_2\}$$

where $\theta, \theta_1, \theta_2 \in \operatorname{Fix} X$, and $|\operatorname{Fix} X| \leq 2$.

Recall that simple algebras are the one whose only congruences are Δ and ∇. Now, using the above theorem, we have:

Lemma 2.7. *Let Cbx be a cyclic finitely supported Cb-set. Then, each simple sub Cb-set of Cbx is a retract of Cbx.*

Proof. Let A be a simple sub Cb-set and $\theta \in \operatorname{Fix} Cb\,x$. Then, by Theorem 6.3 of [3], A is θ-simple and so by Theorem 2.6, $A = \mathrm{Perm_f}(\mathbb{D})x' \cup \{\theta\}$, where $x' \in Cbx$. Take $x' = \pi\delta_0 x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta_0 \in S'_x$. Now, applying Theorem 2.4, it is sufficient to show that the assignment φ mentioned there, is a map. Notice that, if $\pi\delta x \in B$ then $\mathbb{D}_\delta \cap \operatorname{supp} \delta_0 x \neq \emptyset$ and since A is $\mathrm{Perm_f}(\mathbb{D})\, x' \cup \{\theta\}$, we get $\delta\delta_0 x = \theta$. Thus, we have

$$\varphi(a) = \begin{cases} \theta, & \text{if } a \in B \\ \pi\delta_0 x, & \text{if } a = \pi x \text{ or } \pi\delta x \notin B \end{cases}$$

To see that it is well-defined, assume $a = a'$. If $a = a' \in B$, then $\varphi(a) = \theta = \varphi(a')$. Let $a \notin B$. Then, by Remark 2.3(3), $a, a' \in \{\pi x, \pi' x, \pi\delta x, \pi'\delta' x\}$, for some $\pi, \pi' \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta, \delta' \in S_{\delta_0 x}$. So by Remark 1.7(4), we have the following cases:

Case (1): $\pi x = a = a' = \pi' x$.

Case (2): $\pi\delta x = \pi'\delta' x$.

In each case, we must show $\pi\delta_0 x = \pi'\delta_0 x$. To show this, by Theorem 6.4 of [3], it is sufficient to show $\operatorname{supp} \pi\delta_0 x = \operatorname{supp} \pi'\delta_0 x$. Notice that, $\operatorname{supp} \delta_0 x \subseteq \operatorname{supp} \delta x$, for all $\delta \in S_{\delta_0 x}$. This is because, if there exists some $d \in \operatorname{supp} \delta_0 x \setminus \operatorname{supp} \delta x$ then $(0/d)\delta x = \delta x$, and so $\delta x \in B$, which is impossible. We prove case (2). The other case is proved similarly. Suppose $\pi\delta x = \pi'\delta' x$. Let $d' \in \operatorname{supp} \pi\delta x_0$. Then, $\pi^{-1} d' \in \operatorname{supp} \delta_0 x$, and so

$$\pi(0/\pi^{-1}d')\delta x = (0/d')\pi\delta x = (0/d')\pi'\delta' x = \pi'(0/\pi'^{-1}d')\delta' x.$$

Now, since $\pi^{-1}d' \in \operatorname{supp} \delta_0 x$, we get $(0/\pi^{-1}d')\delta x \in B$, and so $(0/\pi'^{-1}d')\delta' x \in B$. Therefore, $\pi'^{-1}d' \in \operatorname{supp} \delta_0 x$. Similarly $\operatorname{supp} \pi'\delta_0 x \subseteq \operatorname{supp} \pi\delta_0 x$, and so the result holds. □

Remark 2.8. Let $Cb\, x_0$ be a non-discrete fix-simple sub Cb-set of $Cb\, x$ with two zero elements $\theta_1, \theta_2 \in \operatorname{Fix} Cb x_0$. Then, by Theorem 2.6,

$$Cb\, x_0 = \mathrm{Perm_f}(\mathbb{D})\delta_0 x \cup \{\theta_1, \theta_2\}.$$

