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Some remarks on Abel-Grassmann's groups

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Abstract. Abel-Grassmann's groupoids or shortly AG-groupoids have been considered in quite a number of papers, although under the different names (left-almost semigroups, left invertive groupoids). Abel-Grassmann's groups (AG-groups) is an Abel-Grassmann's groupoid with left identity in which every element has inverse. In this paper we describe AG-groups by equations. Also, we describe congruences on AG-groups.

1. Introduction

Abel-Grassmann's groupoids, abbreviated as AG-groupoids, are also called left almost semigroups (LA-semigroups in short). They are closely related with commutative semigroup because if an AG-groupoid contains right identity then it becomes a commutative monoid. Although the structure is non-associative and non-commutative, nevertheless, it posses many interesting properties which we usually found in associative and commutative algebraic structures. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e, moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. An idempotent AG-groupoid with left identity is a semilattice [6].

A groupoid $(S, \cdot)$ is called *AG-groupoid*, if it satisfies the *left invertive law*:

$$ab \cdot c = cb \cdot a. \tag{1}$$

Any AG-groupoid satisfies the *medial law*:

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

An AG-groupoid satisfying the identity

$$a \cdot bc = b \cdot ac \tag{3}$$

is called an *AG^{**}-groupoid*. Notice that each AG-groupoid with left identity is an AG^{**}-groupoid [7]. In any AG^{**}-groupoid G holds the *paramedial law*:

$$ab \cdot cd = db \cdot ca. \tag{4}$$

In this paper by G^e we denote the AG-groupoid G with a left identity e.

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2. AG-groups

The element $x \in G^e$ ($y \in G^e$) is a *left* (resp. *right*) *inverse* for $a \in G^e$ if $xa = e$ (resp. $ay = e$), and an element which is both a left and a right inverse is called an *inverse element*. In [5] it has been proved that in G^e left identity and left inverse are uniquely determined and that any left inverse is a right inverse and conversely. Hence, left and right inverses is an inverse and it is unique. An inverse of $a \in G^e$ is denoted by a'. Clearly, for any $a, b \in G^e$, $(a')' = a$, $(ab)' = a'b'$.

Definition 2.1. [5] An AG-groupoid G^e is called an *AG-group* if every $a \in G^e$ has an inverse element a'.

Obviously, any AG-group is an AG^{**}-groupoid. Hence any AG^{**}-group satisfies (1), (2), (3) and (4).

A simple example of an AG-group is an AG-groupoid $der(G,*)$ *derived* from an Abelian group $(G,*)$ i.e., an AG-groupoid with the operation $xy = x^{-1} * y$. In this AG-group we have $x' = x$ for all $x \in G$. But there are AG-groups which are not of this form.

Example 2.2. It is not difficult to see that the groupoid $(G,\cdot)$ defined by the following table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	c	b	a	e
c	b	c	e	a

is an AG-groupoid. It is not a semigroup since, for example, $ba \cdot a \neq b \cdot aa$. The element e is its left identity, $a = a'$, $b' = c$ and $c' = b$. Hence, $(G,\cdot)$ is an AG-group. Obviously this AG-group is not derived from a group. □

Lemma 2.3. *In an AG-group G^e the equation $xa = b$ has an unique solution for every $a, b \in G^e$.*

Proof. Indeed, since for all $a, b \in G^e$ we have

$$b = eb = aa' \cdot b = ba' \cdot a,$$

the element $x = ba' \in G^e$ is a solution of the equation $xa = b$.

