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On prime and primary avoidance theorem for subsemimodules

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Abstract. We study some important results of prime and primary subsemimodules. We also prove the primary avoidance theorem for subsemimodules.

1. Introduction

Prime and primary submodules play crucial role in ring and module theory. These concepts were widely studied in [1], [2], [3], [6], [8], [9]. C. P. Lu in [8], proved the prime avoidance theorem for submodules. El-Atrash and Ashour in [7], proved primary avoidance theorem for submodules. Several authors have studied and explored these concepts in semimodule theory. In this paper, we study the concepts of prime and primary subsemimodules and prove several results analogous to module theory.

By a *semiring*, we mean an algebraic structure $(S, +, 0_S)$ such that $(S, \cdot)$ is a semigroup and $(S, +, 0_S)$ is a commutative monoid in which the multiplication is distributive with respect to the addition both from the left and from the right and 0_S is the additive identity of S and also $0_S x = x0_S = 0_S$ for all $x \in S$. A nonempty subset I of a semiring S is called an *ideal* of S if $a, b \in I$ and $s \in S$, then $a + b \in I$ and $sa, as \in I$. An ideal I of a semiring S is called *subtractive* if $a, a + b \in I, b \in S$, then $b \in I$. An ideal I of a semiring S is called *prime* if $ab \in I$, then either $a \in I$ or $b \in I$. If I is an ideal of S, then the *radical* of I is defined as $Rad(I) = \sqrt{I} = \{a \in S : a^2 \in I\}$. An ideal I of a semiring S is called a *primary ideal* of S if $ab \in I$, then either $a \in I$ or $b \in \sqrt{I}$. Let S be a semiring. A *left S-semimodule* M is a commutative monoid $(M, +)$ which has a zero element 0_M, together with an operation $S \times M \to M$; denoted by $(a, x) \to ax$ such that for all $a, b \in S$ and $x, y \in M$,

1. $a(x + y) = ax + ay$,
2. $(a + b)x = ax + bx$,
3. $(ab)x = a(bx)$,
4. $0_S x = 0_M = a0_M$.

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A proper subsemimodule N of an S-semimodule M is called *subtractive* if $a, a+b \in N$, $b \in M$ then $b \in N$. The *associated ideal* of a subsemimodule N of M is defined as $(N : M) = \{a \in S : aM \subseteq N\}$. A proper subsemimodule N of an S-semimodule M is said to be *strong subsemimodule* if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

We shortly summarize the content of the paper: In the first section, by applying the prime avoidance theorem for subsemimodules [10], we prove the extended version of prime avoidance theorem for subsemimodules. In the second section, we prove some results on primary subsemimodules and by using the technique of efficient covering of subsemimodules, we prove the primary avoidance theorem for subsemimodules.

Throughout this paper, S will always denote a commutative semiring with identity $1 \neq 0$ and S-semimodules means semimodules.

2. Prime subsemimodules

A proper subsemimodule N of an S-semimodule M is called *prime* if whenever $rm \in N$ then $rM \subseteq N$ or $m \in N$.

We start with the following obvious results

Theorem 2.1. *If N is a maximal subsemimodule of an S-semimodule M, then N is a prime subsemimodule of M.* □

Corollary 2.2. *Let M be an S-semimodule and N be a proper subsemimodule of M. If N is a subtractive subsemimodule of M and $m \in M \setminus N$. Then the following statements holds:*

1. *$(N : M)$ is a subtractive ideal of S.*
2. *$(0 : M)$ and $(N : m)$ are subtractive ideals of S.* □

Corollary 2.3. *Let N be a prime subsemimodule of an S-semimodule M. Then for each $m \in M \setminus N$, $(N : M)$ and $(N : m)$ are prime ideals of S.* □

Theorem 2.4. *Let $N_1, N_2, \ldots, N_n$ be subsemimodules of an S-semimodule M and let N be a prime subsemimodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq N$, then there exists an $1 \leqslant i \leqslant n$ such that $N_i \subseteq N$ or $(N_i : M) \subseteq (N : m)$ where $m \in M \setminus N$.*

