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On some generalized ideals in ternary semigroups

Mohammad Yahya Abbasi and *Sabahat Ali Khan*

Abstract. We characterize the relationship between minimal m-right, minimal (p, q)-lateral, minimal n-left ideal and m-right simple, (p, q)-lateral simple, n-left simple ternary semigroups. Further, some existing results of regular ternary semigroups are studied.

1. Preliminaries

The idea of invesigation of n-ary algebras i.e., the sets with one n-ary operation was given by Kasner [5]. Investigation of ideals in ternary semigroup was initiated by Sioson [8]. He also defined regular ternary semigroups.

A non-empty set S with a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1, x_2, x_3]$, is called a *ternary semigroup* if it satisfies the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in S$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [[x_1x_2[x_3x_4x_5]].$$

For any positive integers m and n with $m \leqslant n$ and any elements $x_1, x_2, \ldots, x_{2n+1}$ of a ternary semigroup, we can write

$$[x_1x_2 \ldots x_{2n+1}] = [x_1x_2 \ldots [[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}] \ldots x_{2n+1}].$$

Example 1.1. [1] The set

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a ternary semigroup under matrix multiplication.

Definition 1.2. A non-empty subset I of a ternary semigroup S is called:
- a *left ideal* of S if $SSI \subseteq I$,
- a *lateral ideal* of S if $SIS \subseteq I$,
- a *right ideal* of S if $ISS \subseteq I$,
- a *two-sided ideal* if it is left and right ideal of S,
- an *ideal* of S if I is a left, right and lateral ideal of S.

An ideal I of a ternary semigroup S is called *proper* if $I \neq S$ and *idempotent* if $III = I$.

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Proposition 1.3. *Let S be a ternary semigroup and $a \in S$. Then the principal*

(1) *left ideal generated by $'a'$ is given by $L(a) = SSa \cup \{a\}$*

(2) *right ideal generated by $'a'$ is given by $R(a) = aSS \cup \{a\}$*

(3) *lateral ideal generated $'a'$ is given by $M(a) = SaS \cup SSaSS \cup \{a\}$*

(4) *ideal generated by $'a'$ is given by $I(a) = aSS \cup SaS \cup SSaSS \cup SSa \cup \{a\}$.*

Definition 1.4. An element a in a ternary semigroup S is called *regular* if there exists an element $x \in S$ such that $axa = a$.

2. Main results

Definition 2.1. Let S be a ternary semigroup. Then a ternary subsemigroup
• R is called an *m-right ideal* of S if $RS^{2m} \subseteq R$.
• M is called a *(p, q)-lateral ideal* of S if $(S^pMS^q \cup S^{p+1}MS^{q+1}) \subseteq M$.
• L is called an *n-left ideal* of S if $S^{2n}L \subseteq L$.
where m, n, p, q are positive integers and $p + q$ is an even positive integer.

S is called an m-right (resp. (p, q)-lateral, n-left) *simple* if S is a unique m-right (resp. (p, q)-lateral, n-left) ideal of S.

Example 2.2. Let S be a set of all strictly lower triangular matrices of order 6 over $\mathbb{Z}_0^-$, the set of all non-positive integers, i.e.,

$$S = \{(a_{ij})_{6\times 6} \mid a_{ij} = 0 \; if \; i \leqslant j \; and \; a_{ij} \in \mathbb{Z}_0^- \; if \; i > j\}.$$

Then S is a ternary semigroup under the usual multiplication of matrices over $\mathbb{Z}_0^-$ while S is not a semigroup under the same operation. It is easy to see that

$$M_{gen} = \{(a_{ij}) \in S : a_{43} = a \in \mathbb{Z}_0^- \; and \; a_{ij} = 0 \; otherwise\}.$$

is a ternary subsemigroup of S and M_{gen} is a $(3, 1)$-lateral ideal of S. Now

$$SM_{gen}S = \{(a_{ij}) \,|\, a_{51}, a_{52}, a_{61}, a_{62} \in \mathbb{Z}_0^- \; and \; a_{ij} = 0 \; otherwise\} \nsubseteq M_{gen}.$$

Therefore M_{gen} is not a lateral ideal of S.