Let x_1 and x_2 be solutions of the equation $xa = b$, then

$$\begin{aligned} x_1 &= ex_1 = a'a \cdot x_1 = x_1 a \cdot a' = ba' \\ &= x_2 a \cdot a' = a'a \cdot x_2 = ex_2 = x_2. \end{aligned}$$

Hence, the equation $xa = b$ has an unique solution. □

Theorem 2.4. *On any AG-group G^e we can define an Abelian group $ret(G^e)$ by putting $x \circ y = xe \cdot y$.*

Proof. If G^e is an AG-group, then by (1) the operation $x \circ y = xe \cdot y$ is commutative and e is its neutral element. Moreover, for all $x, y, z \in G^e$ we have

$$(x \circ y) \circ z = (xe \cdot y)e \cdot z \overset{(1)}{=} ze \cdot (xe \cdot y) \overset{(1)}{=} ze \cdot (ye \cdot x) \overset{(4)}{=} xe \cdot (ye \cdot z) = x \circ (y \circ z).$$

So, $(G^e, \circ)$ is a commutative monoid.

Consider the equation $b = x \circ a = xe \cdot a$. By Lemma 2.3 the equation $za = b$ has a unique solution $z_0 \in G^e$. The equation $xe = z_0$ also has a unique solution. Thus for every $a, b \in G^e$ there exists x such that $x \circ a = b$. Hence $ret(G^e) = (G^e, \circ)$ is an Abelian group. In this group $a^{-1} = a'e$, where a' is an inverse element in an AG-group G^e. □

Remark 2.5. For an AG-groupoid G^e derived from an Abelian group $(G, *)$ we have $ret(G^e) = der(G, *)$. So, $ret(der(G, *)) = (G, *)$ and $der(ret(G^e)) = G^e$. Example 2.2 show that the last equality is not true for AG-groups which are not derived from an Abelian group.

From results obtained in [6] it follows that $der(ret(G^e)) = G^e$ holds only for AG-groups satisfying the identity $x^2 = e$.

Theorem 2.6. *Let H, K be two AG-subgroups of an AG-groupoid G. If e_H, e_K are left identities of H and K respectively, then*

$$H \cap K \neq \emptyset \iff e_H = e_K.$$

Proof. Let $H \cap K \neq \emptyset$ and let $a \in H \cap K$, then $aa'_H = a'_H a = e_H$, $aa'_K = a'_K a = e_K$ for some $a'_H \in H$, $a'_K \in K$. Thus

$$e_H e_K = a'_H a \cdot e_K = e_K a \cdot a'_H = aa'_H = e_H.$$

By symmetry, $e_K e_H = e_K$. Now, by (1) and (2), we obtain

$$e_H e_K = aa'_H \cdot e_K = e_K a'_H \cdot a = e_K a'_H \cdot e_K a = e_K e_H \cdot a'_H a = e_K e_H,$$

and so $e_H = e_K$.

The converse statement is trivial. □

3. Congruences on AG-groups

In this section we shall characterize all congruences on an arbitrary AG-group by its normal AG-subgroups.

Lemma 3.1. *If ρ is a congruence on an AG-group G^e, then for all $a, b \in G^e$ we have*

$$a \rho b \iff a' \rho b'.$$

Proof. Indeed, if $a\rho\, b$, then $(b'a)\rho\,(b'b)$, and so $(b'a)\rho\, e$. Therefore, $(b'a \cdot a')\rho\,(ea')$. On the other hand, $b'a \cdot a' = a'a \cdot b' = eb' = b'$. Hence, $b'\rho\, a'$.

The converse implication is obvious, since $(a')' = a$ for every $a \in G^e$. □

Definition 3.2. A nonempty subset K of an AG-group G^e is called

- *self conjugate* if $x \cdot Kx' \subseteq K$ for all $x \in G^e$,
- *inverse closed* if $x' \in K$ for all $x \in K$,
- an *AG-subgroup* if it is an inverse closed subgroupoid of G^e,
- a *normal AG-subgroup* if it is a self conjugate AG-subgroup of G^e.

Obviously, any AG-subgroup of G^e is a subgroup of the group $ret(G^e)$. The converse is not true in general. For example, it is not difficult to see that for an AG-group G^e defined in Example 2.2, $H = \{e, b\}$ is a subgroup of $ret(G^e)$, but it is not an AG-subgroup of G^e.