Proof. Suppose $N_i \nsubseteq N$ and $(N_i : M) \nsubseteq (N : m)$ where $m \in M \setminus N$ and for all $1 \leqslant i \leqslant n$. For particular, $i = k$, we have $N_k \nsubseteq N$, then there exists an $m_k \in M$ such that $m_k \in N_k$ but $m_k \notin N$. Also, there exist $a_i \in (N_i : M)$ such that $a_i \notin (N : m_k)$ for all $i \neq k$. This gives $a_i m_k \in N_i$ and $a_i m_k \notin N$. Therefore, $a_i m_k \in N_i \cap N_k$ for all $i \neq k$. So $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n m_k \in N_1 \cap \ldots \cap N_n \subseteq N$. This implies, $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n \in (N : m_k)$. By Corollary 2.3, $(N : m_k)$ is a prime ideal. Therefore, we have $a_i \in (N : m_k)$ for $i \neq k$, a contradiction. Hence there exists an i such that $N_i \subseteq N$ or $(N_i : M) \subseteq (N : m)$, where $m \in M \setminus N$. □

Theorem 2.5. *Let M be an S-semimodule, N be an arbitrary subsemimodule of M and $N_1, N_2, \ldots, N_n$ be subtractive prime subsemimodules of M. Suppose $(N_j : M) \nsubseteq (N_i : m)$ for all $m \in M \setminus N_i$ with $i \neq j$. If $N \nsubseteq N_i$ for all i, then there exists an element $x \in N$ such that $x \notin \cup N_i$; hence, $N \nsubseteq \cup N_i$.*

Proof. Since $N \nsubseteq N_i$, then there exists $m_i \in N$ such that $m_i \notin N_i$ for all i. By Corollary 2.3, $(N_i : m_i)$ is a prime ideal of S. By the given hypothesis, there exists $r_j \in (N_j : M)$ and $r_j \notin (N_i : m_i)$ for $i \neq j$. Let $s_i = r_1 r_2 \ldots r_{i-1} r_{i+1} \ldots r_n = \prod_{j \neq i} r_j$. Let $x_i = m_i s_i$ for all i. Then $x_i = m_i s_i \in N_j$ for all $j \neq i$. But $x_i \notin N_i$ because, if $x_i \in N_i$ then $m_i s_i \in N_i$, so $s_i \in (N_i : m_i)$, a contradiction. Let $x = x_1 + x_2 + \ldots + x_n$. Then $x = x_i + \sum_{j \neq i} x_j$. Since $\sum_{j \neq i} x_j \in N_i$, therefore $x \notin N_i$ otherwise we would have $x_i \in N_i$ which is a contradiction, so $x \notin \cup N_i$. Also, $m_i \in N$ for all i, therefore $x \in N$ and hence $N \nsubseteq \cup N_i$. □

Let $N_1, N_2, \ldots, N_n$ be subsemimodules of M. Define a covering $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is efficient if no N_i is superfluous for $1 \leqslant i \leqslant n$. In otherwords, we say $N = N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient union if none of the N_i's may be excluded. Any cover or union consisting of subsemimodules of M be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 2.6. (cf. [5]) *Let $N = N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M. Then $\bigcap_{i=1}^{n} N_i = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} N_i$ for any $j \in \{1, 2, \ldots, n\}$.* □

Proposition 2.7. (cf. [10]) *Let $N \subseteq N_1 \cup N_2 \ldots \cup N_n$ be an efficient covering consisting of subtractive subsemimodules of an S-semimodule M, where $n \geqslant 2$. If $(N_j : M) \nsubseteq (N_k : M)$ for every $j \neq k$, then no N_k for $k \in \{1, 2, \ldots, n\}$ is a prime subsemimodule of M.* □

Theorem 2.8. (The prime avoidance theorem, cf. [10])
Let M be an S-semimodule, $N_1, N_2, ..., N_n$ a finite number of subtractive subsemimodules of M and N be a subsemimodule of M such that $N \subseteq N_1 \cup N_2 \ldots \cup N_n$, $(n \geqslant 2)$. Assume that at most two of the N_i's are not prime and that $(N_j : M) \nsubseteq (N_k : M)$ for every $j \neq k$. Then, $N \subseteq N_k$ for some k. □

Now, we come to our main theorem which is a more general form of the above theorem.