Take $\operatorname{supp} \delta_0 x = \{d\}$, $(0/d)\delta_0 x = \theta_1$, and $(1/d)\delta_0 x = \theta_2$. Then,

(1) the sets

$$B_0 = \{\pi\delta x \mid \delta(d) = 0, \delta \in S'_x, \pi \in \mathrm{Perm_f}(\mathbb{D})\}$$

and

$$B_1 = \{\pi\delta x \mid \delta(d) = 1, \delta \in S'_x, \pi \in \mathrm{Perm_f}(\mathbb{D})\}$$

are non-empty sub Cb-sets of $Cb\, x$.

(2) $\delta x \in B_0 \cup B_1$ if and only if $d \in \mathbb{D}_\delta$ if and only if $\mathbb{D}_\delta \cap \operatorname{supp} \delta_0 x \neq \emptyset$.

(3) If $\delta x \in \operatorname{Fix} Cb\, x$ then $\delta x \in B_0 \cup B_1$. This is because, $\operatorname{supp} \delta x = \emptyset$ and so $d \notin \operatorname{supp} \delta x$. Thus $(b/d)\delta x = \delta x$. Now, since $(b/d)\delta x \in B_0 \cup B_1$, we get $\delta x \in B_0 \cup B_1$.

(4) Let $a \notin B_0 \cup B_1$. Then, $a = \pi\delta x$, for some $\delta \in S_{\delta_0 x}$ and $\pi \in \mathrm{Perm_f}(\mathbb{D})$ or $a = \pi x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$.

Theorem 2.9. *Let* $Cb\,x_0 = \mathrm{Perm_f}(\mathbb{D})\delta_0 x \cup \{\theta_1, \theta_2\}$ *be a non-discrete fix-simple sub Cb-set of* $Cb\,x$ *with two zero elements* $\theta_1, \theta_2 \in \mathrm{Fix}\,Cbx_0$*, and* $\mathrm{supp}\,\delta_0 x = \{d\}$*,* $(0/d)\delta_0 x = \theta_1$*,* $(1/d)\delta_0 x = \theta_2$*. Then,*

$Cb\delta_0 x$ is a retract of Cbx if and only if for all $\delta \in Cb$, $\delta x \in \mathrm{Fix}\,Cbx$ implies $d \in \mathbb{D}_\delta$.

Proof. Let $\varphi : Cbx \to Cb\delta_0 x$ be a retraction. Then, by Lemma 2.1, $\varphi(\sigma x) = \sigma\delta_0 x$, for $\sigma \in Cb$. Suppose $\delta x \in \mathrm{Fix}\,Cbx$. We show $d \in \mathbb{D}_\delta$. On the contrary, if $d \notin \mathbb{D}_\delta$ then $\mathbb{D}_\delta \cap \mathrm{supp}\,\delta_0 x = \emptyset$, and so by Remark 1.6(2), $\delta\delta_0 x = \delta_0 x$. Also, notice that since $\delta x \in \mathrm{Fix}\,Cbx$, we get $\mathrm{supp}\,\delta x = \emptyset$, and so $d \notin \mathrm{supp}\,\delta x$. Thus $\delta x = (0/d)\delta x = (1/d)\delta x \in B$. Now,

$$\begin{aligned}\theta_1 &= (0/d)\delta_0 x = (0/d)\delta\delta_0 x = \varphi((0/d)\delta x) = \varphi(\delta x) = \varphi((1/d)\delta x)\\ &= (1/d)\delta\delta_0 x = (1/d)\delta_0 x = \theta_2,\end{aligned}$$

which is impossible.