Lemma 3.3. *Let ρ be a congruence relation defined on an AG-group G^e. Then*

$$\ker \rho = \{x \in G^e : x\,\rho\, e\}$$

is a normal AG-subgroup of G^e.

Proof. Let ρ be a congruence on G^e. Obviously, $e \in \ker\rho$. Moreover, if $a, b \in \ker\rho$, then $a\rho e$, $b\rho e$ and so $ab\,\rho\, e$. Hence, $ab \in \ker\rho$. Thus $\ker\rho$ is a subgroupoid of G^e. It is inverse closed since for every $x \in \ker\rho$ we have $x\rho e$, which by Lemma 3.1 implies $x'\rho\, e$, whence $x' \in \ker\rho$. Hence, $\ker\rho$ is an AG-subgroup of G^e.

Now let $x \in G^e$. Then for every $y \in x \cdot \ker\rho\, x'$ there exists $a \in \ker\rho$ such that $y = a \cdot ax'$. Thus $(x \cdot ax')\rho(x \cdot ex')$, i.e., $(x \cdot ax')\rho e$, which means that $y = x \cdot ax' \in \ker\rho$. So, $\ker\rho$ is a normal AG-subgroup of G^e. □

Theorem 3.4. *Let K be a normal AG-subgroup of an AG-group G^e. Then the relation ρ_K defined by*

$$a\rho_K b \iff a \in Kb \;\wedge\; b \in Ka$$

is the unique congruence on G^e for which $\ker\rho_K = K$.

Proof. Let K be a normal AG-subgroup of G^e. Clearly, the relation ρ_K is reflexive and symmetric. If $a\rho_K b$, $b\rho_K c$, then obviously $a \in Kb$, $b \in Kc$. From this, applying (1) and (4) we obtain

$$a \in Kb \subseteq K \cdot Kc = KK \cdot Kc = cK \cdot KK = cK \cdot K = KK \cdot c = Kc.$$

Dually, $c \in Ka$, whence $a\rho_K c$ and so ρ is a transitive relation. Therefore, ρ is an equivalence on G^e.

Now let $a\rho_K b$ and $c\rho_K d$. Then $ac \in Kb \cdot Kd = KK \cdot bd = K \cdot bd$ and dually $bd \in K \cdot ac$. Hence $(ab)\rho_K(cd)$. Thus ρ_K is a congruence on G^e.

If $a \in \ker \rho_K$, then $a\rho_K e$. Consequently $a \in Ke$ and $e \in Ka$. From the above $ea' \in Ka \cdot a'$, whence $a' \in a'a \cdot K = eK = K$. Now, since K is inverse closed, we have $a \in K$. Hence $\ker \rho_K \subseteq K$. Conversely, if $a \in K$ then $e, a' \in K$ and so

$$e = a'a \in Ka, \quad a \in K = KK = eK \cdot K = KK \cdot e = Ke,$$

whence $a\rho_K e$ and $a \in \ker \rho_K$. Hence $K \subseteq \ker \rho_K$ and so $K = \ker \rho_K$.

To prove that ρ_K is an unique congruence on G^e with the kernel K consider an arbitrary congruence λ on G^e and assume that its kernel also is K. Then for $a\lambda b$ we have $ab'\lambda\, bb'$ and $aa'\lambda\, ba'$. So, $ab', ba' \in \ker \lambda = K$. Thus $ab' \cdot b \in Kb$ and so $a = bb' \cdot a = ab' \cdot b \in Kb$. Analogously we obtain $b = ba' \cdot a \in Ka$. This proves that $a\rho_K b$. Thus $\lambda \subseteq \rho_K$.

