Quasigroups and Related Systems **26** (2018), 45 − 52

Menger algebras of associative and self-distributive n-ary operations

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Abstract. The necessary and sufficient conditions under which a Menger algebra of rank n can be isomorphically represented by associative or (i,j)-associative n-ary operations are given. Also the conditions under which a Menger algebra of rank n can be homomorphically represented by self-distributive n-ary operations are found.

1. Multiplace functions are known to have various applications not only in mathematical analysis, but are also widely used in the theory of many-valued logics, cybernetics and general systems theory. Algebras of such functions (called *Menger algebras*) are studied in various directions [4]. In particular, many authors studied algebras of functions with some additional properties, see for example [4, 6, 7, 8, 9, 10, 11, 12]. Menger algebras of n-place functions closed with respect to certain additional operations and allow fixed exchange of some variables are described in [4].

As it is known semigroups and groups can be isomorphically represented by functions of one variable. Similar results are obtained for $(n+1)$-ary Menger algebras of some types. Namely, it is proved (see for example [4]) that some types of $(n+1)$-ary Menger algebras can be represented by n-place functions.

In this short paper we give necessary and sufficient conditions under which a Menger algebra of rank n can be isomorphically represented by associative or (i,j)-associative n-ary operations defined on some set. We also find conditions under which a Menger algebra of rank n can be homomorphically represented by self-distributive n-ary operations and prove that in the case when this algebra is also reductive then it can be isomorphically represented by these functions.

2. In the whole article, we assume that $n \geqslant 2$ and A is a nonempty set. Any function $f\colon A^n \to A$ defined for each element of the set A^n is called an *n-ary operation on A*. The set of all such operations, for fixed A and $n \geqslant 2$, is denoted by $\Omega_n(A)$. On the set $\Omega_n(A)$ we can consider the *Menger superposition* $\mathcal{O}$ defined by:

$$\mathcal{O}(f,g_1,\ldots,g_n)(a_1,\ldots,a_n) = f(g_1(a_1,\ldots,a_n),\ldots,g_n(a_1,\ldots,a_n)),$$

where $f,g_1,\ldots,g_n \in \Omega_n(A)$, $a_1,\ldots,a_n \in A$.

2010 Mathematics Subject Classification: 20N05, 08A05
Keywords: Algebra of multiplace functions; Menger algebra.

Any subset $\Phi \subset \Omega_n(A)$ closed with respect to this superposition is called a *Menger algebra of n-ary operations* and is denoted by $(\Phi, \mathcal{O})$.

This superposition satisfies (cf. [4]) the following *superassociative law*:

$$\mathcal{O}(\mathcal{O}(f, g_1, \ldots, g_n), h_1, \ldots, h_n) =$$

$$\mathcal{O}(f, \mathcal{O}(g_1, h_1, \ldots, h_n), \ldots, \mathcal{O}(g_n, h_1, \ldots, h_n)),$$

where $f, g_1, \ldots, g_n, h_1, \ldots, h_n \in \Omega_n(A)$.

According to general convention used in theory of n-ary operations, the above superassociative law can be written in the shorted form as:

$$\mathcal{O}(\mathcal{O}(f, g_1^n), h_1^n) = \mathcal{O}(f, \mathcal{O}(g_1, h_1^n), \ldots, \mathcal{O}(g_n, h_1^n)). \tag{1}$$

In the case when $g_1 = g_2 = \ldots = g_k = g$ instead of g_1^k we will write $\overset{k}{g}$.

