Quasigroups and Related Systems **26** (2018), 121 − 138

Torsion-unitary Cayley graph of an R-module as a functor

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Abstract. Let R be a commutative ring with $1 \neq 0$ and $U(R)$ be the set of unit elements. Let M be an R-module and $T(M)$ the set of torsion elements. In this paper, we introduce and investigate the torsion-unitary *Cayley graph* of M, denoted by $\Upsilon_R(M)$. It is a simple graph with vertex set $M \times R$, and two elements $(m,r),(n,s) \in M \times R$ are adjacent if and only if $(m,r)-(n,s) \in T(M) \times U(R)$. We observe that $\Upsilon_R(-)$ acts as a functor on the category of modules. We also introduce the exact sequence of Cayley graphs and determine the properties of functor $\Upsilon_R(-)$.

1. Introduction

The Cayley graph introduced by Arthur Cayley in 1878 is a useful tool for connection between group theory and the theory of algebraic graphs. Let G be an abelian additive group, C be a subset of G. Whenever $0 \notin C$ and $-C = \{-c \mid c \in C\} \subseteq C$, then the Cayley graph $Cay(G,C)$ is the graph with vertex set G and edge set $\{\{a,b\} \mid a-b \in C\}$. The Cayley graphs as a subcategory of category of graphs is denoted by $\mathfrak{C}$. We refer the reader to [8] for general properties of Cayley graphs.

In recent years, for a ring R and M as an R-module, Cayley graphs of the abelian group $(R,+)$ and $(M,+)$ with respect to subsets of R and M have received much attention in the literature. Suppose that $Z(R)$, $U(R)$, $J(R)$ and $Nil(R)$ are the set of zero-divisors, the set of unit elements, the Jacobson radical of R and the ideal of nilpotent elements, respectively. In [4] and [12], the authors obtained some basic properties of $Cay(R,U(R))$, denoted by G_R, which is usually called the unitary Cayley graph. Also in [11], D. Kiani and M. Molla Haji Aghaei show that if $G_R \cong G_S$, then $R/J(R) \cong S/J(S)$ where R and S are finite commutative rings. Moreover, in [13], J. Sato and K. Baba studied the chromatic number of $Cay(R,Z(R)\setminus\{0\})$. In [14], Shekarriz et al. tried to answer the naturally arising question: Under what conditions on a finite commutative ring R, do we have $\tau(R) \cong Cay(R,Z(R)\setminus\{0\})$? where $\tau(R)$ is the total graph defined in [5]. Also G. G. Aalipour and S. Akbari continued to investigate the properties of $Cay(R,Z(R)\setminus\{0\})$ in [1] and [2]. Let M be an R-module where the collection of prime submodules is non-empty. Let N_Λ be an arbitrary union of prime

2010 Mathematics Subject Classification: 05C25, 05C60, 13C60
Keywords: Cayley graph, torsion element, unit element, categorical product of graph.

submodules and $T(M) = \{m \in M \,|\, rm = 0$ for some $0 \neq r \in R\}$ be the set of torsion elements of M. Also, suppose that $c \in U(R)$ such that $c^{-1} = c$. In [3], we define the extended total graph of M as a simple graph $T\Gamma_c(M, N_\Lambda)$ with vertex set M, and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_\Lambda$ and study some graph theoretic results of it. Also In [6], the authors show that if $M \neq T(M)$, then $T(M)$ is a union of prime submodules of M. Hence in [3], we investigate some properties of $T\Gamma_{-1}(M, T(M)) = Cay(M, T(M) \setminus \{0\})$ too. These provide a motivation to introduce a graph over an R-module as a functor from category of modules to subcategory of graphs.

In this paper, we introduce the torsion-unitary Cayley graph of M, denoted by $\Upsilon_R(M)$. It is a simple graph with vertex set $M \times R$, and two elements $(m, r), (n, s) \in M \times R$ are adjacent if and only if $(m, r) - (n, s) \in T(M) \times U(R)$. We show that it acts as a functor over an R-module. Also we introduce two functors, unitary Cayley graph and torsion graph and study some category theoretic properties of them. The motivation is based the fact that any ring homomorphism and R-module homomorphism preserves the unit elements and the torsion elements, respectively. Of course, any ring homomorphism preserves idempotent and nilpotent elements too. But to make a simple graph (without loop), the set of unit elements is used in definition.

In Section 2, we determine some basic properties of $\Upsilon_R(M)$. In section 3, the graph $\Upsilon_R(M)$ will be studied in finite mode. Also in the end of this section, an example will be provided to demonstrate defects of proof given in [14, Theorem 5.2]. It is not counterexample for [14, Theorem 5.2], which is only indicated counting the number of vertices of a maximal clique of $\tau(R)$ is very complicated in this case (a clique in a graph G is a subset of pairwise adjacent vertices). We also show errors underlying their proof. In the last section, we define the functor $\Upsilon_R : \mathfrak{M}_R \to \mathfrak{C}$ with $\Upsilon_R(M) = Cay(M \times R, T(M) \times U(R))$ where $\mathfrak{M}_R$ is the R-module category. Let $\phi : M \to N$ be an R-module homomorphism, then $\Upsilon_R(\phi) : \Upsilon_R(M) \to \Upsilon_R(N)$ given by $\Upsilon_R(\phi)((m, r)) = (\phi(m), r)$ is a homomorphism of graphs. Also let $\mathfrak{R}$ be the category of ring and let $T\Gamma_R(M)$ be $Cay(M, T(M) \setminus \{0\})$ with loop on all vertices. Then $\Upsilon^u : \mathfrak{R} \to \mathfrak{C}$ and $\Upsilon^t : \mathfrak{M}_R \to \mathfrak{C}$ are functors, with $\Upsilon^u(R) = G_R$ and $\Upsilon^t(M) = T\Gamma_R(M)$ respectively. In this section, we investigate the properties of these functors and introduce an exact sequence of cayley graphs.

Throughout this article, all rings are assumed to be commutative with nonzero identity. Let R be an Artinian ring, the structure theorem [7, Theorem 8.7] implies that $R \cong R_1 \times \ldots \times R_t$, where each R_i is a local ring with maximal ideal $\mathfrak{m}_i$; this decomposition is unique up to permutation of factors. We denote by k_i the residue field $R_i/\mathfrak{m}_i$ and $f_i = |k_i|$. We also assume (after appropriate permutation of factors) that $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_t$. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $\mathbb{F}_q$ will denote the integers, rational numbers, integers modulo n, and the finite field with q elements, respectively. R is reduced if $Nil(R) = \{0\}$. For more notations, we refer the reader to [7].

Let G be a graph with the vertex set $V(G)$. A graph G is totally disconnected if no two vertices of G are adjacent. The complement of G is denoted by $\overline{G}$.

For vertices x and y of G, the length of a shortest path from x to y is denoted by $d_G(x,y)$ ($d_G(x,x)=0$ and $d_G(x,y)=\infty$ if there is no such path). The diameter of G is $diam(G)=sup\{d_G(x,y)|x$ and y are vertices of $G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G)=\infty$ if G contains no cycles). The complete graph on n vertices is denoted by K_n. A graph G is called bipartite if its vertex set can be represented as the union of two disjoint sets V_1 and V_2, such that every edge of G connects an element of V_1 with one of V_2. We call V_1, V_2 a bipartition of $V(G)$. The union of two simple graphs (with loop) G and H is the graph $G\cup H$ with the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)$. Also $\bigcup_{i=1}^{t} G$ is denoted by tG. Let $\mathcal{P}=\{V_1,...,V_k\}$ be a partition of the vertex set of G into non-empty classes. The quotient $G/\mathcal{P}$ of G by $\mathcal{P}$ is the graph whose vertices are the sets $V_1,...,V_k$ and whose edges are the pairs $[V_i,V_j]$ such that there are $u_i\in V_i, u_j\in V_j$ with $[u_i,u_j]\in E(G)$. The mapping $\pi_{\mathcal{P}}:V(G)\to V(G/\mathcal{P})$ defined by $\pi_{\mathcal{P}}(u)=V_i$ such that $u\in V_i$, is the natural map for $\mathcal{P}$. Quotients often provide a way of deriving the structure of an object from the structure of a larger one. Observe that $\pi_{\mathcal{P}}$ is a homomorphism and it is automatically faithful. If φ is a homomorphism of graph from X to Y, then the preimages $\varphi^{-1}(y)$ of each vertex y in Y are called the fibres of φ. The fibres of φ determine a partition $\mathcal{K}_\varphi$ of $V(X)$ called the kernel of φ. If Y has no loops, then the kernel is a partition into independent sets. Given a graph X together with a partition $\mathcal{K}_\varphi$ of $V(X)$, define a graph $X/\mathcal{K}_\varphi$ with vertex set the cells of $\mathcal{K}_\varphi$ and with an edge between two cells if there is an edge of X with an endpoint in each cell (and a loop if there is an edge within a cell). The set of finite simple graphs, denoted by Θ. A graph with loop on all vertices, denoted by G°. The set of finite simple graphs in which loops are admitted is denoted by Θ°. The *categorical product* of G and H is the graph, denoted by $G\times H$, and vertex set $V(G)\times V(H)$, such that vertices (g,h) and (g',h') are adjacent precisely if $gg'\in E(G)$ and $hh'\in E(H)$. Other names for the categorical product that have appeared in the literature are *tensor product*, *Kronecker product* or *direct product*. We know that the categorical product is commutative and associative. Let G_1 and G_2 be graphs. Also let G be a subgraph of G_1 and $V\subseteq V(G_2)$ be the set of disjoint vertices, then $G\times V$ is denoted by G^V.