To prove the converse, we show that the assignment φ mentiones in Theorem 2.4, is a map. Notice that, if $a \notin B_0 \cup B_1$, then by Remark 2.8(4), $a = \pi x$ or $a = \pi\delta x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S_{\delta_0 x}$. Thus, we have

$$\varphi(a) = \begin{cases} \theta_1, & \text{if } a \in B_0 \\ \theta_2, & \text{if } a \in B_1 \\ \pi\delta_0 x, & \text{if } a \notin B_0 \cup B_1 \end{cases}$$

To show that φ is well-defined, let $a = a'$. Then, $\mathrm{supp}\,a = \emptyset$ or $\mathrm{supp}\,a \neq \emptyset$. If $\mathrm{supp}\,a = \emptyset$ then by Remark 2.8, $a \in B_0 \cup B_1$, and so $a = \pi'\delta' x$, for some $\pi' \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta' \in S_{\delta_0 x}$. Now, by the assumption, $d \in \mathbb{D}_{\delta'}$. Thus, if $\delta'(d) = 0$ then $a' = a \in B_0$, and so $\varphi(a) = \varphi(a') = \theta_1$. Also, if $\delta'(d) = 1$ then $a' = a \in B_1$, and so $\varphi(a) = \varphi(a') = \theta_2$.

In the case that $\mathrm{supp}\,a \neq \emptyset$ and $a = a' \in B_0 \cup B_1$, it is clear that the result holds.

Let $a \notin B_0 \cup B_1$. Then, by Remark 2.8, $a, a' \in \{\pi x, \pi' x, \pi\delta x, \pi'\delta' x\}$, for some $\pi, \pi' \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta, \delta' \in S_{\delta_0 x}$. So by Remark 1.7(4), we have two following cases:

Case (1): $\pi x = a = a' = \pi' x$;

Case (2): $\pi\delta x = \pi'\delta' x$.

In each case, we must show $\pi\delta_0 x = \pi'\delta_0 x$. To show this, it is sufficient to prove that $\mathrm{supp}\,\pi\delta_0 x = \mathrm{supp}\,\pi'\delta_0 x$. Notice that, $\mathrm{supp}\,\delta_0 x \subseteq \mathrm{supp}\,\delta x$, for all $\delta \in S_{\delta_0 x}$. This is because, if $d \notin \mathrm{supp}\,\delta x$ then $(0/d)\delta x = \delta x$ and so $\delta x \in B_0 \cup B_1$, which is impossible. We prove case (2). The other case is proved similarly. Suppose $\pi\delta x = \pi'\delta' x$. Let $d' \in \mathrm{supp}\,\pi\delta x_0$. Then, $\pi^{-1}d' \in \mathrm{supp}\,\delta_0 x$, and so

$$\pi(0/\pi^{-1}d')\delta x = \pi'(0/\pi'^{-1}d')\delta' x.$$

Now, since $\pi^{-1}d' \in \mathrm{supp}\,\delta_0 x$, we have $(0/\pi^{-1}d')\delta x \in B_0 \cup B_1$, and so $(0/\pi'^{-1}d')\delta' x \in B_0 \cup B_1$. Therefore, $\pi'^{-1}d' \in \mathrm{supp}\,\delta_0 x$. Similarly $\mathrm{supp}\,\pi'\delta_0 x \subseteq \mathrm{supp}\,\pi\delta_0 x$, and the proof is complete. □

Theorem 2.10. *Let $Cb\delta_0 x$ be a non-zero and proper sub Cb-set of Cbx. Also, let the following conditions hold:*

(i) *if $d \in \operatorname{supp}\delta_0 x$ then $(b/d)x \in Cb\delta_0 x$,*

(ii) *if $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})(b_1/d_1)\delta_0 x \cap \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})(b_2/d_2)\delta_0 x \neq \emptyset$, then $d_1 = d_2$.*

Then, $Cb\delta_0 x$ is a retract of Cbx if and only if for all $\delta x \in Cb\delta_0 x$ we have $\delta x = \delta\delta_0 x$.

