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A new characterization of Osborn-Buchsteiner loops

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Abstract. In the study of loops of Bol-Moufang types, a question that quickly comes to mind is this. Since a loop is an extra loop if and only if it is a Moufang loop and a CC-loop(or C-loop), then can one generalize this statement by identifying a "new identity" for a loop which generalizes the C-loop identity such that we can say "An Osborn loop is a Buchsteiner loop if and only if it obeys "certain" identity? A somewhat close answer to this question is the unpublished fact by M. K. Kinyon that "An Osborn loop Q with nucleus N is a Buchsteiner loop if and only if Q/N is a Boolean group" where Q/N being a Boolean group somewhat plays the role of the missing identity. It is proved that an Osborn loop is a Buchsteiner loop if and only if it satisfies the identity $(x \cdot xy)(x^\lambda \cdot xz) = x(x \cdot yz)$. The importance of its emergence which was traced from the facts that Buchsteiner loops generalize extra loops while Osborn loops generalize Moufang loops is the fact that not every Osborn-Buchsteiner loop is an extra loop. An LC-loop obeys this identity. An Osborn-Buchsteiner loop (OBL) is shown to be nuclear square and to obey the identity $x^\rho \cdot xx = xx \cdot x^\lambda = x$. Necessary and sufficient condition for a OBL to be central square is established. It is shown that in an OBL, the cross inverse property and commutativity are equivalent, and the properties: 3-power associativity ($xx \cdot x = x \cdot xx$), self right inverse property ($xx \cdot x^\rho = x$), self left inverse property ($x^\lambda \cdot xx = x$) and $x^\rho = x^\lambda$ are equivalent.

1. Introduction

Let L be a nonempty set with a binary operation denoted by juxtaposition. If the system of equations: $ax = b$, $ya = b$ has unique solutions x and y respectively, then $(L, \cdot)$ is called a *quasigroup*. Furthermore, if there exists a unique element $e \in L$ called the *identity* such that for all $x \in L$, $xe = ex = x$, $(L, \cdot)$ is called a *loop*. For each $x \in L$, the elements $x^\rho = xJ_\rho$, $x^\lambda = xJ_\lambda$ such that $xx^\rho = e = x^\lambda x$ are called the *right*, *left inverses* of x respectively. For any $x, y \in L$, we shall take $(xy)(x, y) = yx$, where $(x, y) \in L$ is called the *commutator* of x and y.

The triple $\alpha = (A, B, C)$ of bijections on a loop $(L, \cdot)$ is called an *autotopism* if and only if

$$xA \cdot yB = (xy)C \quad \text{for all } x, y \in L.$$

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Such triples form a group $AUT(L,\cdot)$ called the *autotopism group* of $(L,\cdot)$. For an overview of the theory of loops, readers may check [10, 19].

Further to reduce of number of brackets we will use dots instead of some brackets. For example, the formula $x((yz)x)$ will be written as $x(yz\cdot x)$.

A loop that satisfies any of the following equivalent identities is called an *Osborn* loop.

$$OS_0 \; : \; x(yz\cdot x) = x(yx^\lambda \cdot x)\cdot zx, \tag{1}$$

$$\mathrm{OS}_1 \; : \; x(yz\cdot x) = [x(yx\cdot x^\rho)]\cdot zx, \tag{2}$$

$$\mathrm{OS}_2 \; : \; x[(x^\lambda y)z\cdot x] = y\cdot zx, \tag{3}$$

$$\mathrm{OS}_3 \; : \; (x\cdot yz)x = xy\cdot[(x^\lambda\cdot xz)\cdot x], \tag{4}$$

$$\mathrm{OS}_4 \; : \; [x\cdot y(zx^\rho)]x = xy\cdot z. \tag{5}$$

Kinyon [17] revived the study of Osborn loops in 2005. The most popularly known varieties of Osborn loops are: VD-loops (Basarab [1]), Moufang loops, CC-loops, universal weak inverse property loops and extra loops. Some recent works on Osborn loops are Jaiyéọlá [11, 12], Jaiyéọlá and Adéníran [13, 14, 15], and Jaiyéọlá, Adéníran and Sòlárìn [16].

The *Buchsteiner* law

$$\mathrm{BL} \; : \; x\backslash(xy\cdot z) = (y\cdot zx)/x$$

was first introduced by Buchsteiner [2]. Its study in loops is on revival by Csörgő et. al. [3, 4, 5] and Drápal et. al. [6, 7, 8]. Buchsteiner loops are G-loops and extra loops belongs to their class.