2. Torsion-unitary Cayley graph

In this section, we define the torsion-unitary Cayley graph of M and we obtain some its basic properties and categorical product. Also, the relationship between the torsion-unitary Cayley graph and the unitary Cayley graph will be expressed.

Definition 2.1. Let R be a commutative ring with nonzero identity and M be an R-module. The *torsion-unitary Cayley graph* of M is a simple graph with vertex set $M\times R$, and two elements $(m,r),(n,s)\in M\times R$ are adjacent if and only if $(m,r)-(n,s)\in T(M)\times U(R)$. This graph is denoted by $\Upsilon_R(M)$.

Definition 2.2. Let R be a commutative ring with nonzero identity and M be an R-module. The *torsion graph* of M, denoted by $T\Gamma_R(M)$, is the graph, whose vertex set is M, and in which $\{m, n\}$ is an edge if and only if $m - n \in T(M)$ (i.e., $T\Gamma_R(M) \cong Cay(M, T(M) \setminus \{0\})^\circ$).

In what follows, the some properties of categorical product is recalled.

Remark 2.3. Let $K_1^\circ \in \Theta^\circ$ denote the graph with exactly one vertex, on which there is a loop. Observe that $K_1^\circ \times G \cong G$ for any $G \in \Theta^\circ$. Therefore, under the operations $\times$ and $+$, the set Θ° is a commutative semiring with unit K_1°. Also if G has no loop at g, then $H^{\{g\}}$ is totally disconnected; whereas if G has a loop at g, then $H^{\{g\}}$ is isomorphic to H. Let $G = G_1 \times G_2 \times \cdots \times G_k = \prod_{i=1}^{k} G_i$. By simple rewording of the definitions, each projection $p_i : G \to G_i$ is a homomorphism. Furthermore, given a graph H and a collection of homomorphisms $\varphi_i : H \to G_i$, for $1 \leqslant i \leqslant k$, observe that the map $\varphi : x \mapsto (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x))$ is a homomorphism $H \to G$. From the two facts just mentioned, we see that every homomorphism $\varphi : H \to G$ has the form $\varphi : x \mapsto (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x))$, for homomorphisms $\varphi_i : H \to G_i$, where $\varphi_i = p_i\varphi$. Clearly φ is uniquely determined by the p_i and φ_i.

Proposition 2.4. [9, Proposition 5.7] *Suppose (g, h) and (g', h') are vertices of a categorical product $G \times H$, and n is an integer for which G has a g, g'-walk of length n and H has an h, h'-walk of length n. Then $G \times H$ has a walk of length n from (g, h) to (g', h'). The smallest such n (if it exists) equals $d_{G\times H}((g, h), (g', h'))$. If no such n exists, then $d_{G\times H}((g, h), (g', h')) = \infty$.*

Proposition 2.5. [9, Proposition 5.8] *Suppose x and y are vertices of $G = G_1 \times G_2 \times \cdots \times G_k$. Then $d_G(x, y) = \min\{n \in \mathbb{N} \mid$ each factor G_i has a walk of length n from $p_i(x)$ to $p_i(y)\}$, where it is understood that $d_G(x, y) = \infty$ if no such n exists.*

Theorem 2.6. (**Weichsel's Theorem**, [9, Theorem 5.9]) *Suppose G and H are connected nontrivial graphs in Θ°. If at least one of G or H has an odd cycle, then $G \times H$ is connected.*

In view of the above theorem, we have the following corollary.

Corollary 2.7. *A categorical product of connected nontrivial graphs is connected if and only if at most one of the factors is bipartite.*

Remark 2.8. (1). $\Upsilon_R(M) \cong T\Gamma_R(M) \times G_R$. Since every vertex $m \in T\Gamma_R(M)$ has a loop, every $G_R^{\{m\}}$ is isomorphic to G_R, also since every vertex $r \in G_R$ has no loop, every $T\Gamma_R(M)^{\{r\}}$ is totally disconnected.

(2). Let R be an Artinian ring and suppose that $f_1 = 2$, then G_R is a bipartite graph. Note that $G_R = \prod G_{R_i}$.

Theorem 2.9. *G_R is a bipartite graph if and only if $\Upsilon_R(M)$ is a bipartite graph.*

Proof. Suppose that G_R is bipartite. Let V_1 and V_2 be bipartition of $V(G_R)$, then $T\Gamma_R(M)^{V_1}$ and $T\Gamma_R(M)^{V_2}$ are bipartition of $V(\Upsilon_R(M))$. Therefore $\Upsilon_R(M)$ is bipartite. Conversely, if G_R is not bipartite, then it has an odd cycle namely O. Hence $O^{\{m\}}$ is an odd cycle in $\Upsilon_R(M)$, since m has a loop in $T\Gamma_R(M)$, a contradiction. Therefore G_R is a bipartite graph. □

Proposition 2.10. *Let R be a commutative ring with identity and let $T(M) \neq \{0\}$, then $gr(\Upsilon_R(M)) \leq 4$. In particular, if $T(M) = \{0\}$, then $\Upsilon_R(M)$ is the union of $|M|$ disjoint G_R's, and $gr(\Upsilon_R(M)) = gr(G_R)$.*

Proof. G_R is not totally disconnected, also since $T(M) \neq \{0\}$, $\Upsilon_R(M)$ is not totally disconnected too. Since $K_2^\circ \times K_2$ is a cycle of length four, $gr(\Upsilon_R(M)) \leqslant 4$. In particular, if $T(M) = \{0\}$, then $\Upsilon_R(M) \cong \bigcup_M K_1^\circ \times G_R \cong \bigcup_M G_R$ and it is clear that $gr(\Upsilon_R(M)) = gr(G_R)$. □

By Remark 2.3, $G_R^{\{m\}}$ is isomorphic to G_R for all $m \in M$. Therefore we have the following corollary in the light of Proposition 2.4.

Corollary 2.11. $gr(\Upsilon_R(M)) \leqslant gr(G_R)$*. In particular,* $gr(\Upsilon_R(M)) = 3$ *if and only if* $gr(G_R) = 3$*. Moreover* $gr(\Upsilon_R(M)) = 4$*, if* $gr(G_R) = 4$*.*

Lemma 2.12. [8, Lemma 3.7.4] *$Cay(G, C)$ is connected if and only if C is a generating set for G.*

Remark 2.13. Let $\mathcal{G}_\mathcal{R} = \{V_1(G_R), \ldots, V_k(G_R)\}$ be a partition of the vertex set of $\Upsilon_R(M)$ where $V_i(G_R) = m_i \times R$ for $m_i \in M$ and $|M| = k$ (k can be infinite). Since m_i has a loop in $T\Gamma_R(M)$, ${G_R}^{\{m_i\}} \cong G_R$ by Remark 2.3. Hence the vertices $V_i(G_R), V_j(G_R) \in \Upsilon_R(M)/\mathcal{G}_\mathcal{R}$ are adjacent if and only if the vertices $m_i, m_j \in T\Gamma_R(M)$ are adjacent since $(m_i, 0) \in V_i(G_R)$ and $(m_j, 1) \in V_j(G_R)$ are adjacent in $\Upsilon_R(M)$ if and only if $m_i - m_j \in T(M)$. Therefore, $\Upsilon_R(M)/\mathcal{G}_\mathcal{R} \cong T\Gamma_R(M)$.