Theorem 2.9. (Extended prime avoidance theorem for subsemimodules)
Let M be an S-semimodules and $N_1, N_2, ..., N_r$ be subtractive prime subsemimodues of M such that $(N_i : M) \nsubseteq (N_j : M)$ for $i \neq j$, $r \geqslant 1$. Let $m \in M$ be such that $mS + N \nsubseteq \bigcup_{i=1}^{r} N_i$. Then there exists $n \in N$ such that $m + n \notin \bigcup_{i=1}^{r} N_i$.

Proof. Suppose that m lies in each of $N_1, \dots, N_k$ but none of $N_{k+1}, N_{k+2}, \dots, N_r$. If $k = 0$, then $m = m+0 \notin \bigcup_{i=1}^{r} N_i$ and so there is nothing to prove. Assume that it is true for $k \geqslant 1$. Now, $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise by prime avoidance theorem for semimodules, we would have a contradiction. Therefore, there exists $p \in N \setminus (N_1 \cup N_2 \cup \dots \cup N_k)$. Also, we have $N_{k+1} \cap \dots \cap N_r \nsubseteq N_1 \cup \dots \cup N_k$. Otherwise, since N_j is a prime subsemimodule, by prime avoidance theorem, we have $N_{k+1} \cap \dots \cap N_r \subseteq N_j$ for some $1 \leqslant j \leqslant k$. This implies $(N_{k+1} \cap \dots \cap N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$, that is, $(N_{k+1} : M) \cap \dots \cap (N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$. Therefore, $(N_i : M) \subseteq (N_j : M)$ where $k+1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$, which contradicts to the hypothesis that $(N_i : M) \nsubseteq (N_j : M)$ for $i \neq j$. Thus, there exists $b \in (N_{k+1} : M) \cap \dots \cap (N_r : M) \setminus (N_1 : M) \cup \dots \cup (N_k : M)$. Let $n = bp \in N$. Also, $n \in \bigcap_{j=k+1}^{r} N_j$ and $n = bp \notin N_1 \cup \dots \cup N_k$ (if $n = bp \in N_1 \cup \dots \cup N_k$, then we have $n \in N_i$ for some $i \in \{1, 2, \dots k\}$, since N_i is prime, either $b \in (N_i : M)$ or $p \in N_i$ for $1 \leqslant i \leqslant k$), a contradiction. Thus, $n \in (N_{k+1} \cap \dots \cap N_r) \setminus (N_1 \cup \dots \cup N_k)$. Consequently, $m + n \notin \bigcup_{i=1}^{r} N_i$. □

Next, we prove that if N is a finitely generated subsemimodule of an S-semimodule M satisfying the assumption of prime avoidance theorem for subsemimodules, then there is a linear combination of the generators of N also avoids $\bigcup_{i=1}^{n} N_i$.

Theorem 2.10. *Let M be an S-semimodule and $N = \langle m_1, m_2, ..., m_r \rangle$ be a finitely generated subsemimodule of M. Let $N_1, N_2, \dots, N_n$ be subtractive prime subsemimodules of M such that $N \nsubseteq N_i$ for each i, $1 \leqslant i \leqslant n$ and $(N_i : M) \nsubseteq (N_n : M)$ for each $i \neq j$. Then there exist $b_2, \dots, b_r \in S$ such that $x = m_1 + b_2 m_2 + ... + b_r m_r \notin \bigcup_{i=1}^{n} N_i$.*

Proof. We prove assertion by induction on n. Without loss of generality, we suppose that $N_i \nsubseteq N_j$ for all $i \neq j$. If $n = 1$, then clearly $x = m_1 + b_2 m_2 + ... + b_r m_r \notin N_1$. So, we have done. Assume that the result is true for $(n-1)$ subtractive prime subsemimodules of M. Then there exist $c_2, c_3, \dots, c_r \in S$ such that $y = m_1 + c_2 m_2 + ... + c_r m_r \notin \bigcup_{i=1}^{n-1} N_i$. If $y \notin N_n$, then there is nothing to prove. So assume that $y \in N_n$. If $m_2, ..., m_r \in N_n$, then from the expression for y, we have $m_1 \in N_n$ (as N_n is a subtractive), which is a contradiction to the fact that $N \nsubseteq N_n$. So for some i, $m_i \notin N_n$. Without loss of generality, suppose $i = 2$. By given hypothesis $(N_i : M) \nsubseteq (N_n : M)$ for $i \neq n$. Therefore, there exists $r_i \in (N_i : M)$ such that $r_i \notin (N_n : M)$ where $i \neq n$. Let $r = r_1 r_2 r_3 \dots r_{n_1}$.