Buchsteiner loops generalize extra loops while Osborn loops generalize Moufang loops. A question that quickly comes to mind is this: since a loop is an extra loop if and only if it is a Moufang loop and a CC-loop (or C-loop), then can one generalize this statement by identifying a new identity that describes a new class of loop which generalizes a C-loop such that we can say "An Osborn loop is a Buchsteiner loop if and only if it is a "certain" loop?" A some what close answer to this question is the unpublished fact by M. K. Kinyon that "An Osborn loop Q with nucleus N is a Buchsteiner loop if and only if Q/N is a Boolean group" where Q/N being a Boolean group some what plays the role of the missing loop variety. It will be shown in this study that this new class of loop is described by the identity

$$(x\cdot xy)(x^\lambda\cdot xz) = x(x\cdot yz). \tag{6}$$

LC-loops fall into this class. It must be noted that when Drápal and Jedlička [6] used nuclear identification to obtain some loop identities, the Osborn and our new loop identities did not feature among such identities. We shall refer to an Osborn loop which obeys the Buchsteiner law as an *Osborn-Buchsteiner* loop.

Theorem 1.1. (Proposition 2.5 in [5]) *Let Q be a CC-loop with nucleus $N(Q)$. Then Q is a Buchsteiner loop if and only if $x^2\in N(Q)$ for every $x\in Q$.* □

We got the following unpublished result from Kinyon through personal contact.

Theorem 1.2. (Kinyon, 2009) *Let Q be a loop with nucleus $N = N(Q)$. Any two of the following implies the third.*
1. *Q is an Osborn loop.*
2. *Q is a Buchsteiner loop.*
3. *N is a normal in Q and Q/N is a Boolean group.* $\square$

We can also say that:

Theorem 1.3. *Let Q be an Osborn loop with nucleus $N = N(Q)$. Then Q is a Buchsteiner loop if and only if Q/N is a Boolean group. Hence, Q is an Osborn loop that is nuclear square.* $\square$

Theorem 1.4. (Theorem 11.3 in [5]) *Let Q be a Buchsteiner loop with nucleus $N = N(Q)$. If $|Q| < 32$, then Q is a CC-loop. If $|Q| < 64$, then Q/N has exponent 2.* $\square$

Theorem 1.5. (Theorem 7.14 in [5]) *Let Q be a Buchsteiner loop with nucleus $N = N(Q)$. Then Q/N is an abelian group of exponent 4.* $\square$

Theorem 1.2 is a generalization of Theorem 1.1.

2. Main Results

Theorem 2.1. *An Osborn loop is a Buchsteiner loop if and only if it obeys identity (6). Hence, it is a nuclear square loop and the loop modulo its nucleus is an abelian group of exponent 2.*

Proof. Using the identities OS_1, OS_2 and OS_3 of an Osborn loop $(L, \cdot)$ and the identity BL of a Buchsteiner loop L, it can be shown that

$$(R_x R_{x^\rho} L_x^2, I, R_x T_{(x)}^{-1} L_x) \in AUT(L). \tag{7}$$

This is done as follows. Take $T_{(x)} = R_x L_x^{-1}$. From equation (2),

$$R_z R_x L_x = R_x R_{x^\rho} L_x R_{zx} \Leftrightarrow R_{zx} = L_x^{-1} R_{x^\rho}^{-1} R_x^{-1} R_z R_x L_x.$$

From the Buchsteiner law, $R_{zx} = L_x R_z L_x^{-1} R_x$. So,

$$\begin{gathered} L_x R_z L_x^{-1} R_x = L_x^{-1} R_{x^\rho}^{-1} R_x^{-1} R_z R_x L_x \Leftrightarrow R_x R_{x^\rho} L_x^2 R_z L_x^{-1} R_x = R_z R_x L_x \Leftrightarrow \\ y R_x R_{x^\rho} L_x^2 R_z L_x^{-1} R_x = y R_z R_x L_x \Leftrightarrow y R_x R_{x^\rho} L_x^2 R_z L_x^{-1} R_x = (yz) R_x L_x \Leftrightarrow \\ y R_x R_{x^\rho} L_x^2 \cdot z = (yz) R_x T_{(x)}^{-1} L_x \Leftrightarrow (R_x R_{x^\rho} L_x^2, I, R_x T_{(x)}^{-1} L_x) \in AUT(L). \end{gathered}$$

Thus, if an Osborn loop L is a Buchsteiner loop, then (7) holds. Doing the reverse of the procedure above, it is also true that if in an Osborn loop L holds (7), then

L is a Buchsteiner loop. So, we have shown that an Osborn loop is a Buchsteiner loop if and only if (7) holds.