As usual, if $A \subseteq M$, then $< A >$ denotes the $\mathbb{Z}$-submodule of M generated by A.

Theorem 2.14. *Let R be a commutative ring and M an R-module. Then $\Upsilon_R(M)$ is connected if and only if $M =< T(M) >$ and $R =< U(R) >$.*

Proof. Let $\Upsilon_R(M)$ be connected. By Lemma 2.12, $M \times R =< T(M) \times U(R) >$ and so $M =< T(M) >$ and $R =< U(R) >$. Conversely, suppose that $M =< T(M) >$ and $R =< U(R) >$. By Lemma 2.12, G_R is connected and also $T\Gamma_R(M)$ is connected with loops. Consider ${G_R}^{\{m_i\}}$ for some $m_i \in M$, then ${G_R}^{\{m_i\}} \cong G_R$ by Remark 2.3. Hence there is a path in $\Upsilon_R(M)$ from (m_i, r) to (m_i, r') for $r, r' \in R$ since G_R is connected. Also since $T\Gamma_R(M)$ is connected, there is a path in $\Upsilon_R(M)/\mathcal{G}_\mathcal{R}$ from $V_i(G_R)$ to $V_j(G_R)$ for every $m_j \in M$ by the above remark. Therefore there is a path from (m_i, r) to (m_j, r') and $\Upsilon_R(M)$ is a connected graph. □

As an applications of the algebraic graph theory in modules theory, the following corollary hold by Lemma 2.12 and the above theorem.

Corollary 2.15. *Let M be an R-module, then $M \times R =< T(M) \times U(R) >$ if and only if $M =< T(M) >$ and $R =< U(R) >$.*

Theorem 2.16. *Let R be a commutative ring and M an R-module. Suppose that $\Upsilon_R(M)$ is a connected graph (i.e., $M \times R =< T(M) \times U(R) >$). If there is k which is a greatest integer i such that $m = n_1 + n_2 + \cdots + n_i$ where, $m \in M \times R$ and $n_1, ..., n_i \in T(M) \times U(R)$ with $n_1 + n_2 + \cdots + n_i$ is a shortest representation of m, then $diam(\Upsilon_R(M)) = k$. Otherwise, $diam(\Upsilon_R(M)) = \infty$. Moreover, if $\Upsilon_R(M)$ is a connected graph, then $diam(\Upsilon_R(M)) = d_{M \times R}(0, m)$.*

Proof. The proof is similar to the proof of [3, Theorem 14]. □

Remark 2.17. Let $u \in U(R)$ and $j \in J(R)$, then $u + j \in U(R)$. Hence, whenever x and y are adjacent vertices in G_R, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$. Moreover, $x + \mathfrak{m}$ is a totally disconnected subgraph of G_R where $\mathfrak{m}$ is a maximal ideal. Therefore $\bigcup_{m \in M}(x + \mathfrak{m})^{\{m\}}$ is a totally disconnected subgraph of $\Upsilon_R(M)$. Also, suppose that $M \neq T(M)$ and $\{N_\lambda\}_{\lambda \in \Omega}$ is the set of all prime submodules of M. We know that $T(M) = \bigcup_{\lambda \in \Lambda} N_\lambda$ for $\Lambda \subseteq \Omega$ as shown in [6]. Let $N^\Lambda = \bigcap_{\lambda \in \Lambda} N_\lambda$, then every element of $m + N^\Lambda$ is adjacent to every element of $n + N^\Lambda$ if m and n are adjacent vertices in $T\Gamma_R(M)$. Furthermore, $m + N_\lambda$ is a clique with loop in $T\Gamma_R(M)$, where $\lambda \in \Lambda$.

Lemma 2.18. *Let R be a commutative ring and M be an R-module. Then :*

(i) *$\Upsilon_R(M)$ is complete graph if and only if $M = 0$ and R is a field,*

(ii) *$\Upsilon_R(M)$ is vertex transitive,*

(iii) *$\Upsilon_R(M)$ is a regular graph of degree $|T(M)| \times |U(R)|$ with isomorphic components.*

Proof. Let $\Upsilon_R(M)$ is complete graph. Then $M = 0$ since (m, r) and (n, r) are not adjacent for every $m, n \in M$ and $r \in R$. Also R is a field since if there exists a nonunit $x \neq 0$ in R, then $(m, 0)$ and (n, x) are not adjacent. Part (ii) holds for every Cayley graph of a group. To prove the last part, note that under an automorphism of graph G, any component of G is isomorphically mapped to another component. Since $\Upsilon_R(M)$ is vertex-transitive, we conclude that the components of $\Upsilon_R(M)$ are isomorphic and so part (iii) is proved. □

3. The case when M and R are finite

In this section, all graphs considered to be finite. It is natural to seek the conditions under which $A \times C \cong B \times C$ implies $A \cong B$. We call this the cancellation problem for the categorical product. In general, cancellation for the categorical product fails dramatically. If C is any bipartite graph, then there are always non-isomorphic graphs A and B for which $A \times C \cong B \times C$. Indeed, just take $A = K_2$ and $B = 2K_1^{\circ}$ (two loops), then $A \times C \cong 2C \cong B \times C$. But we say that cancellation holds for the torsion-unitary Cayley graphs in this section. Finally, we examine the validity of the proof of Theorem 5.2 in [14].

Remark 3.1. Let R be a finite commutative ring, then $\Upsilon_R(R) = \overline{G_R}^{\circ} \times G_R$ since R is an union of zero divisor and unit elements. Therefore if $G_R \cong G_S$, then $\Upsilon_R(R) \cong \Upsilon_S(S)$.

Corollary 3.2. *Let R be a finite commutative reduced ring and let S be a commutative ring. Then $T\Gamma_R(R) \cong T\Gamma_S(S)$ if and only if $R \cong S$.*

Proof. Let R be a finite commutative ring, then $\overline{G_R}^{\circ} \cong T\Gamma_R(R)$. By [11, Corollary 5.4], $R \cong S$ if and only if $T\Gamma_R(R) \cong T\Gamma_S(S)$. □

Theorem 3.3. *Suppose that R and S are commutative ring and let M be an R-S-bimodule. Then $\Upsilon_S(M) \cong \Upsilon_R(M)$ if and only if $G_S \cong G_R$ where $\Upsilon_S(M) \in \Theta$.*

Proof. It is clear by [9, Proposition 9.6]. □

Corollary 3.4. *Suppose that R and S are commutative reduced ring. let M be an R-S-bimodule such that $\Upsilon_R(M) \in \Theta$. Then $\Upsilon_S(M) \cong \Upsilon_R(M)$ if and only if $R \cong S$.*

Proof. This follow directly from [11, Corollary 5.4] and the above theorem. □

Theorem 3.5. *Suppose that there is a ring homomorphism $\psi : S \to R$ and $\Upsilon_R(M), \Upsilon_S(M) \in \Theta$. Also let M and N be R-modules. If $\Upsilon_R(M) \cong \Upsilon_R(N)$, then $\Upsilon_S(M) \cong \Upsilon_S(N)$.*

Proof. It is clear by [9, Proposition 9.9]. □

By Theorem 2.9 and [9, Proposition 9.10], if $\Upsilon_R(M) \in \Theta$ and it has an odd cycle, then $\Upsilon_R(M) \cong \Upsilon_R(N)$ if and only if $T\Gamma_R(M) \cong T\Gamma_R(N)$. Also by Lemma 2.18$(iii)$, if M is a torsion or torsion-free module, then $\Upsilon_R(M) \cong \Upsilon_R(N)$ if and only if $T\Gamma_R(M) \cong T\Gamma_R(N)$ since $T\Gamma_R(M)$ and $T\Gamma_R(N)$ have loop on all vertices and minimum and maximum degree of $T\Gamma_R(M)$ and $T\Gamma_R(N)$ equal two and $|T(M)|+1$ respectively (a loop is incident to only one vertex, when measuring the degree of such a vertex, the loop is counted twice). By the following theorem, the condition that $\Upsilon_R(M)$ has an odd cycle can be omitted.