Example 2.3. Let S be a set of all strictly upper triangular matrices of order 7 over $\mathbb{Z}_0^-$, i.e.,

$$S = \{(a_{ij})_{7\times 7} \,|\, a_{ij} = 0 \; if \; i \geqslant j \; and \; a_{ij} \in \mathbb{Z}_0^- \; if \; i < j\}.$$

Then S is a ternary semigroup under the usual multiplication of matrices over $\mathbb{Z}_0^-$ while S is not a semigroup under the same operation. Then it is easy to see that

$$\mathcal{M} = \{(a_{ij}) \in S \,|\, a_{45} \in \mathbb{Z}_0^- \; and \; a_{ij} = 0 \; otherwise\}$$

is a ternary subsemigroup of S and $\mathcal{M}$ is a $(3,3)$-lateral ideal of S. Now

$$S\mathcal{M}S = \{(a_{ij}) \in S \mid a_{16}, a_{17}, a_{26}, a_{27}, a_{36}, a_{37} \in \mathbb{Z}_0^- \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M,$$

$$S^2\mathcal{M}S^2 \cup S^3\mathcal{M}S^3 = \{(a_{ij}) \in S \mid a_{17}, a_{27} \in \mathbb{Z}_0^- \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M,$$

$$S^3\mathcal{M}S \cup S^4\mathcal{M}S^2 = \{(a_{ij}) \in S \mid a_{16}, a_{17} \in \mathbb{Z}_0^- \textit{ and } a_{ij} = 0 \textit{ otherwise}\} \nsubseteq M.$$

Therefore $\mathcal{M}$ is not an $(1,1)$-lateral, $(2,2)$-lateral and $(3,1)$-lateral ideal of S.

Remark 2.4. We know that for a right ideal R, a lateral ideal M and a left ideal L of a ternary semigroup S, $RML \subseteq R \cap M \cap L$. But this result is not true for an m-right ideal R, an (p,q)-lateral ideal M and an n-left ideal L of a ternary semigroup S.

Lemma 2.5. *Let S be a ternary semigroup.*

(1) *Let $\{R_i : i \in I\}$ be a family of m-right ideals of S. Then $\bigcap_{i\in I} R_i$ is also an m-right ideal of S if $\bigcap_{i\in I} R_i \neq \emptyset$.*

(2) *Let $\{M_i : i \in I\}$ be a family of (p,q)-lateral ideals of S. Then $\bigcap_{i\in I} M_i$ is also a (p,q)-lateral ideal of S if $\bigcap_{i\in I} M_i \neq \emptyset$.*

(3) *Let $\{L_i : i \in I\}$ be a family of n-left ideals of S. Then $\bigcap_{i\in I} L_i$ is also an n-left ideal of S if $\bigcap_{i\in I} L_i \neq \emptyset$.*

Theorem 2.6. *Let S be a ternary semigroup. Then*

(1) *Every m-right ideal is an $(m+m_1)$-right ideal of S, where m_1 is a non-negative integer.*

(2) *Every (p,q)-lateral ideal is a $(p+p_1, q+q_1)$-lateral ideal of S, where p_1 and q_1 are non-negative integers and p_1+q_1 is even.*

(3) *Every n-left ideal is an $(n+n_1)$-left ideal of S, where n_1 is a non-negative integer.*

Proof. (2). We have

$$\begin{aligned} S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1+1}MS^{q+q_1+1} &\subset S^{p+p_1-2}MS^{q+q_1-2} \cup S^{p+p_1-1}MS^{q+q_1-1} \\ &\subset \ldots \subset S^{p+1}MS^{q+1} \cup S^pMS^q \subset M, \end{aligned}$$

if p_1, q_1 are odd, and

$$S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1+1}MS^{q+q_1+1} \subset \ldots \subset S^pMS^q \cup S^{p+1}MS^{q+1} \subset M,$$

if p_1, q_1 are even.

Hence in all the two cases, M is a $(p+p_1, q+q_1)$-lateral ideal of S.