Then $c = m_1 + (c_2 + r)m_2 + \ldots + c_r m_r \notin \bigcup_{i=1}^{n} N_i$, which is a contradiction to our assumption. □

3. The primary avoidance theorem

In this section, we study some properties of primary subsemimodules and prove primary avoidance theorem for subsemimodules.

Definition 3.1. A proper subsemimodule N of an S-semimodule M is called *primary* if whenever $am \in N$ for some $a \in S$ and $m \in M$, then $m \in N$ or $a \in \sqrt{(N : M)}$, where $\sqrt{(N : M)} = \{a \in S : a^t M \subseteq N,$ for some $t \in Z^+\}$.

Theorem 3.2. *If N is a primary subsemimodule of M and $m \in M \setminus N$, then $\sqrt{(N : m)} = \{r \in S : r^n m \in N,$ for some $n \in \mathbb{Z}^+\}$ is a prime ideal of S.*

Proof. Let $rs \in \sqrt{(N : m)}$ for some $r, s \in S$. Then $(rs)^n \in (N : m)$ for some positive integer n. Therefore, $r^n(s^n m) \in N$. Since N is primary, we have either $r^n \in (N : M)$ or $s^n m \in N$. Thus, $r \in \sqrt{(N : M)}$ or $s \in \sqrt{(N : m)}$. Since $\sqrt{(N : M)} \subseteq \sqrt{(N : m)}$, we get $r \in \sqrt{(N : m)}$ or $s \in \sqrt{(N : m)}$. Hence $\sqrt{(N : m)}$ is a prime ideal of S. □

Theorem 3.3. *Let N be a primary subsemimodule of an S-semimodule M. Then $(N : M)$ is a primary ideal of S, and hence $\sqrt{(N : M)}$ is a prime ideal of S.*

Proof. The proof is easy and hence omitted. □

Definition 3.4. Let N be a primary subsemimodule of an S-semimodule M. Then N is called a *P-primary subsemimodule* of M, when $P = \sqrt{(N : M)}$ is a prime ideal of S.

Proposition 3.5. *Let M be an S-semimodule and N be a strong subsemimodule of M and suppose $a \in S$. If P is a prime ideal of S, $a \notin P$ such that $Q = (N : a)$ is a P-primary in M, then $N = Q \cap (N + aM)$. Furthermore, N is a P-primary in $N + aM$, where $(N : a) = \{m \in M : am \in N\}$.*

Proof. Clearly, $N \subseteq Q \cap (N + aM)$. Let $x \in (N + aM) \cap Q$. Then $x = n + am$ where $n \in N$ and $m \in M$. Since N is strong, there exists $n_1 \in N$ such that $n + n_1 = 0$. Now, $x = n + am$ implies $x + n_1 = (n + n_1) + am = 0 + am$. Thus, we have $x + n_1 = am \in Q$, as x and n_1 both are in Q. Since Q is a P-primary and $a \notin P$, we have $m \in Q$, which implies $am \in N$. Therefore, $x = n + am \in N$. Hence, $(N + aM) \cap Q \subseteq N$.

Next, we show that N is a P-primary in $(N + aM)$. Let $rx \in N$ for some $r \in S$ and $x \in (N + aM) \setminus N$. Then $x = n + am$ for some $n \in N$ and $m \in M$. Since N is a strong subsemimodule of M, therefore there exist $n_1 \in N$ such that $n + n_1 = 0$. Now, adding n_1 on both sides, we have $x + n_1 = n + n_1 + am$. This implies, $rx + rn_1 = ram$ where $r \in S$. Since $ram \in N$ gives $rm \in (N : a) = Q$ and Q is P-primary. If $m \in Q$, then $x = n + am \in N$, which is a contradiction. Hence, $m \notin Q$. Therefore, $r \in P$. Therefore, N is a P- primary in $(N + aM)$. $\square$

The following theorem can be proved easily.