Theorem 3.6. *Let M and N be R-modules and let $\Upsilon_R(M) \in \Theta$, then*

$$\Upsilon_R(M) \cong \Upsilon_R(N) \text{ if and only if } T\Gamma_R(M) \cong T\Gamma_R(N).$$

Proof. Suppose that $\Upsilon_R(M) \cong \Upsilon_R(N)$. Since $\Upsilon_R(M) = T\Gamma_R(M) \times G_R$ and $\Upsilon_R(N) = T\Gamma_R(N) \times G_R$, $|T(M)| = |T(N)|$ by Lemma 2.18(iii). Hence

$$T\Gamma_R(M) \times G_R/\mathcal{G}_\mathcal{R} \cong T\Gamma_R(N) \times G_R/\mathcal{G}_\mathcal{R},$$

where $\mathcal{G}_\mathcal{R}$ is as mentioned in Remark 2.13. Therefore $T\Gamma_R(M) \cong T\Gamma_R(N)$ by Remark 2.13. □

Also, by the similar proof, the following corollary is obtained in the cancellation for the categorical product.

Corollary 3.7. *Let $A, B, C \in \Theta^\circ$. Suppose that A and B have loop on all vertices and C has at least one edge. Then $A \times C \cong B \times C$ if and only if $A \cong B$.*

Shekarriz et al. answered the isomorphic question in [14, Theorem 5.2]: Let R be a finite commutative ring, then $\tau(R) \cong Cay(R, Z(R) \setminus \{0\})$ if and only if at least one of the following conditions is true: (a) $R \cong R_1 \oplus \cdots \oplus R_k$, where $k \geq 1$ and each R_i is a local ring of an even order; (b) $R \cong R_1 \oplus \cdots \oplus R_k$, where $k \geq 2$ and each R_i is a local ring and $f_1 = 2$. But, they have errors in its proof when they conclude $\tau(R) \ncong Cay(R, Z(R) \setminus \{0\})$, supposed (a) and (b) do not hold for a finite commutative ring R. In the following, an example will be provided to demonstrate defects of proof given in [14, Theorem 5.2], and we investigate the method of proof too. The equivalence class $Z(R_i) + a_i$, is denoted by $[a_i]$.

Example 3.8. Let $R = \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{Z}_3$ and $(1, 1, 1), (0, 0, -1) \in R$, denoted by 1 and x, respectively. Then $\tau(\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{Z}_3)$ has five maximal cliques, all containing the edge $\{1, x\}$, which are given separately as follows:

(a). Let $c_1 = ([1], [0], \mathbb{Z}_3)$, $c_2 = (\mathbb{F}_4, [0], [-1])$ and $c_3 = ([1], \mathbb{F}_4, [1])$, then $c_1 \cup c_2 \cup c_3$ forms a maximal clique, where $|c_1 \cup c_2 \cup c_3| = |c_1| + |c_2| + |c_3| - |c_1 \cap c_2| - |c_1 \cap c_3| - |c_2 \cap c_3| + |c_1 \cap c_2 \cap c_3| = 3 + 4 + 4 - 1 - 1 - 0 + 0 = 9$.

By permuting the first two components, a new maximal clique will be generated: $([0], [1], \mathbb{Z}_3) \cup ([0], \mathbb{F}_4, [-1]) \cup (\mathbb{F}_4, [1], [1])$. Since, $|R|/f_1 = |R|/f_2$, these two cliques will be equal in size. Moreover, in these maximal cliques, vertices 1 and x are already counted.

(b). Let $c_1 = ([1], \mathbb{F}_4, [1])$ and $c_2 = ([0], \mathbb{F}_4, [-1])$, then $c_1 \cup c_2$ forms a maximal clique, where $|c_1 \cup c_2| = |c_1| + |c_2| - |c_1 \cap c_2| = 4 + 4 - 0 = 8$. By permuting the first two components, a new maximal clique will be generated:

$$(\mathbb{F}_4, [1], [1]) \cup (\mathbb{F}_4, [0], [-1]).$$

Since, in this example, $|R|/f_1 = |R|/f_2$, these two cliques will be equal in size. Moreover, in these maximal cliques, vertices 1 and x are already counted.

(c). Let $c_1 = ([1], [0], [0])$, $c_2 = ([0], [1], [0])$, $c_3 = ([1], [1], [1])$ and $c_4 = ([0], [0], [-1])$, then $c_1 \cup c_2 \cup c_3 \cup c_4$ forms a clique of maximal size 4. It should be noted that, the mutual intersection of every pair of c_i's is empty, for $i = 1, \ldots, 4$, and vertices 1 and x are already counted.

Remark 3.9. Note that this example is not contra example for [14, Theorem 5.2], this is an example which determines the method of counting the number of vertices of a maximal clique of $\tau(R)$ is not true. Let $R = R_1 \oplus R_2 \oplus R_3$ where R_1 and R_2 are even such that $R_i/Z(R_i) \cong \mathbb{F}_{2^t}$, for $i = 1, 2$ and $t \geqslant 2$, and R_3 is odd. Then the layouts of equivalence classes of maximal cliques containing the edge $\{1, x\}$ are as the above example.

Now, let us return to the main subject concerning the flaws in the proof of [14, Theorem 5.2].

The findings discussed in the proof are well-reasoned until they were going to show that for $i = 1, \ldots, k$, the edge $\{1, x\}$ does not belong to a maximal $(|R|/f_i)$-clique in $\tau(R)$. In that proof, it is supposed that $\{y_s | s \in S\}$ is a set of elements of R of maximal size which are adjacent to both 1 and x and also to themselves. It is also cited that if $\{y_s \,|\, s \in S\} \cup \{1, x\}$ forms a clique of maximal size $|R|/f_i$, then there must be $1 \leqslant m_1 < m_2 < \cdots < m_q \leqslant k$; $0 \leqslant q \leqslant k$ such that all y_s's belong to

$$R_1 \oplus \cdots \oplus R_{m_1-1} \oplus [a_{m_1}] \oplus R_{m_1+1} \oplus \cdots \oplus R_{m_q-1} \oplus [a_{m_q}] \oplus R_{m_q+1} \oplus \cdots \oplus R_k. \quad (1)$$

Now, according to this direct sum and ambiguity in the assumption, y_s's could be chosen in three following ways:

(1) y_s's belong to (1) in which a_{m_i} and m_i are fixed for all $i = 1, \ldots, q$. Based on maximal cliques in the example 3.8(a), 3.8(b) and 3.8(c), $\{y_s|s \in S\} \cup \{1, x\}$ is not a maximal clique. It shows that the argument can not be true.

(2) y_s's belong to (1) in which only m_i are fixed for all $i = 1, \ldots, q$. Now, example 3.8(a) shows that $\{y_s|s \in S\} \cup \{1, x\}$ is not a maximal clique.

(3) y_s's belong to (1) in such away a_{m_i}, m_i and q can vary. Thus q will be replaced with q_λ in (1), for some $\lambda \in \Lambda$ such that $1 \leqslant q_\lambda \leqslant k$, and $y_{S_\lambda} = \{y_s|s \in S_\lambda\}$'s are contained in the representation (1), where $S_\lambda \subseteq S$ such that for all $s \in S_\lambda$, the elements of y_{S_λ} in (1) have a fixed representation (i.e., m_{i_λ} and q_λ are fixed). In Example 3.8, y_{S_λ} is the set of vertices of a clique c_i. Based on deduction in [14, Theorem 5.2], $q_\lambda \neq 1$. If $q_\lambda \geqslant 2$, then $|y_{S_\lambda}| = \dfrac{|R|}{\prod_{i=1}^{q_\lambda} f_{m_i}}$, and the required number is calculated by $|\bigcup y_{S_\lambda}|$ as in Example 3.8.

