Quasigroups and Related Systems **20** (2012), 211 − 218

# Construction of subdirectly irreducible SQS-skeins of cardinality $n2^m$

*Enas M. A. Elzayat*

**Abstract.** We give a construction for subdirectly irreducible SQS-skeins of cardinality $n2^m$ having a monolith with a congruence class of cardinality $2^m$ for each integer $m \geqslant 2$. Moreover, the homomorphic image of the constructed SQS-skein modulo its atom is isomorphic to the initial SQS-skein. Consequently, we will construct an $SK(n2^m)$ with a derived $SL(n2^m)$ such that $SK(n2^m)$ and $SL(n2^m)$ are subdirectly irreducible and have the same congruence lattice. Also, we may construct an $SK(n2^m)$ with a derived $SL(n2^m)$ in which the congruence lattice of $SK(n2^m)$ is a proper sublattice of the congruence lattice of $SK(n2^m)$.

## 1. Introduction

A *Steiner quadruple* (*triple*) *system* is a pair $(S; B)$ where $S$ is a finite set and $B$ is a collection of 4-subsets (3-subsets) called*blocks* of $S$ such that every 3-subset (2-subset) of $S$ is contained in exactly one block of $B$ (see [8] and [11]). Let $\mathbf{SQS}(m)$ denote a Steiner quadruple system (briefly quadruple system) of cardinality $m$ and $\mathbf{STS}(n)$ denote Steiner triple system (briefly triple system) of cardinality $n$. It is well-known that $\mathbf{SQS}(m)$ exists iff $m \equiv 2$ or 4 (mod 6) and $\mathbf{STS}(n)$ exists if and only if $n \equiv 1$ or 3 (mod 6) [8] and [11]. Let $(S; B)$ be an **SQS.** If one considers $S_a = S - \{a\}$ for any point $a \in S$ and deletes that point from all blocks which contain it then the resulting system $(S_a; B(a))$ is a triple system, where $B(a) = \{b - \{a\} \, \forall b \in B, a \in b\}$. Now, $(S_a; B(a))$ is called a *derived triple system* (or briefly **DTS**) of $(S; B)$ (cf. [8] and [11]).

A *sloop* (briefly **SL**) $L = (L; \cdot, 1)$ is a groupoid with a neutral element 1 satisfying the identities:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad x \cdot (x \cdot y) = y.$$

A sloop $L$ is called *Boolean* if it satisfies the associative law. The cardinality of the Boolean sloop is equal $2^m$.

2010 Mathematics Subject Classification: 05B30, 05B07, 20N05, 08A99
Keywords: Steiner triple system, Steiner loop, sloop, subdirectly irreducible sloop, Steiner quadruple systems, Steiner skein, SQS-skein, subdirectly irreducible SQS-skein.
This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (18-857-D1432).

There is one to one correspondence between **STS**s and Steiner loops (sloops) [8].

An ***SQS**-skein* (briefly an **SK**) $(Q; q)$ is an algebra with a unique ternary operation $q$ satisfying:

$$q(x, y, z) = q(x, z, y) = q(z, x, y), \quad q(x, x, y) = y, \quad q(x, y, q(x, y, z)) = z.$$

An **SQS**-skein $(Q; q)$ is called *Boolean* if it satisfies in addition the identity:

$$q(a, x, q(a, y, z)) = q(x, y, z).$$

There is one to one correspondence between **SQS**s and **SQS**-skeins (cf. [8] and [11]).

The sloop associated with a derived triple system is also called *derived*. A derived sloop of an **SQS**-skein $(Q; q)$ with respect to $a \in Q$ is the sloop $(Q_a; \cdot, a)$ with the binary operation $\cdot$ defined by $x \cdot y = q(a, x, y)$.

A subsloop $N$ of $L$ (sub-**SQS**-skein of $Q$) is called *normal* if and only if $N = [1]\theta$ ($N = [a]\theta$ ) for a congruence $\theta$ on $L$ ($Q$) (cf. [8] and [12]). The associated congruence $\theta$ with the normal subsloop (sub-**SQS**-skein) $N$ is given by:

$$\theta = \{(x, y) : x \cdot y \quad (\text{or } q(a, y, z)) \in N\}.$$

Quackenbush in [12] and similarly Armanious in [1] have proved that the congruences of sloops (**SQS**-skeins) are permutable, regular and uniform. Also, we may say that the congruence lattice of each of sloops and **SQS**-skeins is modular. Moreover, they proved that a maximal subsloop (sub-**SQS**-skein) has the same property as in groups.