Proofs of (1) and (3) are similar. □

Corollary 2.7. *Let S be a ternary semigroup and A be its ternary subsemigroup. If A is a (p,q)-lateral ideal of S. Then, for any positive integer n:*

(1) *A will be an (np, nq)-lateral ideal of S.*

(2) *A will be a (p^n, q^n)-lateral ideal of S.*

Lemma 2.8. *For any non-empty subset A of a ternary semigroup S*

(1) AS^{2m} *is an m-right ideal of S,*

(2) $S^pAS^q \cup S^{p+1}AS^{q+1}$ *is a (p,q)-lateral ideal of S,*

(3) $S^{2n}A$ *is an n-left ideal of S.*

Lemma 2.9. *For any non-empty subset A of a ternary semigroup S*

(1) $(A \cup A^3 \cup A^5 \cup \ldots \cup A^{2m-1}) \cup AS^{2m}$ *is the smallest m-right ideal of S containing A,*

(2) $(A \cup A^3 \cup A^5 \cup \ldots \cup A^{p+q-1}) \cup (S^{p+1}AS^{q+1} \cup S^pAS^q)$ *is the smallest (p,q)-lateral ideal of S containing A,*

(3) $(A \cup A^3 \cup A^5 \cup \ldots \cup A^{2n-1}) \cup S^{2n}A$ *is the smallest n-left ideal of S containing A,*

where m, n, p, q are positive integers and $p+q$ is an even positive integer.

Proof. (1). Let $R = (\bigcup_{i=1}^{m} A^{2i-1}) \cup AS^{2m}$ and $x, y, z \in R$. Clearly $A \subseteq R$.

If $x, y, z \in \bigcup_{i=1}^{m} A^{2i-1}$, then $xyz \in A^r$. So, $xyz \in \bigcup_{i=1}^{m} A^{2i-1}$ for $r \leqslant 2m-1$, and we have $xyz \in AS^{2m}$ for $r > 2m-1$.

If $x, y, z \in AS^{2m}$, then obviously $xyz \in AS^{2m}$. Therefore R is a ternary subsemigroup of S.

To show R is an m-right ideal of S. We have

$$RS^{2m} = ((\bigcup_{i=1}^{m} A^{2i-1}) \cup AS^{2m})S^{2m} = (\bigcup_{i=1}^{m} A^{2i-1})S^{2m} \cup (AS^{2m})S^{2m} \subseteq AS^{2m} \subseteq R.$$

Finally it remains to prove that R is the smallest m-right ideal of S containing A. For this suppose that R_1 is an m-right ideal of S containing A. Then

$$R = (\bigcup_{i=1}^{m} A^{2i-1}) \cup AS^{2m} \subseteq (\bigcup_{i=1}^{m} R_1^{2i-1}) \cup R_1S^{2m} \subseteq R_1 \cup R_1 = R_1.$$

Hence R is the smallest m-right ideal of S containing A.

(2). Let $M = (\bigcup_{i=1}^{m} A^{2i-1} \cup (S^pAS^q) \cup S^{p+1}AS^{q+1})$, where $p+q = 2m$, and $x, y, z \in M$. Clearly $A \subseteq M$. Now we have following two cases:

Case 1: $x, y, z \in \bigcup_{i=1}^{m} A^{2i-1}$, then $xyz \in A^n$. If $n \leqslant p+q-1$, then we have $xyz \in (\bigcup_{i=1}^{m} A^{2i-1})$. If $n > p+q-1$, then $xyz \in (S^pAS^q \cup S^{p+1}AS^{q+1})$.

Case 2: $x, y, z \in S^pAS^q \cup S^{p+1}AS^{q+1}$. Then, as it is easy to show, $xyz \in S^pAS^q \cup S^{p+1}AS^{q+1}$.

Therefore M is a ternary subsemigroup of S. It is easy verify that M is a (p,q)-lateral ideal of S.

Finally it remains to prove that M is the smallest (p,q)-lateral ideal of S containing A. For this suppose that M_1 is a (p,q)-lateral ideal of S containing A. Then

$$M = (\bigcup_{i=1}^{p+q-1} A^i \cup (S^pAS^q \cup S^{p+1}AS^{q+1})) \subseteq (\bigcup_{i=1}^{p+q-1} M_1^i \cup (S^pM_1S^q \cup S^{p+1}M_1S^{q+1})) \subseteq M_1.$$

Hence M is the smallest (p,q)-lateral ideal of S containing A.