Proof. If $Cb\delta_0 x$ is a retract of Cbx then applying Corollary 2.2, for all $\delta x \in Cb\delta_0 x$ we have $\delta x = \delta\delta_0 x$. To prove the converse, let $\delta x = \delta\delta_0 x$, for all $\delta x \in Cb\delta_0 x$. Then to get the result, using Theorem 2.4, we show that φ is a map. First, we prove

$$a \in B \Rightarrow a \in Cb\delta_0 x \qquad (*).$$

Since $a \in B$, $a = \pi\delta x$, for some $\pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta \in S'_{\delta_0 x}$. So $\mathbb{D}_\delta \cap \operatorname{supp}\delta_0 x \neq \emptyset$. Thus there exists some $d \in \mathbb{D}$ with $d \in \mathbb{D}_\delta \cap \operatorname{supp}\delta_0 x$, and so by (i), $\delta x \in Cb\delta_0 x$.

Let $a = a' \in B$. Then, $\pi\delta x = a = a' = \pi'\delta' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta, \delta' \in S'_{\delta_0 x}$. So by $(*)$, $a, a' \in Cb\delta_0 x$. Thus $a = \pi_1\delta_1\delta_0 x$ and $a' = \pi_2\delta_2\delta_0 x$, for some $\pi_1, \pi_2 \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta_1, \delta_2 \in S'_{\delta_0 x}$. Now,

$$\begin{aligned}\varphi(a) &= \varphi(\pi\delta x) = \varphi(\pi_1\delta_1\delta_0 x) = \pi_1\delta_1\delta_0\delta_0 x = \pi_1\delta_1\delta_0 x = a = a' = \pi_2\delta_2\delta_0 x \\ &= \pi_2\delta_2\delta_0\delta_0 x = \varphi(\pi_2\delta_2\delta_0 x) = \varphi(a').\end{aligned}$$

Let $a = a' \notin B$. Then, we have the following cases:

Case (1): $\pi x = a = a' = \pi' x$.

Case (2): $\pi\delta x = a = a' = \pi'\delta' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta, \delta' \in S'_x$.

In each case, we show $\pi\delta_0 x = \pi'\delta_0 x$. We prove case (1). The other case is proved similarly. Let $d \in \operatorname{supp}\delta_0 x$. Then $(b/\pi d)\pi x = (b/\pi d)\pi' x$. So $\pi(b/d)x = \pi'(b/\pi'^{-1}\pi d)x$. Since $d \in \operatorname{supp}\delta_0 x$, by (i), we get $(b/d)x \in Cb\delta_0 x$. So $(b/\pi'^{-1}\pi d)x \in Cb\delta_0 x$, and we have

$$\begin{aligned}\pi(b/d)\delta_0 x &= \pi(b/d)x \quad \text{(by the assumption)} \\ &= \pi'(b/\pi'^{-1}\pi d)x \\ &= \pi'(b/\pi'^{-1}\pi d)\delta_0 x \quad \text{(by the assumption).}\end{aligned}$$

Now, if $\pi'^{-1}\pi d \notin \operatorname{supp}\delta_0 x$ then $\pi(b/d)\delta_0 x = \pi'\delta_0 x$, which is impossible, since in this case, by Lemma 3.4 of [3], $|\operatorname{supp}(b/d)\delta_0 x| < |\operatorname{supp}\delta_0 x|$. Therefore $\pi'^{-1}\pi d \in \operatorname{supp}\delta_0 x$, and so by (ii), $\pi'^{-1}\pi d = d$. Thus for all $d \in \operatorname{supp}\delta_0 x$, we have $\pi'^{-1}\pi d = d$ which implies that $\pi'^{-1}\pi\delta_0 x = \delta_0 x$. □

Theorem 2.11. *Let Cbx be a cyclic finitely supported Cb-set and $Cb\delta_0 x$ be a proper sub Cb-set such that for all $z, z' \in \mathrm{Cb}\,\delta_0 x$, $\operatorname{supp} z = \operatorname{supp} z'$ implies $z = z'$. Also, suppose for all $\delta, \delta' \in S'_x$ with $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})\delta\, x \cap \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})\delta'\, x \neq \emptyset$ we have $|\mathbb{D}_\delta \cap \operatorname{supp}\delta_0 x| = |\mathbb{D}_{\delta'} \cap \operatorname{supp}\delta_0 x|$. Then, there exists a retraction from Cbx to $Cb\delta_0 x$.*

Proof. Applying Theorem 2.4, we show that φ is a map. If $a = a' \notin B$ then by Remark 2.8, we have the following cases:

Case (1) $\pi x = a = a' = \pi' x$.