From equation (3), $(L_{x^\lambda}, R_x^{-1}, L_x^{-1}R_x^{-1}) \in AUT(L)$ while from equation (4), $(L_x, L_xL_{x^\lambda}R_x, L_xR_x) \in AUT(L)$. Thus,

$$(L_xL_{x^\lambda}, L_xL_{x^\lambda}, L_xR_xL_x^{-1}R_x^{-1}) = (L_xL_{x^\lambda}, L_xL_{x^\lambda}, L_xT_{(x)}R_x^{-1}) \in AUT(L).$$

Therefore, in an Osborn loop L, keeping in mind that $L_xL_{x^\lambda}R_xR_{x^\rho} = I$,

$$(L_xL_{x^\lambda}, L_xL_{x^\lambda}, L_xT_{(x)}R_x^{-1})(R_xR_{x^\rho}L_x^2, I, R_xT_{(x)}^{-1}L_x) =$$
$$(L_xL_{x^\lambda}R_xR_{x^\rho}L_x^2, L_xL_{x^\lambda}, L_x^2) = (L_x^2, L_xL_{x^\lambda}, L_x^2) \in AUT(L)$$
$$\Leftrightarrow (x \cdot xy)(x^\lambda \cdot xz) = x(x \cdot yz).$$

Thus, we have shown that an Osborn loop which is also a Buchsteiner loop obeys the identity (6). Assuming the identity (6) is true in the Osborn loop L and doing the reverse of the process above, it will be observed that (7) holds, hence by the earlier fact, L is a Buchsteiner loop. Recall that (7) implies $yR_xR_{x^\rho}L_x^2 \cdot z = (yz)R_xT_{(x)}^{-1}L_x$ for all $y, z \in L$. Substituting $z = e$, we have

$$R_xR_{x^\rho}L_x^2 = R_xT_{(x)}^{-1}L_x \text{ for all } x \in L. \tag{8}$$

So, (7) implies $yR_xR_{x^\rho}L_x^2 \cdot z = (yz)R_xR_{x^\rho}L_x^2$ for all $y, z \in L$. Substituting $y = e$, we see that $x^2z = zR_xR_{x^\rho}L_x^2$ implies $L_{x^2} = R_xR_{x^\rho}L_x^2$ for all $x \in L$. Thus, $(L_{x^2}, I, L_{x^2}) \in AUT(L)$ which means that $x^2 \in N$ for all $x \in L$. That is, L is nuclear square. Thus, by Theorem 1.5, L/N is a Boolean group. □

From Theorem 2.1 we can deduce that in Theorem 1.2 conditions 1. and 2. imply 3. The proof of Theorem 2.1 was carried out without the knowledge of Theorem 1.2.

Corollary 2.2. *Let Q be an Osborn loop with nucleus $N = N(Q)$. The following are equivalent:*
1. *Q is a Buchsteiner loop,*
2. *Q/N is a Boolean group,*
3. *Q obeys* (6).

Hence, Q is an Osborn loop that is nuclear square.

Proof. The proof follows from Theorem 1.3 and Theorem 2.1. □

Lemma 2.3. *Let $(Q, \cdot)$ be an Osborn loop that is nuclear square. Then*
1. $x^\rho \cdot xx = xx \cdot x^\lambda = x$.
2. *The following are equivalent: $xx \cdot x^\rho = x$, $x^\lambda \cdot xx = x$, $x^\rho = x^\lambda$ and $xx \cdot x = x \cdot xx$. Hence, $(x^2, x^\rho) = (x^2, x^\lambda) = e$.*
3. *L is central square if and only if $x \cdot (x^\lambda y \cdot x)x = x(x \cdot yx^\rho) \cdot x$.*