The proof of (3) is analogous. □

Furthermore, for any $a \in S$ we have:

$R(a) = aS^{2m} \cup \{a, a^3, a^5, \dots, a^{2m-1}\}$ is an m-right ideal generated by a;

$M(a) = (S^{p+1}aS^{q+1} \cup S^paS^q) \cup \{a, a^3, a^5, \dots, a^{p+q-1}\}$ is a (p,q)-lateral ideal generated by a;

$L(a) = S^{2n}a \cup \{a, a^3, a^5, \dots, a^{2n-1}\}$ is an n-left ideal generated by a.

Theorem 2.10. *Let A and B be ternary subsemigroups of S such that $A \subseteq B$ and $B^3 = B$. If A is a (p,q)-lateral ideal of S, then it is a lateral ideal of B.*

Proof. Suppose A and B are two ternary subsemigroups of S such that $A \subseteq B$ and $B^3 = B$. If A is an (p,q)-lateral ideal of S, then $S^{p+1}AS^{q+1} \cup S^pAS^q \subseteq A$. Now we have $BAB \cup B^2AB^2 = BAB^3 \cup B^3BABB^3$ or $B^3AB \cup B^3BABB^3$. Proceed in this way, we get $BAB \cup B^2AB^2 = B^pAB^q \cup B^{p+1}AB^{q+1}$. Now

$$BAB \cup B^2AB^2 = B^pAB^q \cup B^{p+1}AB^{q+1} \subseteq S^pAS^q \cup S^{p+1}AS^{q+1} \subseteq A.$$

This shows that A is a lateral ideal of B. □

Corollary 2.11. *If S is a ternary semigroup such that $S^3 = S$, then every its (p,q)-lateral ideal is its lateral ideal.*

Corollary 2.12. *An idempotent (p,q)-lateral ideal of a ternary semigroup S is its lateral ideal.*

Theorem 2.13. *Let S be a ternary semigroup. Then:*

(1) *An m-right ideal R is minimal if and only if $aS^{2m} = R$ for all $a \in R$.*

(2) *A (p,q)-lateral ideal M is minimal if and only if $(S^paS^q \cup S^{p+1}aS^{q+1}) = M$ for all $a \in M$.*

(3) *An n-left ideal L is minimal if and only if $S^{2n}a = L$ for all $a \in L$.*

Proof. (2) Suppose that a (p,q)-lateral ideal M is minimal. Let $a \in M$. Then $S^paS^q \cup S^{p+1}aS^{q+1} \subseteq S^pMS^q \cup S^{p+1}MS^{q+1} \subseteq M$. By Lemma 2.8(2), we have $S^paS^q \cup S^{p+1}aS^{q+1}$ is a (p,q)-lateral ideal of S. As M is minimal (p,q)-lateral ideal of S therefore $S^paS^q \cup S^{p+1}aS^{q+1} = M$.

Conversely, suppose that $S^paS^q \cup S^{p+1}aS^{q+1} = M$ for all $a \in M$. Let $M^{'}$ be a (p,q)-lateral ideal of S contained in M. Let $m \in M^{'}$. Then $m \in M$. By assumption, we have $S^pmS^q \cup S^{p+1}mS^{q+1} = M$ for all $m \in M$. Now $M = S^pmS^q \cup S^{p+1}mS^{q+1} \subseteq S^pM^{'}S^q \cup S^{p+1}M^{'}S^{q+1} \subseteq M^{'}$. This implies $M \subseteq M^{'}$. Thus, $M = M^{'}$. Hence, M is a minimal (p,q)-lateral ideal of S.

Proofs of (1) and (3) are similar. □

Theorem 2.14. *Let S be a ternary semigroup. Then:*

(1) *S is an m-right simple if and only if $aS^{2m} = S$ for all $a \in S$.*

(2) *S is a (p, q)-lateral simple if and only if $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$.*

(3) *S is an n-left simple if and only if $S^{2n}a = S$ for all $a \in S$.*

Proof. (2) Assume that S is a (p, q)-lateral simple, we have that S is a minimal (p, q)-lateral ideal of S. By the Theorem 2.13(2), $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$.

Conversely, suppose that $S^p aS^q \cup S^{p+1}aS^{q+1} = S$ for all $a \in S$. By the Theorem 2.13(2), S is a minimal (p, q)-lateral ideal of S, and therefore S is a (p, q)-lateral simple.