Any nonempty set G with an $(n+1)$-ary superassociative operation o is called an $(n+1)$*-ary Menger algebra* or a *Menger algebra of rank* n and is denoted by (G, o). If in a Menger algebra (G, o) of rank n from the fact that the equation $o(g_1, x_1^n) = o(g_2, x_1^n)$ is valid for all $x_1, \ldots, x_n \in G$ it follows $g_1 = g_2$, then this Menger algebra is called *reductive* [4]. An element $e \in G$ is called a *left* (*right*) *diagonal unit* of a Menger algebra (G, o) of rank n if $o(e, \overset{n}{x}) = x$ (respectively $o(x, \overset{n}{e}) = x$) holds for all $x \in G$. If e is both left and right diagonal unit, then it is called a *diagonal unit*. If a Menger algebra has an element that is a left diagonal unit and an element that is a right diagonal unit, then these elements are equal and no other elements which are left or right diagonal units. A left (right) diagonal unit of (G, o) is a left (right) neutral element of a *diagonal semigroup* of (G, o), i.e., a semigroup $(G, \cdot)$ with the operation $x \cdot y = o(x, \overset{n}{y})$. It is clear that a Menger algebra with right diagonal unit is reductive. Also a Menger algebra (G, o) of rank n in which there are $x_1, \ldots, x_n$ such that $o(g, x_1^n) = g$ for all $g \in G$ is reductive.

3. We say that an n-ary operation $f \in \Omega_n(A)$ is

- (i, j)*-associative*, where $1 \leqslant i < j \leqslant n$, if it satisfies the identity

$$f(a_1^{i-1}, f(a_i^{i+n-1}), a_{i+n}^{2n-1}) = f(a_1^{j-1}, f(a_j^{j+n-1}), a_{j+n}^{2n-1}), \tag{2}$$

- *associative*, if it is (i, j)-associative for all $1 \leqslant i < j \leqslant n$,

- *superassociative*, if it satisfies the identity

$$f(f(a, b_1^{n-1}), c_1^{n-1}) = f(a, f(b_1, c_1^{n-1}), \ldots, f(b_{n-1}, c_1^{n-1})). \tag{3}$$

- *self-distributive* or *autodistributive*, if for all $1 \leqslant i \leqslant n$, it satisfies the identity

$$f(a_1^{i-1}, f(b_1^n), a_{i+1}^n) = f(f(a_1^{i-1}, b_1, a_{i+1}^n), \ldots, f(a_1^{i-1}, b_n, a_{i+1}^n)). \tag{4}$$

It is clear that the operation $f \in \Omega_n(A)$ is associative if and only if it is $(1, j)$-associative for all $j = 2, 3, \ldots, n$.

Unfortunately, the set of all associative ((i, j)-associative, superassociative) n-ary operations defined on the set A may not be closed with respect to the Menger superposition. Indeed, ternary operations $f(x, y, z) = x \wedge y \wedge z$ and $g(x, y, z) = x \vee y \vee z$ defined on a lattice L are associative and superassociative, but their Menger superposition $h = \mathcal{O}(f, g, g, g)$ is neither associative nor superassociative.

We will say that an $(n+1)$-ary operation $o \in \Omega_{n+1}(A)$ is

- *quasi-(i, j)-associative*, where $1 \leqslant i < j \leqslant n$, if it satisfies the identity

$$o(a, b_1^{i-1}, o(a, b_i^{i+n-1}), b_{i+n}^{2n-1}) = o(a, b_1^{j-1}, o(a, b_j^{j+n-1}), b_{j+n}^{2n-1}), \qquad (5)$$

- *quasi-associative*, if it is quasi-(i, j)-associative for all $1 \leqslant i < j \leqslant n$,
- *quasi-superassociative*, if it satisfies the identity

$$o(a, o(a, b, c_1^{n-1}), d_1^{n-1}) = o(a, b, o(a, c_1, d_1^{n-1}), \ldots, o(a, c_{n-1}, d_1^{n-1})), \qquad (6)$$

- *quasi-self-distributive* or *quasi-autodistributive*, if for all $1 \leqslant i \leqslant n$ it satisfies the identity

$$o(a, b_1^{i-1}, o(a, c_1^n), b_{i+1}^n) = o(a, o(a, b_1^{i-1}, c_1, b_{i+1}^n), \ldots, o(a, b_1^{i-1}, c_n, b_{i+1}^n)). \qquad (7)$$