Case (2) $\pi\delta x = a = a' = \pi'\delta' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta, \delta' \in S'_x$.

In each case, we must show $\pi\delta_0 x = \pi'\delta_0 x$. By the assumption, it is sufficient to show that $\mathrm{supp}\,\pi\delta_0 x = \mathrm{supp}\,\pi'\delta_0 x$. We prove case (1). The other case is proved similarly. Let $d \in \mathrm{supp}\,\pi\delta_0 x$. Then $(b/d)\pi x = (b/d)\pi' x$. So $\pi(b/\pi^{-1}d)x = \pi'(b/\pi'^{-1}d)x$. Since $\pi^{-1}d \in \mathrm{supp}\,\delta_0 x$, by the assumption, $\pi'^{-1}d \in \mathrm{supp}\,\delta_0 x$, and so $d \in \mathrm{spp}\,\pi'\delta_0 x$. Similarly, $\mathrm{supp}\,\pi'\delta_0 x \subseteq \mathrm{supp}\,\pi\delta_0 x$. Thus $\mathrm{supp}\,\pi\delta_0 x = \mathrm{supp}\,\pi'\delta_0 x$, and so by the assumption, $\pi\delta_0 x = \pi'\delta_0 x$.

Now, suppose $a = a' \in B$. Then, $a = \pi\delta x$ and $a' = \pi'\delta' x$, for some $\pi, \pi' \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ and $\delta, \delta' \in S'_{\delta_0 x}$. We show that $\mathrm{supp}\,\pi\delta\delta_0 x = \mathrm{supp}\,\pi'\delta'\delta_0 x$, and so by the assumption, $\pi\delta\delta_0 x = \pi'\delta'\delta_0 x$. First, notice that $\mathrm{supp}\,\delta\delta_0 x \subseteq \mathrm{supp}\,\delta\, x$. To show this, suppose $d \in \mathrm{supp}\,\delta\delta_0\, x$. So $d \in \mathrm{supp}\,\delta_0\, x$. If $d \notin \mathrm{supp}\,\delta\, x$, then $(b/d)\delta\, x = \delta\, x$, which is impossible. Let $d \in \mathrm{supp}\,\pi\delta\delta_0 x$. Then, $d \in \mathrm{supp}\,\pi\delta x$. Now, $\pi(0/\pi^{-1}d)\delta x = (0/d)\pi\delta x = (0/d)\pi'\delta' x = \pi'(0/\pi'^{-1}d)\delta' x$. Thus by the assumption, $\pi'^{-1}d \in \mathrm{supp}\,\delta_0 x$. Now, if $d \notin \mathrm{supp}\,\pi'\delta'\delta_0 x$ then $\pi'(0/\pi'^{-1}d)\delta'\delta_0 x = (0/d)\pi'\delta'\delta_0 x = \pi'\delta'\delta_0 x$, which is a contradiction. □

Theorem 2.12. *Let Z be a finitely supported Cb-set, and $Y = Cb\,x \cup Z$, where $Cb\,x \cap Z = \emptyset$ or $Cb\,x \cap Z = \{\theta\}$ for $\theta \in \mathrm{Fix}\,Cb\,x \cap \mathrm{Fix}\,Z$. Then, there exists a retraction from Y to $Cb\,x$.*