Proofs of (1) and (3) are analogous. □

Lemma 2.15. *If R is an m-right ideal of S and T is a ternary subsemigroup of S and if T is an m-right simple such that $T \cap R \neq \emptyset$, then $T \subseteq R$.*

Proof. Assume that T is an m-right simple such that $T \cap R \neq \emptyset$. Let $a \in T \cap R$. By Lemma 2.8, we have $aT^{2m} \cap T$ is an m-right ideal of T. This implies that $aT^{2m} \cap T = T$. Hence $T \subseteq aT^{2m} \subseteq RS^{2m} \subseteq R$, so $T \subseteq R$. □

Lemma 2.16. *If M is a (p, q)-lateral ideal of S and T is a ternary subsemigroup of S and if T is a (p, q)-lateral simple such that $T \cap M \neq \emptyset$, then $T \subseteq M$.*

Proof. Proof is similar to the Lemma 2.15. □

Lemma 2.17. *If L is an n-left ideal of S and T is a ternary subsemigroup of S and if T is an n-left simple such that $T \cap L \neq \emptyset$, then $T \subseteq L$.*

Proof. Proof is similar to the Lemma 2.15. □

Theorem 2.18. *Let S be a ternary semigroup. Then:*

(1) *If an m-right ideal R of S is an m-right simple ternary semigroup, then R is a minimal m-right ideal of S.*

(2) *If a (p, q)-lateral ideal M of a ternary semigroup S is a (p, q)-lateral simple ternary semigroup, then M is a minimal (p, q)-lateral ideal of S.*

(3) *If an n-left ideal L of a ternary semigroup S is an n-left simple ternary semigroup, then L is a minimal n-left ideal of S.*

Proof. (2) Assume that M is a (p, q)-lateral simple. Let A be a (p, q)-lateral ideal of S such that $A \subseteq M$. Then $A \cap M \neq \emptyset$, it follows from Lemma 2.16, that $M \subseteq A$. Hence $A = M$, so M is a minimal (p, q)-lateral ideal of S.

(1) and (3) can be proved analogously. □

Theorem 2.19. *Let S be a regular ternary semigroup. Then:*

(1) *Every m-right ideal is a right ideal.*

(2) *Every (p,q)-lateral ideal is a lateral ideal.*

(3) *Every n-left ideal is a left ideal.*

Proof. (2) Let M be a (p,q)-lateral ideal of S and $a \in SMS \cup SSMSS$. Then there exists $x_1, x_2, x_3, x_4, x_5, x_6 \in S$ and $m_1, m_2 \in M$ such that $a = x_1m_1x_2$ or $a = x_3x_4m_2x_5x_6$. Since S is regular, for any $m_1, m_2 \in M$ there exists $x_7, x_8 \in S$ such that $m_1 = m_1x_7m_1$ or $m_2 = m_2x_8m_2$. Hence $a = x_1m_1x_7m_1x_2$ or $a = x_3x_4m_2x_8m_2x_5x_6$. Therefore $a \in SMSMS \subseteq S^3MS$ or $a \in S^2MSMS^2 \subseteq S^4MS^2$. Thus, by the property of regularity, we see that $a \in S^pMS^q$ or $a \in S^{p+1}MS^{q+1}$ implies $a \in S^pMS^q \cup S^{p+1}MS^{q+1}$. As M is a (p,q)-lateral ideal, it implies $a \in S^pMS^q \cup S^{p+1}MS^{q+1} \subseteq M$. Therefore $SMS \cup S^2MS^2 \subseteq M$ and hence M is a lateral ideal of S.

Proofs of (1) and (3) are similar. □

Theorem 2.20. *If a ternary semigroup S is an m-right and an n-left simple. Then it is regular.*

Proof. Suppose that S is an m-right and an n-left simple. Let $a \in S$. Then by the Theorem 2.14(1) and (3), $aS^{2m} = S$ and $S^{2n}a = S$. Now

$$a \in S = aS^{2m} = aS^{2(m-1)}S^2 = aS^{2(m-1)}SS^{2n}a = aS^{2(m-1)}S^3S^{2(n-1)}a \subseteq aSa.$$

This shows that $a \in aSa$. Hence for any $a \in S$ there exists $x \in S$ such that $a = axa$. Therefore a is regular. Hence S is regular. □

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Department of Mathematics, Jamia Millia Islamia, New Delhi-110 025, India

E-mail: yahya_alig@yahoo.co.in, khansabahat361@gmail.com