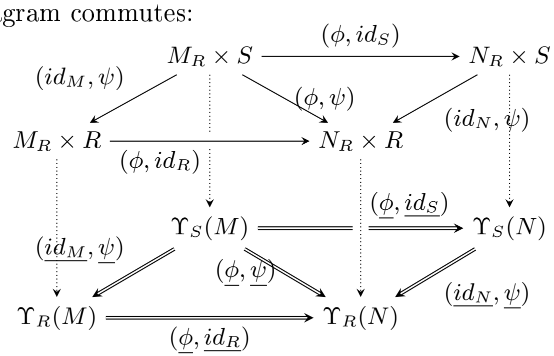
The counting method given in [14, Theorem 5.2] implies that the authors have considered either conditions (1) or (2). Moreover, in the proof, where it is supposed that $2 \leqslant q \leqslant k$, if $[a_{m_p}] = [-1_{m_p}]$ and $[a_{m_v}] = [-x_{m_v}]$ for some $v \neq p$, $1 \leqslant p \leqslant j$ and $j+1 \leqslant v \leqslant k$, then 1 may belong to $\{y_s|s \in S\}$. Correspondingly, if $1 \leqslant v \leqslant j$ and $j+1 \leqslant p \leqslant k$, then x may belong to $\{y_s|s \in S\}$. Therefore, it is generally incorrect to add 2 in counting the total number of vertices of maximal cliques.

4. Torsion-unitary Cayley functor

In this section, we define torsion-unitary Cayley functor and determine some of its categorical properties.

Definition 4.1. Let R be a commutative ring with nonzero identity and M be an R-module. The functor $\Upsilon_R : \mathfrak{M}_R \to \mathfrak{C}$ with $\Upsilon_R(M) = Cay(M \times R, T(M) \times U(R))$ is a covariant functor. It is easily verified that if $\phi : M \to N$ is an R-module homomorphism, then $\Upsilon_R(\phi) : \Upsilon_R(M) \to \Upsilon_R(N)$ given by $\Upsilon_R(\phi)((m, r)) = (\phi(m), r)$ is a homomorphism of graph.

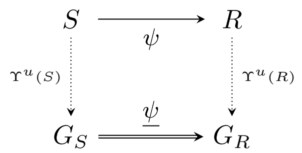
Remark 4.2. In general, let R and S be commutative rings, $\psi : S \to R$ a ring homomorphism. Suppose that M_R and N_R are R-modules and $\phi : M \to N$ is an R-module homomorphism. Then $M_R \times S$ and $N_R \times S$ are S-modules, $\Upsilon_R(\underline{\phi}, \underline{\psi}) : \Upsilon_S(M_R) \to \Upsilon_R(N_R)$ given by $\Upsilon_S(\underline{\phi}, \underline{\psi})((m, r)) = (\underline{\phi}(m), \underline{\psi}(r))$ is a homomorphism of graph $((\underline{\phi}, \underline{id_R})$ replace by $(\underline{\phi}, \underline{\psi})$ in the above definition) and the following diagram commutes:



where, $(\longrightarrow)$ denotes S-module homomorphisms, $(\Longrightarrow)$ denotes homomorphisms of graph and $(\dashrightarrow)$ denotes functors.

By Remark 2.3, if $R = M = 0$, then $G_R = T\Gamma_R(M) = K_1^{\circ}$. So the followings hold:

(a) Let $M = 0$, then $M \times R \cong R$, $\Upsilon_R(M) \cong G_R$, $\Upsilon_-(0)$ is a functor from category of rings to unitary Cayley graphs as a subcategory of graphs category, denoted by $\Upsilon^u(-)$, and the following diagram commutes:



(b) Let M be an R-module, then $\Upsilon_0(-)$ is a functor from $\mathfrak{M}_R$ to torsion graphs as a subcategory of category of graphs, denoted by $\Upsilon^t(-)$. Note that, in this case, graphs are not simple such that every vertex has a loop.

We say that a functor $F : C \to D$ preserves a property $\mathfrak{P}$ of a morphism f in C if $F(f)$ in D also has the property $\mathfrak{P}$. We say that F reflects a property $\mathfrak{P}$ if f has $\mathfrak{P}$ in C whenever $F(f)$ has $\mathfrak{P}$ in D. Analogous definitions can be made with respect to properties of objects. It is clear that every functor preserves commutative diagrams. A homomorphism f from G to $f(G) \subseteq H$ is called a retraction if there exists an injective homomorphism g from $f(G)$ to G such that $fg = id_{f(G)}$. In this case $f(G)$ is called a retract of G, and then G is called a coretract of $f(G)$ while g is called a coretraction. According to the definition of the functor Υ, we have the following corollary.

Corollary 4.3. *The functor Υ preserves and reflects injective mappings and surjective mappings. It preserves retractions and coretractions.*

A homomorphism $\varphi : G \to H$ is called $faithful$ if $\varphi(G)$ is an induced subgraph of H. It will be called $full$ if $\{g, g'\} \in E(G)$ if and only if $\{\varphi(g), \varphi(g')\} \in E(H)$. Let G be a simple graph and φ a full homomorphism, then $\varphi^{-1}(h) \cup \varphi^{-1}(h')$ induces a complete bipartite graph whenever $\{h, h'\} \in E(H)$.

Corollary 4.4. *Let S be a commutative ring and M be an R-module. Suppose that $\psi' : S \to S/J(S)$ and $\phi' : M \to M/N^{\Lambda}$ are the canonical homomorphism, where N^{Λ} is as mentioned in Remark* 2.17. *Then $\Upsilon^u(\psi')$ and $\Upsilon^t(\phi')$ are full homomorphism of graph.*

Let $m' \in T(M)$, then $\underline{\sigma_{m'}} : G_R \to T\Gamma_R(M)$ given by $\underline{\sigma_{m'}}(r) = rm'$ is a homomorphism since $Im(\underline{\sigma_{m'}})$ is a complete graph with loop.

Proposition 4.5. *Let $m \in M \setminus T(M)$ such that $U(R) = R \setminus (T(M) : m)$, then $\underline{\phi_m} : G_R \to \overline{T\Gamma_R(M)}$ given by $\underline{\phi_m}(r) = rm$ is a full homomorphism. In particular, if R is a finite commutative ring, then $\underline{\phi_m}$ is a full homomorphism for all $m \in M \setminus T(M)$.*

Proof. It is clear that $\underline{\phi_m}$ is a homomorphism of graphs. Suppose that $\{r_1 m, r_2 m\}$ is an edge in $\overline{T\Gamma_R(M)}$ for some $r_1, r_2 \in R$, then $u = r_2 - r_1 \in U(R)$ since $um \in T(M)$ if and only if $u \in R \setminus U(R) = (T(M) : m)$ for $m \in M \setminus T(M)$. Therefore $\underline{\phi_m}$ is full. For the "in particular" statement, suppose that R is finite. Hence $U(R) = R \setminus (T(M) : m)$ for all $m \in M \setminus T(M)$ since every regular element of a finite commutative ring is a unit. □

Remark 4.6. In Remark 4.2, $\underline{\psi}$ is a faithful homomorphism if and only if $\psi^{-1}(\psi(s)) \cap U(S) \neq \varnothing$ for all $\psi(s) \in U(R)$ because if $\{\underline{\psi}(s_1), \underline{\psi}(s_2)\}$ is an edge in G_R for some $s_1, s_2 \in S$, then $(s_2 + k_2) - (s_1 + k_1) \in U(S)$ for some $k_1, k_2 \in Ker(\psi)$. According to the same reason, $\underline{\phi}$ is a faithful homomorphism if and only if $\phi^{-1}(\phi(s)) \cap T(M) \neq \varnothing$ for all $\phi(s) \in T(N)$. Also, $\underline{\psi}$ is a full homomorphism if and only if $\psi^{-1}(\psi(s)) \subseteq U(S)$ for all $\psi(s) \in U(R)$ because if $\psi(s) \in U(R)$, then $\{\psi(s), \psi(0)\}$ is a edge in G_R and so $s - 0 \in U(S)$ since $\underline{\psi}$ is a full homomorphism. According to the same reason, $\underline{\phi}$ is a full homomorphism if and only if $\phi^{-1}(\phi(m)) \subseteq T(M)$

for every $\phi(m) \in T(N)$. Moreover, the homomorphism $(\underline{\phi}, \underline{\psi})$ is faithful (full) if and only if each of $\underline{\phi}$ and $\underline{\psi}$ is faithful (full).