Proof. Let $i : Cb\,x \hookrightarrow Y$ be the inclusion map and $\theta \in Cb\,x$, which exists by Remark 1.6(7). Then, $g : Y \to Cb\,x$ which is defined by

$$g(z) = \begin{cases} z & \text{if} \quad z \in Cb\,x \\ \theta & \text{if} \quad z \in Z \end{cases}$$

is a retraction. □

Here, to have a better scenery, we summarize the results of this section. In Lemma 2.1, assuming the existence of a retraction from Cbx to a sub Cb-set, we found some necessary conditions to have a retraction. We gave a characterization of retracts of cyclic finitely supported Cb-sets in Theorem 2.4. In Lemma 2.7, we showed that all simple sub Cb-sets of a cyclic finitely supported Cb-sets are retracts. Further, in Theorem 2.9, a sufficient and necessary condition for a fix-simple finitely supported Cb-set with two zero elements is stated to make it into a retract of a cyclic Cb-set.

3. Retractable finitely supported Cb-sets

In this section, we consider retractable cyclic finitely supported Cb-set.

Definition 3.1. A finitely supported Cb-set X is called *retractable* if for every non-empty sub Cb-set Y of X, there exists a retraction from X to Y.

Example 3.2. (1) Discrete Cb-sets are retractable. The converse is not correct.
(2) Each fix-simple Cb-set with a unique zero is retractable.

Remark 3.3. Every sub Cb-set of a retractable Cb-set is retractable.

Lemma 3.4. *A retractable cyclic finitely supported Cb-set has a unique zero.*

Proof. Take $X = Cb\,x$, for some $x \in X$. If $\operatorname{supp} x = \emptyset$ then X is a singleton, and so the result holds. Suppose $\operatorname{supp} x \neq \emptyset$. By Remark 1.6(7), X has a zero element. We show X has a uniuqe zero element. On the contrary, suppose $\theta_1 \neq \theta_2 \in \operatorname{Fix} X$. Since X is retractable, there exists an equivariant map $f : X \to \{\theta_1, \theta_2\}$ with $fi = id$, where $i : \{\theta_1, \theta_2\} \hookrightarrow X$ is an inclusion arrow. Now, $f(x) \in \{\theta_1, \theta_2\}$. If $f(x) = \theta_1$ then $f(Cb\,x) \subseteq \{\theta_1\}$. In particular, $\theta_2 = f(\theta_2) = \theta_1$, which is impossible. Similarly, $f(x) = \theta_2$ is impossible. Thus X has a unique zero element. □

Lemma 3.5. *Every non-trivial cyclic sub Cb-set of a non-discrete retractable Cb-set X has a unique infinite θ-simple sub Cb-set.*

Proof. Let $Cb\,x$ be a non-trivial sub Cb-set of X. Then, by Remark 3.3, $Cb\,x$ is retractable, and so by Lemma 3.4, $Cb\,x$ has a unique zero θ. Also, by Lemma 7.6 of [3], $Cb\,x$ has a θ-simple sub Cb-set, say B. Now, if B, B' are two θ-simple sub Cb-sets of $Cb\,x$ then $B \cap B' = \{\theta\}$ or $B = B'$. Suppose $B \cap B' = \{\theta\}$. Since Cbx is retractable, there exists a retraction $f : Cb\,x \to B \cup B'$, which is impossible, because $f(x) \in B$ or $f(x) \in B'$, and so $f(Cb\,x) \subseteq B$ or $f(Cb\,x) \subseteq B'$. Now, since f is a retraction, we get $f(B \cup B') = B \cup B'$, and so $B \cup B' \subseteq B$. Thus $B' \subseteq B$, which is impossible. □

Theorem 3.6. *Let $Cb\,x = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x \cup A$, where $A = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})\delta_0 x \cup \{\theta\}$ is a simple sub Cb-set of X, $\operatorname{Fix} Cb\,x = \{\theta\}$ and $\delta_0 \in S'_x$. Then,*

(i) *the non-empty sub Cb-sets of Cbx are $\{\theta\}$, A, and $Cb\,x$;*

(ii) *Cbx is retractable;*

(iii) *$(b/d)x = \theta$, for all $d \in \operatorname{supp} \delta_0 x$.*

Proof. (i) Let C be a non-empty non-trivial proper sub Cb-set of $Cb\,x$. Then $x \notin C$, and so $C \subseteq A$. Now, since $\delta_0 x \in C$, $A \subseteq C$, and so $C = A$.