Theorem 3.6. *Let M and M' be S-semimodules, $f : M \longrightarrow M'$ be an epimorphism and N is a proper subsemimodue of M'. Then N is a primary subsemimodule of M' if and only if $f^{-1}(N)$ is a primary subsemimodule of M.* $\square$

Theorem 3.7. *Let M and M' be S-semimodules, $f : M \longrightarrow M'$ be an epimorphism such that $f(0) = 0$ and N be a subtractive strong subsemimodule of M. If N is a primary subsemimodule of M with $kerf \subseteq N$, then $f(N)$ is a primary subsemimodule of M'*

Proof. Let N be a primary subsemimodule of M and $ax \in f(N)$ for some $a \in S$ and $x \in M'$. Since $ax \in f(N)$, there exists an element $x' \in N$ such that $ax = f(x')$. Since f is an epimorphism and $x \in M'$, then there exists $y \in M$ such that $f(y) = x$. As $x' \in N$ and N is a strong subsemimodule of M, therefore there exists $x'' \in N$ such that $x' + x'' = 0$, which gives $f(x' + x'') = 0$. Therefore, $ax + f(x'') = 0$ or $f(ay) + f(x'') = 0$ implies $ay + x'' \in kerf \subseteq N$. Thus, we have $ay \in N$, since N is a subtractive subsemimodule of M. Since N is a primary, we conclude that $a \in \sqrt{(N : M)}$ or $y \in N$. Thus, $a \in f(\sqrt{(N : M)}) \subseteq \sqrt{f(N : M)}$ or $f(y) \in f(N)$ and hence $a \in \sqrt{(f(N) : M')}$ or $x \in f(N)$. Thus, $f(N)$ is a primary subsemimodule of M'. $\square$

Theorem 3.8. *Let $N_1, N_2, \ldots, N_n$ be subsemimodule of an S-semimodule M and let N be a primary subsemimodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq N$, then there exists an $1 \leqslant i \leqslant n$ such that $N_i \subseteq N$ or $(N_i : M) \subseteq \sqrt{(N : m)}$ where $m \in M \setminus N$.*

Proof. Suppose $N_i \nsubseteq N$ and $(N_i : M) \nsubseteq \sqrt{(N : m)}$ where $m \in M \setminus N$ and for all $1 \leqslant i \leqslant n$. For, $i = k$, we have $N_k \nsubseteq N$, then there exists an $m_k \in M$ such that $m_k \in N_k$ but $m_k \notin N$. Also, there exist $a_i \in (N_i : M)$ such that $a_i \notin \sqrt{(N : m_k)}$ for all $i \neq k$. This gives $a_i m_k \in N_i$ and for every positive integer p_i, $a_i^{p_i} m_k \notin N$. Therefore, $a_i^{p_i} m_k \in N_i \cap N_k$ for all $i \neq k$. So $(a_1^{p_1} a_2^{p_2} \ldots a_{k-1}^{p_{k-1}} a_{k+1}^{p_{k+1}} \ldots a_n^{p_n}) m_k \in N_1 \cap \ldots N_n \subseteq N$. Let $l = max\{p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n\}$. Therefore, $(a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n)^l m_k \in N$. This implies, $(a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n)^l \in (N : m_k)$ and hence $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n \in \sqrt{(N : m_k)}$. By Theorem 3.2, $\sqrt{(N : m_k)}$ is a prime ideal. Therefore, we have $a_i \in \sqrt{(N : m_k)}$ for $i \neq k$, a contradiction. Hence there exists an i such that $N_i \subseteq N$ or $(N_i : M) \subseteq \sqrt{(N : m)}$ where $m \in M \setminus N$. $\square$

Theorem 3.9. *Let N be a P-primary subsemimodule of M. Then $(N : r)$ is a P-primary subsemimodule of M containing N for all $r \in \sqrt{(N : M)} \setminus (N : M)$.*