Proposition 4.7. *Let $\psi : S \to R$ be a ring homomorphism such that the induced map $Spec(R) \to Spec(S)$ is surjective. Then $\underline{\psi} : G_S \to G_R$ is a full homomorphism.*

Proof. Let $\psi(x)$ is an unit, then $\psi(x) \notin \mathfrak{q}$ for all $\mathfrak{q} \in Spec(R)$. Hence $x \notin \psi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in Spec(R)$. Since induced map is surjective, x is an unit and $\underline{\psi}$ is full homomorphism by the above remark. □

Theorem 4.8. *In Remark 4.2, $\underline{\psi}$ is a surjective full homomorphism of graph if and only if ψ is surjective and $Ker(\psi) \subseteq J(S)$. In particular, if ψ is a surjective ring homomorphism and S is a local commutative ring, then $\underline{\psi}$ is a full surjective homomorphism.*

Proof. Suppose that $\underline{\psi}$ is full. Hence, $\psi^{-1}(\psi(s)) \subseteq U(S)$ for all $\psi(s) \in U(R)$ by Remark 4.6. Let $s \in Ker(\psi)$. Then $\psi(1+ss') = 1$ for all $s' \in S$, hence $1+ss'$ has inverse and it follows that $s \in J(S)$. Therefore $Ker(\psi) \subseteq J(S)$ and ψ is a surjective ring homomorphism by Corollary 4.3. Conversely, let $\psi(s) \in U(R)$, then there is $s' \in S$ such that $(s'+Ker(\psi))(s+Ker(\psi)) = 1+Ker(\psi)$ since $S/Ker(\psi) \cong R$. Hence $ss' - 1 \in Ker(\psi)$ and so $(ss' - 1) \in J(S)$ since $Ker(\psi) \subseteq J(S)$. Therefore $ss' \in U(S)$ and so $s \in U(S)$ since $1 + J(R) \subseteq U(R)$ and $U(S)$ is a saturated multiplicatively closed subset of S. Moreover, if ψ is surjective, then $\underline{\psi}$ is surjective too by Corollary 4.3. The "in particular" statement is clear since $Ker(\psi) \subseteq J(S) = \mathfrak{m}_S$, where $\mathfrak{m}_S$ is a maximal ideal. □

Corollary 4.9. *Let $\psi : S \to R$ be a surjective ring homomorphism. Then $\underline{\psi} : G_S \to G_R$ is a full homomorphism if and only if the map $\psi^* \colon Max(R) \to Max(S)$ is surjective.*

Proof. Let $\underline{\psi}$ be a surjective full homomorphism. Then $Ker(\psi) \subseteq J(S)$ by the above theorem. Now, ψ^* is a surjective map because if $Ker(\psi)$ contained in the every maximal ideal and ψ is surjective, then $\psi(\mathfrak{m}_S)$ and $\psi^{-1}(\mathfrak{m}_R)$ are maximal ideals for $\mathfrak{m}_S \in Max(S)$ and $\mathfrak{m}_R \in Max(R)$. Conversely, by the proof of Proposition 4.7, $\underline{\psi}$ is a full homomorphism. □

Recall that a ring homomorphism $S \to R$ is called flat (faithfully flat) if R is flat(faithfully flat) as an S-module.

Theorem 4.10. *Let $\psi : S \to R$ be a surjective flat homomorphism. Then $\underline{\psi} : G_S \to G_R$ is full if and only if ψ is faithfully flat.*

Proof. Let $\underline{\psi}$ be a surjective full homomorphism, then $Ker(\psi) \subseteq J(S)$, by Theorem 4.8. Also, $\psi^* : Max(R) \to Max(S)$ is surjective and so for all $\mathfrak{m} \in Max(S)$, $R/\psi(\mathfrak{m})$ is nonzero by Corollary 4.9. Therefore, by [10, Lemma 10.38.15], $\psi : S \to R$ is faithfully flat. Conversely, the induced map on $Spec$ is surjective by [10, Lemma 10.38.16]. Therefore, by Proposition 4.7, $\underline{\psi}$ is a full homomorphism. □

Lemma 4.11. *If ϕ in Remark* 4.2 *is an injective homomorphism of modules, then $\underline{\phi}$ is a full injective homomorphism of graphs. Moreover, $\Upsilon_R(\phi)$ is a full injective homomorphism of graphs too.*

Proof. It is clear by Corollary 4.3 and Remark 4.6. □

Theorem 4.12. *Let M and N be R-modules where R is an integral domain and let $\phi : M \to N$ be an R-module homomorphism. Then $Ker(\phi) \subseteq T(M)$ if and only if $\Upsilon^t(\phi) = \underline{\phi} : T\Gamma_R(M) \to T\Gamma_R(N)$ is a full homomorphism of graph.*

Proof. Suppose $\phi(m) \in T(N)$ for some $m \in M$. Then $r\phi(m) = \phi(rm) = 0$ for some $r \in R$. Hence $rm \in T(M)$ since $Ker(\phi) \subseteq T(M)$. Therefore $m \in T(M)$ since R is an integral domain. Conversely, if $\underline{\phi}$ is full, then inverse map of any torsion elements of N is a torsion element in M by Remark 4.6. Hence, $\phi^{-1}(0) = Ker(\phi) \subseteq T(M)$. □

Remark 4.13. A homomorphism φ of a graph G into H gives rise to an equivalence relation $\equiv_\varphi$. In other words, the kernel of φ, defined on V by $u \equiv_\varphi v$ if and only if $\varphi(u) = \varphi(v)$. Therefore, a homomorphism of graphs $\varphi : G \to H$ is surjective and faithful if and only if $\omega : G/\mathcal{K}_\varphi \to H$ is an isomorphism.

Theorem 4.14. *According to the assumptions of Remark* 4.2*, let $\underline{\psi}$ and $\underline{\phi}$ be faithful homomorphisms of graphs. Then*

(1) $\Upsilon^u(S/Ker(\psi)) \cong G_S/\mathcal{K}_{\underline{\psi}}$,

(2) $\Upsilon^t(M/Ker(\phi)) \cong T\Gamma_R(M)/\mathcal{K}_{\underline{\phi}}$,

(3) $\Upsilon_R(M/Ker(\phi)) \cong \Upsilon_R(M)/\mathcal{K}_{\underline{\phi}\times\underline{id}}$.

Proof. (1). By the above remark, if $\underline{\psi} : G_S \to G_R$ is a faithful homomorphism, then $G_S/\mathcal{K}_{\underline{\psi}} \cong \underline{\psi}(G_S)$. Since the diagram commutes in Remark 4.2(a), $\underline{\psi}(\Upsilon^u(S)) = \Upsilon^u(\underline{\psi(S)})$. Therefore $G_S/\mathcal{K}_{\underline{\psi}} \cong \underline{\psi}(G_S) \cong \Upsilon^u(S/Ker(\psi))$.

(2) The proof is similar to the proof of part (1).

(3) Let $\underline{\phi}$ be faithful and $\underline{\phi}\times\underline{id} : \Upsilon_R(M) \to \Upsilon_R(N)$ be the graph homomorphism induced by ϕ. Then $\underline{\phi} \times \underline{id}$ is faithful by Remark 4.6. Hence

$$(\underline{\phi} \times \underline{id})(\Upsilon_R(M)) \cong \Upsilon_R(M)/\mathcal{K}_{\underline{\phi}\times\underline{id}},$$

by the above remark. Since the diagram is commutative in Remark 4.2,

$$(\underline{\phi} \times \underline{id})(\Upsilon_R(M)) = \Upsilon_R(\phi(M)) \cong \Upsilon_R(M/Ker(\phi)).$$

Therefore $\Upsilon_R(M/Ker(\phi)) \cong \Upsilon_R(M)/\mathcal{K}_{\underline{\phi}\times\underline{id}}$. □

Corollary 4.15. *Let $\mathcal{I}$ and $\mathcal{N}$ be the partitions of ring S and R-module M which generated by the equivalence relation modulo I as an ideal of S and N as a submodule of M, respectively. Let $\underline{\psi} : G_S \to G_{S/I}$ and $\underline{\phi} : T\Gamma_R(M) \to T\Gamma_R(M/N)$ be faithful. Then*

(1) $\Upsilon^u(S/I) \cong G_S/\mathcal{I}$,

(2) $\Upsilon^t(M/N) \cong T\Gamma_R(M)/\mathcal{N}$,

(3) $\Upsilon_R(M/N) \cong T\Gamma_R(M)/\mathcal{N} \times G_R$.