(ii) It is sufficient to show that A is a retract of $Cb\,x$. Applying Theorem 2.4, we show that $\varphi : Cb\,x \to A$ is a map. First, we show $\varphi = \psi$, where

$$\psi(a) = \begin{cases} a, & \text{if } a \in Cb\delta_0 x \\ \pi\delta_0 x, & \text{if } a \notin Cb\delta_0 x \end{cases}$$

Let $a \in A$. Then, $a = \theta = (0/d)\delta_0 x$, where $d \in \operatorname{supp} \delta_0 x$ or $a = \pi\delta_0 x$, for some $\pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$. If $a = \theta = (0/d)\delta_0 x$ then $\theta \in B$, and so $\varphi(\theta) = \theta = \psi(\theta)$. Also, if $a = \pi\delta_0 x$ then $a \notin B$, and so

$$\varphi(\pi\delta_0 x) = \pi\delta_0 x = \psi(\pi\delta_0 x).$$

Let $a \notin A$. Then, $a = \pi x$, for some $\pi \in \mathrm{Perm_f}(\mathbb{D})$, and so $a \notin B$. Thus $\varphi(a) = \pi\delta_0 x = \psi(a)$.

Now, we show that ψ is well-defined. Let $a = a' \notin A = Cb\delta_0 x$. Then $\pi x = a = a' = \pi' x$, for some $\pi, \pi' \in \mathrm{Perm_f}(\mathbb{D})$. Take $\pi'^{-1}\pi = \pi_1$. We must show that $\pi_1\delta_0 x = \delta_0 x$. First, notice that, since A is simple, it is sufficient to show that $\mathrm{supp}\,\pi_1\delta_0 x = \mathrm{supp}\,\delta_0 x$. To prove this, let $d \in \mathrm{supp}\,\pi_1\delta_0 x$, then $d \in \mathrm{supp}\,\pi_1 x$, and so $(0/d)x = (0/d)\pi_1 x = \pi_1(0/\pi_1^{-1}d)x$. Now, since $\pi_1^{-1}d \in \mathrm{supp}\,\delta_0 x$, $(0/\pi^{-1}d)x \in B$, and so $(0/d)x \in B$. Thus $d \in \mathrm{supp}\,\delta_0 x$. Similarly, $\mathrm{supp}\,\delta_0 x \subseteq \mathrm{supp}\,\pi_1\delta_0 x$, and so $\mathrm{supp}\,\pi_1\delta_0 x = \mathrm{supp}\,\delta_0 x$.

Now, since ψ is a map, we get that φ is a retraction.

(iii) Let $d \in \mathrm{supp}\,\delta_0 x$. Then, $Cb(b/d)\,x$ is a proper sub Cb-set of $Cb\,x$, for $b \in 2$. Since otherwise, if $Cb\,x = Cb(b/d)x$ then $x = \sigma(b/d)x$, and so by Remark 2.4(4) of [3], $|\mathrm{supp}\,x| = |\mathrm{supp}\,\sigma(b/d)x| \leq |\mathrm{supp}\,(b/d)x| < |\mathrm{supp}\,(b/d)x|$, which is impossible. Therefore, $Cb(b/d)x = A$ or $Cb(b/d)x = \{\theta\}$, and so $(b/d)x \in A$. Since Cbx is retractable, there exists a retraction $\varphi : Cbx \to A$. Applying Lemma 2.1(ii), $\varphi(x) = \delta_0 x$, $(b/d)x = \varphi((b/d)x) = (b/d)\delta_0 x = \theta$. $\square$

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