Note that an $(n+1)$-ary operation $o \in \Omega_{n+1}(A)$ is quasi-associative if and only if all n-ary operations $f_a \in \Omega_n(A)$ defined by

$$f_a(x_1^n) = o(a, x_1^n) \qquad (8)$$

are associative in the above sense. Thus any algebra (A, o) with one $(n+1)$-ary quasi-associative operation o can be characterized by the algebra $(A, \mathbb{F})$ with the family $\mathbb{F} = \{f_a \,|\, a \in A\}$ of n-ary associative operations defined by (8), i.e., by the class of n-ary semigroups (A, f_a), where $a \in A$. Analogously a quasi-superassociative (quasi-self-distributive) algebra (A, o) can be characterized by the class of n-ary superassociative (self-distributive) algebras (A, f_a). A group-like Menger algebra (A, o) of rank n (cf. [2] or [4]) which is simultaneously quasi-associative and quasi-self-distributive can be characterized by self-distributive n-ary groups (A, f_a), i.e., by self-distributive associative n-ary quasigroups (cf. [1]). This means that these Menger algebras can be described by commutative groups and one of their automorphisms (cf. [3, Theorem 3]).

Example 1. On a commutative semigroup $(G, +)$ with the property $nx = x$ for each $x \in G$ and fixed $n \geqslant 2$ we define the $(n+1)$-ary operation

$$o(x_0, x_1^n) = \varphi(x_0) + x_1 + x_2 + \cdots + x_n + b,$$

where $b \in G$ is fixed and φ is an idempotent endomorphism of $(G,\cdot)$ such that $\varphi(b) = b$. Direct computations show that (G, o) is a quasi-associative Menger algebra of rank n. If φ is not the identity map, then this Menger algebra is not associative. So, the class of quasi-associative Menger algebras contains non-trivial non-associative algebras.

Example 2. Let $(G,+)$ be a commutative semigroup in which $nx = x$ for all $x \in G$ and fixed $n \geqslant 2$. Then for every $a \in G$ the set G with the operation

$$f_a(x_0, x_1^n) = x_0 + x_1 + x_2 + \cdots + x_n + a$$

is a Menger algebra of rank n that is both associative, quasi-associative and self-distributive. Moreover, the set $\Phi_G = \{f_a \,|\, a \in G\}$ is closed with respect to the Menger superposition $\mathcal{O}$. Thus, $(\Phi_G, \mathcal{O})$ is a Menger algebra of associative, quasi-associative, quasi-superassociative and self-distributive $(n+1)$-ary operations.

4. Now we present a characterization of quasi-associative Menger algebras by algebras of associative n-place functions.

First, we characterize quasi-(i,j)-associative Menger algebras.

Theorem 1. *A Menger algebra (G, o) of rank n is isomorphically represented by (i,j)-associative n-ary operations if and only if it is quasi-(i,j)-associative.*

Proof. Necessity. Let $(\Phi, \mathcal{O})$ be a Menger algebra of (i,j)-associative n-ary operations defined on the set A. Then, for all $f, g_1, \ldots, g_{2n-1} \in \Phi$ and all elements $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}
&\mathcal{O}(f, g_1^{i-1}, \mathcal{O}(f, g_i^{i+n-1}), g_{i+n}^{2n-1})(a_1^n) = \\
&f(g_1(a_1^n), \ldots, g_{i-1}(a_1^n), \mathcal{O}(f, g_i^{i+n-1})(a_1^n), g_{i+n}(a_1^n), \ldots, g_{2n-1}(a_1^n)) = \\
&f(g_1(a_1^n), \ldots, g_{i-1}(a_1^n), f(g_i(a_1^n), \ldots, g_{i+n-1}(a_1^n))(a_1^n), g_{i+n}(a_1^n), \ldots, g_{2n-1}(a_1^n)) = \\
&f(g_1(a_1^n), \ldots, g_{j-1}(a_1^n), f(g_j(a_1^n), \ldots, g_{j+n-1}(a_1^n))(a_1^n), g_{j+n}(a_1^n), \ldots, g_{2n-1}(a_1^n)) = \\
&f(g_1(a_1^n), \ldots, g_{j-1}(a_1^n), \mathcal{O}(f, g_j^{j+n-1})(a_1^n), g_{j+n}(a_1^n), \ldots, g_{2n-1}(a_1^n)) = \\
&\mathcal{O}(f, g_1^{j-1}, \mathcal{O}(f, g_j^{j+n-1}), g_{j+n}^{2n-1})(a_1^n).
\end{aligned}$$