Conversely, if $a\rho_K b$, then $a \in Kb$, $b \in Ka$, and consequently

$$ab' \in Kb \cdot b' = b'b \cdot K = eK = K = \ker \lambda,$$

whence $ab'\lambda e$. This implies $(ab' \cdot b)\lambda eb$, i.e., $(ab' \cdot b)\lambda b$. But $ab' \cdot b = bb' \cdot a = a$, so $a\lambda b$. Hence $\rho_K \subseteq \lambda$. Thus $\rho_K = \lambda$. This means that ρ_K is an unique congruence on G^e with kernel K. □

Corollary 3.5. *For any congruence λ on an AG-group G^e we have $\rho_{\ker \lambda} = \lambda$.*

Proof. Indeed, by Lemma 3.3, $\ker \lambda$ is a normal AG-subgroup of G^e, and in a view of Theorem 3.4 we have $\ker \rho_{\ker \lambda} = \ker \lambda$. This implies $\rho_{\ker \lambda} = \lambda$. □

As a consequence of results proved in [3] we obtain the following proposition which will be used later.

Proposition 3.6. *The lattice of congruences on an AG-group is modular.* □

4. Congruences on AG**-groupoids

An AG^{**}-groupoid G in which for every $x \in G$ there exists uniquely determined element $x^{-1} \in G$ such that

$$x = xx^{-1} \cdot x, \quad x^{-1} = x^{-1}x \cdot x^{-1} \tag{5}$$

and

$$xx^{-1} = x^{-1}x \tag{6}$$

is called *completely inverse*.

Obviously any AG-group is a completely inverse AG^{**}-groupoid. Moreover, in this case $x^{-1} = x'$.

One can prove (cf. [1]) that an AG^{**}-groupoid (satisfying (5)) satisfies (6) if and only if xx^{-1} and $x^{-1}x$ are idempotents. Thus a completely inverse AG^{**}-groupoid containing only one idempotent is an AG-group (cf. [3]).

Let E_G denote the set of idempotents of a completely inverse AG^{**}-groupoid G. Then E_G is a semilattice (cf. [1]) and the relation $\leqslant$ defined on G by

$$a \leqslant b \Longleftrightarrow a \in E_G b$$

is the *natural partial order* on G.

The following result can be deduced from [9].

Lemma 4.1. *In any completely inverse AG^{**}-groupoid G, the relation $\leqslant$ is a compatible partial order on G. Also, $a \leqslant b$ implies $a^{-1} \leqslant b^{-1}$ for all $a, b \in G$.* $\square$

Definition 4.2. For any nonempty subset B of a completely inverse AG^{**}-groupoid G, the set

$$B\omega = \{a \in G : \exists\ (b \in B)\ b \leqslant a\}$$

is called the *closure of B in G*.

If $B = B\omega$, then we shall say that B is *closed in G*. Clearly, $B\omega$ is closed in G.

It is clear that a subgroupoid B of a completely inverse AG^{**}-groupoid G is itself a completely inverse AG^{**}-groupoid if and only if $b \in B$ implies $b^{-1} \in B$ for every $b \in B$. A subgroupoid with this property is called a *completely inverse AG^{**}-subgroupoid* of G.

Definition 4.3. A nonempty subset B of a completely inverse AG^{**}-groupoid G is called:

- *full* if $E_G \subseteq B$,
- *symmetric* if $xy \in B$ implies $yx \in B$ for all $x, y \in G$,
- *normal* if it is full, closed and symmetric.

Denote the set of AG-group congruences on an arbitrary completely inverse AG^{**}-groupoid G by $\mathcal{GC}(G)$, and denote by σ the least such a congruence on G. Then $\mathcal{GC}(G) = [\sigma, G \times G]$ is a complete sublattice of the lattice $\mathcal{C}(G)$ of all congruences on G. Notice that $\mathcal{GC}(G) \cong \mathcal{C}(G/\sigma)$ and so the lattice $\mathcal{GC}(G)$ is modular (by Proposition 3.6). Furthermore, let $\mathcal{N}(G)$ be the set of all normal completely inverse AG^{**}-subgroupoids of G. Obviously, $E_G\omega \subseteq N$ for every normal completely inverse AG^{**}-subgroupoid N of G. If $\emptyset \neq \mathcal{F} \subseteq \mathcal{N}(G)$, then $\bigcap \mathcal{F} \in \mathcal{N}(G)$. Consequently, $\mathcal{N}(G)$ is a complete lattice.