Proof. Let $\psi : S \to S/I$ and $\phi : M \to M/N$ are ring and module homomorphism, respectively. Then $\mathcal{I} = \mathcal{K}_{\underline{\psi}}$ and $\mathcal{N} = \mathcal{K}_{\underline{\phi}}$. Hence the three parts are clear by the above theorem. □

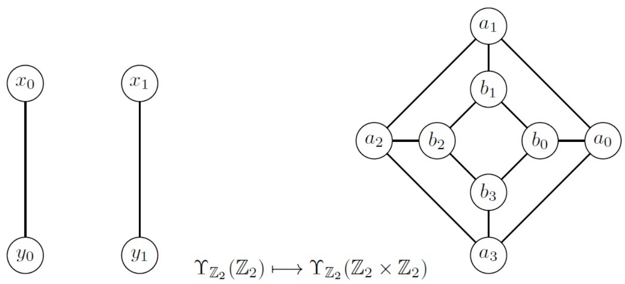
Example 4.16. (a). Let $n \geq 4$ be an integer and $\psi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be a ring homomorphism. Then $\psi^{-1}(\bar{m}) \cap U(\mathbb{Z}) = \varnothing$, where $-1, 1 \neq \bar{m} \in U(\mathbb{Z}_n)$. Hence $\underline{\psi} : G_{\mathbb{Z}} \to G_{\mathbb{Z}_n}$ is not faithful by Remark 4.6.

(b). Let $\theta : \mathbb{Z}_6 \to \mathbb{Z}_6/3\mathbb{Z}_6$ be the canonical homomorphism of rings. Then $\theta^{-1}(\bar{m}) \cap U(\mathbb{Z}_6) \neq \varnothing$ and $\theta^{-1}(\bar{m}) \nsubseteq U(\mathbb{Z}_6)$ for $m = 1, 2$. Hence the graph homomorphism $\underline{\theta} : G_{\mathbb{Z}_6} \to G_{\mathbb{Z}_3}$ is faithful, but is not full by Remark 4.6. Also, consider θ as a homomorphism of $\mathbb{Z}_6$-modules, then $\theta^{-1}(\bar{m}) \cap T(\mathbb{Z}_6) \neq \varnothing$ and $\theta^{-1}(\bar{m}) \nsubseteq T(\mathbb{Z}_6)$ for $m = 0, 1, 2$. Therefore, $\underline{\theta} : T\Gamma_{\mathbb{Z}_6}(\mathbb{Z}_6) \to T\Gamma_{\mathbb{Z}_6}(\mathbb{Z}_3)$ is faithful, but is not full by Remark 4.6. Moreover, let $\underline{\theta} \times \underline{id_{\mathbb{Z}_6}} : \Upsilon_{\mathbb{Z}_6}(\mathbb{Z}_6) \to \Upsilon_{\mathbb{Z}_6}(\mathbb{Z}_3)$. Since $\underline{id_{\mathbb{Z}_6}}$ is a full homomorphism of graph and $\underline{\theta}$ is faithful, $\underline{\theta} \times \underline{id_{\mathbb{Z}_6}}$ is a faithful homomorphism of graph by Remark 4.6.

(c). Let R be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$, where a_n is nilpotent and Let $\mathcal{R}$ be the partition of ring $R[[x]]$ which generated by the equivalence relation modulo $Nil(R)$ as an ideal of nilpotent elements. Then $\{f | a_n \in Nil(R)\} = Nil(R[[x]]) \subseteq J(R[[x]]) = \sum_{n=0}^{\infty} b_n x^n$ where $b_0 \in J(R)$ by Exercise 2 in [7, p. 84] and Exercise 5 in [7, p. 11]. Therefore $\underline{\psi} : G_{R[[x]]} \to G_{R[[x]]}/\mathcal{R}$ is a full homomorphism of graphs by Theorem 4.8 and Corollary 4.15.

Let C and D be categories. A covariant functor $F : C \to D$ is said to be $faithful$ if the mapping $Hom_C(A, A') \to Hom_D(F(A), F(A'))$ is injective for all $A, A' \in C$, and it will be called $full$ if this mapping is surjective.

Example 4.17. Let $\phi : \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be an $\mathbb{Z}_2$-module homomorphism and let $\varphi : \Upsilon_{\mathbb{Z}_2}(\mathbb{Z}_2) \to \Upsilon_{\mathbb{Z}_2}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ be a graph homomorphism with $\varphi(x_0) = a_0$, $\varphi(x_1) = a_2$ and $\varphi(y_i) = a_1$ for $i = 1, 2$ by the following figure:



then there is not a module homomorphism such that $\underline{\phi} \times \underline{id_{\mathbb{Z}_2}} = \varphi$ since $Im(\phi)$ is a submodule of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Corollary 4.18. *The functor* $\Upsilon_R : \mathfrak{M}_R \to Cay(G, C)$ *is faithful. But is not full.*

Proof. The first part follows directly from the definition. By the above example, a homomorphism of graph is not a module homomorphism in general. Therefore the functor Υ is not full. □

Let R_i be a commutative ring for $1 \leqslant i \leqslant t$. The element $(u_1, u_2, \ldots, u_t)$ is a unit of $\bigoplus R_i$ if and only if each u_i is a unit element in R_i. Hence $G_{\bigoplus R_i} \cong \prod G_{R_i}$.

Remark 4.19. (1). Note that unlike in group theory, the inverse of a bijective homomorphism of graph need not be a homomorphism. For example, any bijective homomorphism from $\overline{K}_n$ to K_n. A faithful bijective homomorphism is an isomorphism of graphs.

(2). Since $T(N \oplus M) \subseteq T(N) \times T(M)$, the map

$$i : T\Gamma_R(N \oplus M) \to T\Gamma_R(N) \times T\Gamma_R(M)$$

is a graph homomorphism.

Proposition 4.20. *Let R be an integral domain and let M and N be R-modules. Then* $\Upsilon_R(N \oplus M) \cong T\Gamma_R(N) \times \Upsilon_R(M)$.

Proof. Consider the map $\iota : \Upsilon_R(N \oplus M) \to T\Gamma_R(N) \times \Upsilon_R(M)$ given by $\iota(n, m, r) = (n, (m, r))$. Hence by Remark 4.19(2), it is a bijective homomorphism of graph. Since R is an integral domain, $(n, m) \in T(N \oplus M)$ if and only if $n \in T(N)$ and $m \in T(M)$. Therefore ι is faithful and $\Upsilon_R(N \oplus M) \cong T\Gamma_R(N) \times \Upsilon_R(M)$. □

Definition 4.21. Suppose that $\{G_i\}_{i \in \mathbb{Z}}$ is a family of groups where e_i is the identity element of G_i. A sequence of Cayley graphs

$$\cdots \to Cay(G_{i-1}, C_{i-1}) \xrightarrow{\varphi_{i-1}} Cay(G_i, C_i) \xrightarrow{\varphi_i} Cay(G_{i+1}, C_{i+1}) \to \cdots, \qquad (2)$$

is called exact if $\varphi_i^{-1}(e_{i+1}) = Im(\varphi_{i-1})$ and $\varphi_j(C_j) \subseteq C_{j+1}$ for all $i, j \in \mathbb{Z}$. In particular, the short exact sequence of Cayley graph is an exact sequence in the form

$$Cay(G_1, C_1) \xrightarrow{\varphi_1} Cay(G_2, C_2) \xrightarrow{\varphi_2} Cay(G_3, C_3),$$

such that φ_1 and φ_2 are injective and surjective, respectively.

The above definition may be extended to the Cayley graph with loop on all vertices (i.e., $e_i \in C_i$).

Remark 4.22. By the above definition and Corollary 4.3, the functors Υ^u and Υ^t are exact. Let

$$\cdots \to M_{i-1} \xrightarrow{\phi_{i-1}} M_i \xrightarrow{\phi_i} M_{i+1} \to \cdots$$

is an exact sequence of R-modules and R-homomorphisms, then

$$\cdots \to \Upsilon_R(M_{i-1}) \xrightarrow{\underline{\phi_{i-1}}} \Upsilon_R(M_i) \xrightarrow{\underline{\phi_i}} \Upsilon_R(M_{i+1}) \to \cdots$$

is not the exact sequence since $Ker(\underline{\phi_i}) \subsetneqq Im(\underline{\phi_{i-1}})$, where $\underline{\phi} = (\phi, id)$. Also if (2) is the sequence of Cayley graphs such that $Im(\varphi_{i-1}) = \varphi_i^{-1}(g_{i+1})$ for some $g_{i+1} \in G_{i+1}$ and every $i \in \mathbb{Z}$, then it can be turned into an exact sequence whenever φ_i's replace with $\sigma_{i+1}\varphi_i\sigma_i^{-1}$, where σ_i is an automorphism of vertex transitive graph $Cay(G_i, C_i)$ with $\sigma_i(g_i) = e_{i+1}$, for all $i \in \mathbb{Z}$.