Thus, $\mathcal{O}(f, g_1^{i-1}, \mathcal{O}(f, g_i^{i+n-1}), g_{i+n}^{2n-1})(a_1^n) = \mathcal{O}(f, g_1^{j-1}, \mathcal{O}(f, g_j^{j+n-1}), g_{j+n}^{2n-1})(a_1^n)$, i.e., a Menger algebra $(\Phi, \mathcal{O})$ is quasi-(i,j)-associative.

Sufficiency. Let (G, o) be a quasi-(i,j)-associative Menger algebra of rank n. For every $g \in G$ we define on the set $G_0 = G \cup \{e, c\}$, where $e \neq c$ and $e \notin G$, $c \notin G$, the n-ary operation ω_g by putting:

$$\omega_g(x_1^n) = \begin{cases} o(g, x_1^n), & \text{if } x_1, \ldots, x_n \in G, \\ g, & \text{if } x_1 = \ldots = x_n = e, \\ c, & \text{otherwise.} \end{cases}$$

To prove that this operation is associative, we must consider a few cases.

First, we consider the case when $x_1,\ldots,x_{2n-1}\in G$. In this case

$$\omega_g(x_1^{i-1},\omega_g(x_i^{i+n-1}),x_{i+n}^{2n-1})=o(g,x_1^{i-1},o(g,x_i^{i+n-1}),x_{i+n}^{2n-1})=$$
$$o(g,x_1^{j-1},o(g,x_j^{j+n-1}),x_{j+n}^{2n-1})=\omega_g(x_1^{j-1},\omega_g(x_j^{j+n-1}),x_{j+n}^{2n-1}).$$

If $x_1=\ldots=x_{2n-1}=e$, then, according to the definition, we have

$$\omega_g(\overset{i-1}{e},\omega_g(\overset{n}{e}),\overset{n-i}{e})=\omega_g(\overset{i-1}{e},g,\overset{n-i}{e})=c$$

and

$$\omega_g(\overset{j-1}{e},\omega_g(\overset{n}{e}),\overset{n-j}{e})=\omega_g(\overset{j-1}{e},g,\overset{n-j}{e})=c.$$

Thus,

$$\omega_g(\overset{i-1}{e},\omega_g(\overset{n}{e}),\overset{n-i}{e})=\omega_g(\overset{j-1}{e},\omega_g(\overset{n}{e}),\overset{n-j}{e}).$$

In all other cases

$$\omega_g(x_1^{i-1},\omega_g(x_i^{i+n-1}),x_{i+n}^{2n-1})=c=\omega_g(x_1^{j-1},\omega_g(x_j^{j+n-1}),x_{j+n}^{2n-1}).$$

So, in any case the operation ω_g is (i,j)-associative.

Now we show that $P\colon g\mapsto\omega_g$ is an isomorphism between (G,o) and $(\Phi_G,\mathcal{O})$, where $\Phi_G=\{\omega_g\mid g\in G\}$. To prove this fact we also must consider a few cases.

Let $g,g_1,\ldots,g_n\in G$.