The following theorem describes the AG-groups congruences on a completely inverse AG^{**}-groupoid in the terms of its normal completely inverse AG^{**}-subgroupoids.

Theorem 4.4. *Let N be a normal completely inverse AG^{**}-subgroupoid of a completely inverse AG^{**}-groupoid G. Then the relation*

$$\rho_N = \{(a,b) \in G \times G : ab^{-1} \in N\}$$

is the unique AG-group congruence ρ on G for which $\ker(\rho) = N$.

Proof. Clearly, ρ_N is reflexive. Further, if $ab^{-1} \in N$, then $b^{-1}a \in N$, so $ba^{-1} \in N$, therefore, ρ_N is symmetric. Also, if $ab^{-1}, bc^{-1} \in N$, then $ab^{-1} \cdot c^{-1}b \in N$. Hence $ac^{-1} \cdot b^{-1}b \in N$, so $b^{-1}b \cdot ac^{-1} \in N$, that is, $b^{-1}b \cdot ac^{-1} = n$ for some $n \in N$ and so $n \leqslant ac^{-1}$. Thus $ac^{-1} \in N\omega = N$. Consequently, ρ_N is an equivalence relation on G. Moreover, let $(a,b) \in \rho_N$ and $c \in G$. Then

$$ac \cdot (bc)^{-1} = ac \cdot b^{-1}c^{-1} = ab^{-1} \cdot cc^{-1} \in NE_G \subseteq NN \subseteq N$$

and similarly $ca \cdot (cb)^{-1} \in N$, therefore, ρ_N is a congruence on G. Furthermore, since $ef^{-1} = ef \in E_G \subseteq N$ for all $e, f \in E_G$, then S/ρ_N is an AG-group. Finally, if $a \in N\omega$, then $ea \in N$ for some $e \in E_G$. Hence $ae = ae^{-1} \in N$ and so $(a,e) \in \rho_N$. Thus we have $a \in \ker(\rho)$. Conversely, if $a \in \ker(\rho)$, then $aa \cdot a^{-1} = a^{-1}a \cdot a \in N$. Hence $a \in N\omega = N$. Consequently, $\ker(\rho_N) = N$. It is easy to see that an arbitrary AG-group congruence on G is uniquely determined by its kernel, so ρ_N is a unique AG-group congruence with $\ker(\rho_N) = N$. □

Theorem 4.5. *If ρ is a group congruence on a completely inverse AG^{**}-groupoid G, then* $\ker(\rho) \in \mathcal{N}(G)$ *and* $\rho = \rho_{\ker(\rho)}$.

Proof. Indeed, $N = \ker(\rho)$ is a normal completely inverse AG^{**}-subgroupoid of G, so that $\rho = \rho_N$. □

Corollary 4.6. *The map $\varphi : \mathcal{N}(G) \to \mathcal{GC}(G)$ given by $\varphi(N) = \rho_N$, where G is a completely inverse AG^{**}-groupoid, is a complete lattice isomorphism of $\mathcal{N}(G)$ onto $\mathcal{GC}(G)$. In particular, the lattice $\mathcal{N}(G)$ is modular.* □

More interesting facts concerning certain fundamental congruences on a completely inverse AG^{**}-groupoid one can find in [2] and [3]. In [3] are determined, for example, the maximum idempotent-separating congruence, the least AG-group and the least E-unitary congruence. In particular, the congruences on completely inverse AG^{**}-groupoids are described by their kernel and trace.

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