Proof. 1. By OS_1, $x(yz \cdot x) = [x(yx \cdot x^\rho)] \cdot zx$. Substituting $z = x$, we have $x(yx \cdot x) = [x(yx \cdot x^\rho)] \cdot xx \Rightarrow (yx \cdot x) = (yx \cdot x^\rho)(xx) \Rightarrow x^\rho \cdot xx = x$. Doing a similar thing with OS_3, we get $xx \cdot x^\lambda = x$.
2. Using OS_0 the way OS_1 was used above, we get $yx \cdot x = (yx^\lambda \cdot x)(xx)$. Taking $y = x$, it is easy to see that $xx \cdot x = x \cdot xx$ if and only if $x^\rho = x^\lambda$. In Lemma 3.20 of [16], the equivalence of the first three identities was proved in an Osborn loop. Hence, the equivalence of the four identities follows.
3. In OS_2, $x[(x^\lambda y)z \cdot x] = y \cdot zx$, making $z = x$, we get $R_{x^2} = L_{x^\lambda} R_x^2 L_x$. Doing a similar thing with OS_4, we have $L_{x^2} = R_{x^\rho} L_x^2 R_x$. So, L is central square if and only if $R_{x^2} = L_{x^2} \Leftrightarrow L_{x^2} = R_{x^\rho} L_x^2 R_x = R_{x^\rho} L_x^2 R_x \Leftrightarrow x \cdot (x^\lambda y \cdot x)x = x(x \cdot yx^\rho) \cdot x$. □

Lemma 2.4. *Let $(Q, \cdot)$ be an Osborn-Buchsteiner loop. Then*

1. *the following are equivalent: $xx \cdot x^\rho = x$, $x^\lambda \cdot xx = x$, $x^\rho = x^\lambda$, $xx \cdot x = x \cdot xx$ and $(x \cdot xy)x = x(x \cdot yx)$. Hence, $(x, y) = e$ if and only if $(x, x \cdot xy) = e$ or $(x, x \cdot yx) = e$.*
2. *$(x \cdot yx^\rho)x = xy$.*
3. *L is a cross inverse property loop if and only if L is commutative.*

Proof. 1. The equivalence of the first four identities follows from 1. of Lemma 2.3. From identity (6), $(x \cdot xy)(x^\lambda \cdot xz) = x(x \cdot yz)$, so taking $z = x$, $(x \cdot xy)(x^\lambda \cdot xx) = x(x \cdot yx)$, so $x^\lambda \cdot xx = x \Leftrightarrow (x \cdot xy)x = x(x \cdot yx)$.
2. Recall that from (8), $R_x R_{x^\rho} L_x^2 = R_x T_{(x)}^{-1} L_x$, for all $x \in L$. Putting $T_{(x)} = R_x L_x^{-1}$, we get $R_{x^\rho} L_x R_x = L_x \Leftrightarrow (x \cdot yx^\rho)x = xy$.
3. This follows from 2. above. □

Not all Osborn-Buchsteiner loops are extra loops. A loop is said to be *nuclear square* if the square of each element is nuclear (i.e. in the nucleus). It is well known from Fenyves [9] that extra loops are nuclear square loops. In Table 2 of the last section of [18], the authors established the fact that there exists a non-extra CC-loop that is nuclear square by constructing a power associative CC-loop of order 16 that is nuclear square. Thus, by Theorem 1.1, such a loop is a Buchsteiner loop, hence an Osborn-Buchsteiner loop. This fact can also be corroborated with Theorem 1.4 following the fact that $|Q| < 32$.

Furthermore, in [Page 7, [4]], it was observed that not every Buchsteiner loop Q with nucleus N such that Q/N is a Boolean group has to be a CC-loop. Hence, since Q/N is a Boolean group implies Q is nuclear square, then there exist nuclear square Buchsteiner loops that are not CC-loops.

As shown in Corollary 2.1, Theorem 2.1 is another characterization of Osborn-Buchsterner loops in identity form relative to the group-structural characterization form of Q modulo N being a Boolean group. The importance of this characterization can be linked to the fact that Buchsteiner [2] originally claimed that in a Buchsteiner loop Q, Q/N is a Boolean group, while [5] clarified this statement by showing in Theorem 1.5 that Q/N is actually an abelian group of exponent 4.

Kinyon in personal correspondence went further to show that a Buchsteiner loop Q for which Q/N is a Boolean group must be an Osborn loop. So, a single identity to describe a Osborn-Buchsterner loop Q for which Q/N is a Boolean group is (6).

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