1) If $x_1,\ldots,x_n\in G$, then

$$\begin{aligned}\omega_{o(g,g_1^n)}(x_1^n)&=o(o(g,g_1^n),x_1^n)=o(g,o(g_1,x_1^n),\ldots,o(g_n,x_1^n))\\&=\omega_g(\omega_{g_1}(x_1^n),\ldots,\omega_{g_n}(x_1^n))=\mathcal{O}(\omega_g,\omega_{g_1},\ldots,\omega_{g_n})(x_1^n).\end{aligned}$$

2) If $x_1=x_2=\ldots x_n=e$, then, according to the definition of ω_g, we obtain

$$\mathcal{O}(\omega_g,\omega_{g_1},\ldots,\omega_{g_n})(\overset{n}{e})=\omega_g(\omega_{g_1}(\overset{n}{e}),\ldots,\omega_{g_n}(\overset{n}{e}))=\omega_g(g_1^n)=o(g,g_1^n)$$

and $\omega_{o(g,g_1^n)}(\overset{n}{e})=o(g,g_1^n)$. Thus, $\omega_{o(g,g_1^n)}(\overset{n}{e})=\mathcal{O}(\omega_g,\omega_{g_1},\ldots.\omega_{g_n})(\overset{n}{e})$.

3) In other cases we have $\omega_{o(g,g_1^n)}(x_1^n)=c$ and

$$\mathcal{O}(\omega_g,\omega_{g_1},\ldots,\omega_{g_n})(x_1^n)=\omega_g(\omega_{g_1}(x_1^n),\ldots,\omega_{g_n}(x_1^n))=\omega_g(\overset{n}{c})=c.$$

Thus, in any case $\omega_{o(g,g_1^n)}=\mathcal{O}(\omega_g,\omega_{g_1},\ldots,\omega_{g_n})$ This means that $P(o(g,g_1^n))=\mathcal{O}(P(g),P(g_1),\ldots,P(g_n))$. Hence, P is a homomorphism.

Obviously P is onto $(\Phi_G,\mathcal{O})$. Moreover, if $P(g_1)=P(g_2)$, for some $g_1,g_2\in G$, then also $\omega_{g_1}(x_1^n)=\omega_{g_2}(x_1^n)$ for $x_1,\ldots x_n\in G_0$. In particular, $\omega_{g_1}(\overset{n}{e})=\omega_{g_2}(\overset{n}{e})$, which gives $g_1=g_2$. So, P is an isomorphism. □

In the same way, using the same construction of the operations ω_g, we can prove the following two theorems.

Theorem 2. *A Menger algebra (G,o) of rank n is isomorphically represented by associative n-ary operations if and only if it is quasi-associative.*

Theorem 3. *Any quasi-superassociative Menger algebra (G,o) of rank n can be isomorphically represented by suprassociative n-ary opertations defined on some set.*

From the above results we can deduce the following corollary.

Corollary 1. *If a Menger algebra of rank n is at the same time quasi-(i,j)-associative and quasi-superassociative, then it can be isomorphically represented by n-ary operations which are at the same time (i,j)-associative and superassociative.*

Analogous result is valid for Menger algebras which are at the same time quasi-associative and quasi-superassociative.

Problem A. *Find necessary and sufficient conditions under which a Menger algebra of rank n can be isomorphically represented by superassociative n-ary operations defined on some set.*

5. We will now consider a Menger algebra of self-distributive n-ary operations.

Theorem 4. *A Menger algebra (G,o) of rank n is homomorphically represented by self-distributive n-ary operations if and only if it is quasi-self-distributive.*

Proof. NECESSITY. Let $(\Phi,\mathcal{O})$ be a Menger algebra of self-distributive n-ary operations defined on the set A. Then for all $f,g_i,h_i\in\Phi$, $i=1,2,\ldots,n$, and all $a_1,\ldots,a_n\in A$ we have