Proof. Let $r \in \sqrt{(N:M)} \setminus (N:M)$. Clearly, $N \subseteq (N:r)$. Let $s \in S$ and $m \in M$ be such that $sm \in (N:r)$. Therefore, $srm \in N$. Since N is primary, we have either $s \in \sqrt{(N:M)}$ or $rm \in N$, that is $s^n M \subseteq N$ or $m \in (N:r)$ for some positive integer n. Hence $s^n \in ((N:r):M)$ or $m \in (N:r)$ for some positive integer n. Thus, $(N:r)$ is a primary ideal of M. Next, we show that $\sqrt{(N:M)} = \sqrt{(N:r):M}$. Since, $N \subseteq (N:r)$, we have $(N:M) \subseteq ((N:r):M)$ and therefore, $\sqrt{(N:M)} \subseteq \sqrt{((N:r):M)}$. Let $s \in \sqrt{((N:r):M)}$. Therefore, $s^n \in ((N:r):M)$, for some positive integer n. This gives, $rs^n \subseteq (N:M)$. Since N is a primary subsemimodule of M, $(N:M)$ is a primary ideal of S. Therefore, $rs^n \subseteq (N:M)$ implies $s \in \sqrt{(N:M)}$, since $r \notin (N:M)$. Thus, $\sqrt{(N:r):M} \subseteq \sqrt{(N:M)}$. Hence, $\sqrt{(N:M)} = \sqrt{(N:r):M}$. □

Theorem 3.10. *Let N be a subsemimodule of an S-semimodule M such that $N \subseteq N_1 \cup N_2$ for some subtractive subsemimodules N_1, N_2 of M. Then either $N \subseteq N_1$ or $N \subseteq N_2$.*

Proof. The proof is straightforward. □

Now, by using Theorem 2.6, we prove the following proposition.

Proposition 3.11. *Let $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ be an efficient union of subtractive subsemimodules of an S-semimodule M, where $n > 1$. If $\sqrt{(N_j:M)} \nsubseteq \sqrt{(N_k:M)}$ for every $j \neq k$, then no N_k for $k \in \{1, 2, \ldots, n\}$ is a primary subsemimodule of M.*

Proof. Suppose that N_k is a primary subsemimodule of M for some $1 \leqslant k \leqslant n$. Since $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$ is an efficient covering, $N = (N \cap N_1) \cup (N \cap N_2) \cup \ldots \cup (N \cap N_n)$ is an efficient union, otherwise for some $i \neq j$, $N \cap N_i \subseteq N \cap N_j$ and this imply $N = (N \cap N_1) \cup \ldots \cup (N \cap N_{i-1}) \cup (N \cap N_{i+1}) \cup \ldots (N \cap N_n)$ and thus we get $N \subseteq N_1 \cup \ldots \cup N_{i-1} \cup N_{i+1} \cup \ldots \cup N_n$, a contradiction. Hence for every $k \in \{1, 2, ..., n\}$ there exists an element $\ell_k \in N \setminus N_k$. Also, by Theorem 2.6, we have $\bigcap_{j \neq k} (N \cap N_j) \subseteq N \cap N_k$. Since N_k is a primary subsemimodule of M, by Theorem 3.2, we have $\sqrt{(N_k:M)}$ is a prime ideal of S. By hypothesis, if $j \neq k$, $\sqrt{(N_j:M)} \nsubseteq \sqrt{(N_k:M)}$ so there exists an $s_j \in \sqrt{(N_j:M)} \setminus \sqrt{(N_k:M)}$. Now, $s = \prod_{j \neq k} s_j \in \sqrt{(N_j:M)}$ but $s = \prod_{j \neq k} s_j \notin \sqrt{(N_k:M)}$. Since $s = \prod_{j \neq k} s_j \in \sqrt{(N_1:M)}\sqrt{(N_2:M)}...\sqrt{(N_{k-1}:M)}\sqrt{(N_{k+1}:M)}...\sqrt{(N_n:M)}$ but $s = \prod_{j \neq k} s_j \notin \sqrt{(N_k:M)}$, where $s_j \in \sqrt{(N_j:M)}$, where $1 \leqslant j \leqslant n$. Therefore, for some positive integers $m_1, m_2, ...m_n$, we have $s_1^{m_1} \in (N_1:M), s_2^{m_2} \in (N_2:M), ..., s_n^{m_n} \in (N_n:M)$. Let $l = max\{m_1, m_2, ..., m_n\}$. Then for $j \neq k$, $s^l \in (N_j:M)$ but $s^l \notin (N_k:M)$. Therefore, $s^l l_k \in N \cap N_j$ for every $j \neq k$ but $s^l l_k \notin (N \cap N_k)$ because if $s^l l_k \in (N \cap N_k)$, then $sl_k \in N_k$. This gives, $l_k \in N_k$ or $s \in \sqrt{(N_k:M)}$, since N_k is primary. Therefore, $s^l l_k \notin (N \cap N_k)$, which is a contradiction to the fact that $\bigcap_{j\neq k}(N\cap N_j)\subseteq N\cap N_k$. Therefore, no N_k is primary subsemimodule of M. □

Now, we come to our main theorem of this paper.