Theorem 4.23. *Let R be a commutative ring and*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\phi_1} & M_2 & \xrightarrow{\phi_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\
0 & \longrightarrow & M_1' & \xrightarrow{\phi_1'} & M_2' & \xrightarrow{\phi_2'} & M_3' & \longrightarrow & 0
\end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Consider the commutative diagram:

$$\begin{array}{ccccc}
TUC_R(M_1) & \xrightarrow{(\underline{\phi_1}, \underline{id_R})} & TUC_R(M_2) & \xrightarrow{(\underline{\phi_2}, \underline{id_R})} & TUC_R(M_3) \\
{\scriptstyle (\underline{\eta_1}, \underline{id_R})}\downarrow & & \downarrow{\scriptstyle (\underline{\eta_2}, \underline{id_R})} & & \downarrow{\scriptstyle (\underline{\eta_3}, \underline{id_R})} \\
TUC_R(M_1') & \xrightarrow{(\underline{\phi_1'}, \underline{id_R})} & TUC_R(M_2') & \xrightarrow{(\underline{\phi_2'}, \underline{id_R})} & TUC_R(M_3')
\end{array}.$$

(1) *If η_1 and η_3 are injective then so is $(\underline{\eta_2}, \underline{id_R})$.*

(2) *If η_1 and η_3 are surjective then so is $(\underline{\eta_2}, \underline{id_R})$.*

(3) *If η_1 and η_3 are isomorphism of module then $(\underline{\eta_2}, \underline{id_R})$ is an isomorphism of graph.*

Proof. Parts (1) and (2) follow directly from Corollary 4.3 and Short Five Lemma (Note that by the above remark, rows of the second diagram in this theorem is not the short exact sequences of Cayley graphs).

(3). This follows directly from parts above, Lemma 4.11 and Remark 4.6. □

Let R be a ring and let $0 \to M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \to 0$ be a short exact sequence of R-modules. The sequence is said to be split if $\phi_1(M_1)$ is a direct summand of M_2. Up to isomorphism, one has $M_2 = M_1 \oplus M_3$.

Theorem 4.24. *Let*

$$0 \to M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \to 0 \tag{3}$$

be a split short exact sequence of R-modules and let $T(M_2)$ be a submodule of M_2. Then

$$\Upsilon_R(M_2) \cong T\Gamma_R(M_1) \times \Upsilon_R(M_3) \cong T\Gamma_R(M_3) \times \Upsilon_R(M_1).$$

Proof. Since (3) is a split short exact sequence of R-module, there are R-module homomorphisms $\psi_1 : M_2 \to M_1$ and $\psi_2 : M_3 \to M_2$ such that $\psi_1 \circ \phi_1 = id_{M_1}$ and $\phi_2 \circ \psi_2 = id_{M_3}$. Consider map $\varphi : \Upsilon_R(M_2) \to T\Gamma_R(M_1) \times \Upsilon_R(M_3)$ given by $\varphi(m_2, r) = (\underline{\psi_1}(m_2), \underline{\phi_2} \times \underline{id_R}(m_2, r))$. Since $\underline{\psi_1}$ and $\underline{\phi_2} \times \underline{id_R}$ are homomorphisms of graph, so is φ. Let $\varphi(m_2, r) = \varphi(m_2', r')$, then $\underline{\psi_1}(m_2) = \underline{\psi_1}(m_2')$ and $\underline{\phi_2} \times \underline{id_R}(m_2, r) = \underline{\phi_2} \times \underline{id_R}(m_2', r')$. So $\psi_1(m_2 - m_2') = 0$, $m_2 - m_2' \in Ker(\phi_2)$ and $r = r'$ since $(\phi_2(m_2), r) = (\phi_2(m_2'), r')$. Hence $m_2 - m_2' \in Im(\phi_1)$ since (3) is a short exact sequence of R-modules. So $m_2 = m_2'$ since $\psi_1 \circ \phi_1 = id_{M_1}$ and $\psi_1(m_2 - m_2') = 0$. Therefore $(m_2, r) = (m_2', r')$ and φ is injective. Moreover, φ is a surjective homomorphism of graph because if $(m_1, (m_3, r)) \in T\Gamma_R(M_1) \times \Upsilon_R(M_3)$, then $\varphi(\phi_1(m_1) + \psi_2(m_3) - \phi_1 \circ \psi_1 \circ \psi_2(m_3), r) = (m_1, (m_3, r))$ since $\phi_2 \circ \phi_1 = 0$, $\psi_1 \circ \phi_1 = id_{M_1}$ and $\phi_2 \circ \psi_2 = id_{M_3}$. Also we need to prove that φ is faithful for being an isomorphism of graphs. Suppose that vertices $a = (\underline{\psi_1}(m_2), \underline{\phi_2} \times \underline{id_R}(m_2, r))$ and $b = (\underline{\psi_1}(m_2'), \underline{\phi_2} \times \underline{id_R}(m_2', r'))$ are adjacent in $T\Gamma_R(M_1) \times \Upsilon_R(M_3)$, then $m_1' = \psi_1(m_2 - m_2') \in T(M_1)$ and $m_3' = \phi_2(m_2 - m_2') \in T(M_3)$. Since $T(M_2)$ is a submodule of M_2, $(\phi_1(m_1') + \psi_2(m_3') - \phi_1 \circ \psi_1 \circ \psi_2(m_3') \in T(M_2)$. Therefore the vertices $\varphi^{-1}(a) = (\phi_1 \circ \psi_1(m_2) + \psi_2 \circ \phi_2(m_2) - \phi_1 \circ \psi_1 \circ \psi_2 \circ \phi_2(m_2), r)$ and $\varphi^{-1}(b) = (\phi_1 \circ \psi_1(m_2') + \psi_2 \circ \phi_2(m_2') - \phi_1 \circ \psi_1 \circ \psi_2 \circ \phi_2(m_2'), r')$ are adjacent in $\Upsilon_R(M_2)$. □

Corollary 4.25. *Let* (3) *be a split short exact sequence of R-module and $T(M_2)$ is a submodule of M_2. Then*

$$\Upsilon_R(M_2) \cong \Upsilon_R(M_1 \oplus M_3) \cong T\Gamma_R(M_1) \times \Upsilon_R(M_3) \cong T\Gamma_R(M_3) \times \Upsilon_R(M_1).$$

Proof. By Theorem 4.23 and the above theorem, it is clear. □

Example 4.26. Let $T(M)$ be a proper submodule of R-module M such that $|T(M)| = \alpha$ and $|M/T(M)| = \beta$. If R is a principal ideal domain, then the short exact sequence of R-modules

$$0 \to T(M) \to M \to M/T(M) \to 0 \tag{4}$$

splits, so $M \cong T(M) \oplus M/T(M)$ as a direct sum of a torsion module and a free module. Then

$$\Upsilon_R(M) \cong T\Gamma_R(T(M)) \times T\Gamma_R(M/T(M)) \times G_R = \beta K_\alpha^\circ \times G_R$$

by the above corollary and [3, Theorem 7(1)]. But if ring R is not a domain, then $M/T(M)$ is torsion by [6, Theorem 2.8]. By [3, Theorem 7(1)], $T\Gamma_R(M) =$

$K^\circ_\alpha \times \beta K^\circ_1 = \beta K^\circ_\alpha$. Hence $\Upsilon^t(M) \ncong T\Gamma_R(T(M)) \times T\Gamma_R(M/T(M))$ because if ring R is not a domain, then $T\Gamma_R(T(M)) \times T\Gamma_R(M/T(M)) = K^\circ_\alpha \times K^\circ_\beta$ (let $K^\circ_\alpha \times K^\circ_\beta = K^\circ_\alpha \times \beta K^\circ_1$. By [9, Proposition 9.6], $K^\circ_\beta = \beta K^\circ_1$, so $\beta = 1$ and $M = T(M)$).

As an applications of the algebraic graph theory in modules theory, the following corollary hold by the above example.

Corollary 4.27. *Suppose that the short exact sequence of R-modules* (4) *splits, then R is a domain.*

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Received November 27, 2017

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