$$\begin{aligned}
&\mathcal{O}(f,g_1^{i-1},\mathcal{O}(f,h_1^n),g_{i+1}^n)(a_1^n)=\\
&f(g_1(a_1^n),\ldots,g_{i-1}(a_1^n),f(h_1(a_1^n),\ldots,h_n(a_1^n)),g_{i+1}(a_1^n),\ldots g_n(a_1^n))\stackrel{(4)}{=}\\
&f(f(g_1(a_1^n),\ldots,g_{i-1}(a_1^n),h_1(a_1^n),g_{i+1}(a_1^n),\ldots g_n(a_1^n)),\ldots,\\
&\qquad\qquad f(g_1(a_1^n),\ldots,g_{i-1}(a_1^n),h_n(a_1^n),g_{i+1}(a_1^n),\ldots g_n(a_1^n)))=\\
&\mathcal{O}(f,\mathcal{O}(f,g_1^{i-1},h_1,g_{i+1}^n),\ldots,\mathcal{O}(f,g_1^{i-1},h_n,g_{i+1}^n))(a_1^n).
\end{aligned}$$

Thus, $(\Phi,\mathcal{O})$ is a quasi-self-distributive Menger algebra.

SUFFICIENCY. Let (G,o) be a quasi-self-distributive Menger algebra of rank n. For every $g\in G$ we define on G the n-ary operation ω_g by putting

$$\omega_g(x_1^n)=o(g,x_1^n).$$

Then for all $x_i, y_i \in G$ and $i = 1, \ldots, n$, we obtain:

$$\omega_g(x_1^{i-1}, \omega_g(y_1^n), x_{i+1}^n) = o(g, x_1^{i-1}, o(g, y_1^n), x_{i+1}^n) \overset{(7)}{=}$$
$$o(g, o(g, x_1^{i-1}, y_1, x_{i+1}^n), \ldots, o(g, x_1^{i-1}, y_n, x_{i+1}^n)) =$$
$$\omega_g(\omega_g(x_1^{i-1}, y_1, x_{i+1}^n), \ldots, \omega_g(x_1^{i-1}, y_n, x_{i+1}^n)).$$

So, the operation ω_g is self-distributive.

Now we show that $P\colon g \mapsto \omega_g$ is a homomorphism between (G, o) and $(\Phi_G, \mathcal{O})$, where $\Phi_G = \{\omega_g \,|\, g \in G\}$. Indeed, for all $g, g_1, \ldots, g_n, x_1, \ldots, x_n \in G$ we have:

$$\begin{aligned}\omega_{o(g,g_1^n)}(x_1^n) &= o(o(g, g_1^n), x_1^n) = o(g, o(g_1, x_1^n), \ldots, o(g_n, x_1^n)) \\ &= o(g, \omega_{g_1}(x_1^n), \ldots, \omega_{g_n}(x_1^n)) = \omega_g(\omega_{g_1}(x_1^n), \ldots, \omega_{g_n}(x_1^n)) \\ &= \mathcal{O}(\omega_g, \omega_{g_1}, \ldots, \omega_{g_n})(x_1^n),\end{aligned}$$

which means that

$$P(o(g, g_1^n)) = \mathcal{O}(P(g), P(g_1), \ldots, P(g_n)).$$

This completes the proof. □

Note that the homomorphism $P\colon g \mapsto \omega_g$ may not be one-to-one, but if a Menger algebra (G, o) is reductive, then it is one-to-one, and consequently, it is an isomorphism. Thus the following result is valid.

Corollary 2. *If a quasi-self-distributive Menger algebra of rank n is reductive, then it can be isomorphically represented by self-distributive n-ary operations defined on some set.*

In a similar way we can prove

Theorem 5. *Any quasi-associative (quasi-(i, j)-associative, quasi-superassociative) Menger algebra (G, o) of rank n satisfying (7) can be homomorphically represented by self-distributive associative (respectively, (i, j)-associative, superassociative) n-ary operations defined on some set.*

Problem B. *Find necessary and sufficient conditions under which a Menger algebra of rank n can be isomorphically represented by self-distributive n-ary operations defined on some set.*

Problem C. *Find necessary and sufficient conditions under which a quasi-(i, j)-associative (quasi-associative, quasi-superassociative) Menger algebra of rank n satisfying (7) can be isomorphically represented by self-distributive (i, j)-associative (associative, superassociative) n-ary operations defined on some set.*

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Received February 20, 2018

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