Theorem 3.12. (The Primary Avoidance Theorem)
Let $N_1, N_2, ..., N_n$ be subtractive subsemimodules of an S-semimodule M and let N be a subsemimodule of M such that $N \subseteq N_1 \cup N_2 \cup \ldots \cup N_n$. Suppose that at most two of N_k's are not primary subsemimodule of M and $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.

Proof. Assume that the covering is efficient. Then $n \neq 2$. Also by Proposition 3.12, $n < 2$ (as $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$). Therefore, $n = 1$. Hence $N \subseteq N_k$ for some k. □

Theorem 3.13. (Extended Version of Primary Avoidance Theorem)
Let M be an S-semimodules and $N_1, N_2, ..., N_r$ subtractive primary subsemimodues of M such that $\sqrt{(N_i : M)} \nsubseteq \sqrt{(N_j : M)}$ for $i \neq j$, $r \geqslant 1$. Let $m \in M$ be such that $mS + N \nsubseteq \bigcup_{i=1}^{r} N_i$. Then there exists $n \in N$ such that $m + n \notin \bigcup_{i=1}^{r} N_i$.

Proof. Suppose that m lies in each of $N_1, N_2, \ldots, N_k$ but in none of $N_{k+1}, N_{k+2}, \ldots, N_r$. If $k = 0$, we have $m = m+0 \notin \bigcup_{i=1}^{r} N_i$ and so there is nothing to prove. Assume that it is true for $k \geqslant 1$. Now, $N \nsubseteq \bigcup_{i=1}^{k} N_i$, for otherwise by primary avoidance theorem for semimodules, we would have a contradiction. Therefore, there exists $p \in N \setminus (N_1 \cup N_2 \cup \ldots \cup N_k)$. Thus, we have $N_{k+1} \cap \ldots \cap N_r \nsubseteq N_1 \cup \ldots \cup N_k$. Otherwise, since $N_j's$ are primary subsemimodules, by primary avoidance theorem, we have $N_{k+1} \cap \ldots \cap N_r \subseteq N_j$ for some $1 \leqslant j \leqslant k$. This implies $(N_{k+1} \cap \ldots \cap N_r : M) \subseteq (N_j : M)$ for some $1 \leqslant j \leqslant k$, gives $\sqrt{(N_{k+1} : M) \cap \ldots \cap (N_r : M)} \subseteq \sqrt{(N_j : M)}$ for some $1 \leqslant j \leqslant k$. This gives, $\sqrt{(N_{k+1} : M)} \cap \ldots \cap \sqrt{(N_r : M)} \subseteq \sqrt{(N_j : M)}$ for some $1 \leqslant j \leqslant k$. Therefore, $\sqrt{(N_i : M)} \subseteq \sqrt{(N_j : M)}$, (since $\sqrt{(N_i : M)}$'s are subtractive prime ideals for all i) where $k + 1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$, which contradicts to the hypothesis that $\sqrt{(N_i : M)} \nsubseteq \sqrt{(N_j : M)}$ for $i \neq j$. Thus, there exists $b \in (N_{k+1} : M) \cap \ldots \cap (N_r : M) \setminus (N_1 : M) \cup \ldots \cup (N_k : M)$. Let $n = bp$, then $n \in N$. Also, $n \in \bigcap_{j=k+1}^{r} N_j$ and $n = bp \notin N_1 \cup \ldots \cup N_k$ (since if $n = bp \in N_1 \cup \ldots \cup N_k$, then $n = bp \in N_i$ for some $1 \leqslant i \leqslant k$ and since N_i is primary, either $b \in \sqrt{(N_i : M)}$ or $p \in N_i$ for $1 \leqslant i \leqslant k$). Thus, $n \in (N_{k+1} \cap \ldots \cap N_r) \setminus (N_1 \cup \ldots \cup N_k)$. Also, $m \in N_1, N_2, \ldots N_k$, it follows that $m + n \notin \bigcup_{i=1}^{r} N_i$. □

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