**Theorem 1.** (cf. [1] and [8]) *Every subsloop (sub-SQS-skein) of a finite sloop $(L; \cdot, 1)$ (**SQS**-skein $(Q; q)$) with cardinality $\frac{1}{2}|L|$ ($\frac{1}{2}|Q|$) is normal.* □

A Boolean sloop is a Boolean group. If $(G; +)$ is a Boolean group, then $(G; q(x, y, z) = x + y + z)$ is a Boolean **SQS**-skein [1].

Guelzow [10] and Armanious [2], [3] gave generalized doubling constructions for nilpotent subdirectly irreducible **SQS**-skeins and sloops of cardinality $2n$. In [6] the authors gave recursive construction theorems as $n \to 2n$ for subdiredtly irreducible sloops and **SQS**-skeins. All these constructions supplies us with subdirectly irreducible **SQS**-skeins having a monolith $\theta$ satisfying $|[x]\theta| = 2$ (the minimal possible order of a proper normal **SQS**-skeins). Also in these constructions, the authors begin with a subdirectly irreducible $\mathbf{SK}(n)$ to construct a subdirectly irreducible $\mathbf{SK}(2n)$ satisfying the property that the cardinality of the congruence class of its monolith is equal 2. Armanious [5] has given another construction of a subdirectly irreducible $\mathbf{SK}(2n)$. He begins with a finite simple $\mathbf{SK}(n)$ to costruct a subdirectly irreducible $\mathbf{SK}(2n)$ having a monolith $\theta$ with $|[x]\theta| = n$ (the maximal possible order of a proper normal sub-**SQS**-skein).

In [7] the authors begin with an arbitrary $\mathbf{SL}(n)$ to construct subdirectly irreducible $\mathbf{SL}(n2^m)$ for each possible integers $n \geqslant 4$ and $m \geqslant 2$.

In this article, we begin with an arbitrary $\mathbf{SK}(n)$ for each possible value $n \geqslant 4$ to construct subdirectly irreducible $\mathbf{SK}(n2^m)$ for each integer $m \geqslant 2$. This construction enables us to construct subdirectly irreducible $\mathbf{SQS}$-skein having a monolith $\theta$ satisfying that its congruence class is $\mathbf{SK}(2^m)$. Moreover, its homomorphic image modulo $\theta$ is isomorphic to $Q$.

We will show that our construction supplies us with construction of an $\mathbf{SK}(n2^m)$ with a derived $\mathbf{SL}(n2^m)$ such that the congruence lattices of $\mathbf{SK}(n2^m)$ and $\mathbf{SL}(n2^m)$ are the same for each possible case. Moreover, we may construct an $\mathbf{SK}(n2^m)$ with a derived $\mathbf{SL}(n2^m)$ such that the congruence lattices of $\mathbf{SK}(n2^m)$ is a proper sublattice of the congruence lattice of $\mathbf{SL}(n2^m)$.

## 2. Subdirectly irreducible SQS-skeins $Q \times_\alpha B$

Let $Q := (Q; q)$ be an $\mathbf{SK}(n)$ and $B := (B; \bullet, 1)$ be a Boolean $\mathbf{SL}(2^m)$, where $Q = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ and $B = \{1, a_1, a_2, \dots, a_{2^m-1}\}$. In this section we extend the $\mathbf{SQS}$-skein $Q$ to a subdireclty irreducible $\mathbf{SQS}$-skein $Q \times_\alpha B$ of cardinality $n2^m$ having $Q$ as a homomorphic image.

We divide the set of elements of the direct product $Q \times B$ into two subsets $\{x_0, x_1\} \times B$ and $\{x_2, \dots, x_{n-1}\} \times B$. Consider the cyclic permutation $\alpha = (a_1 a_2 \dots a_{2^m-1})$ on the set $\{1, a_1, a_2, \dots, a_{2^m-1}\}$ and the characteristic function $\chi$ from the direct product $Q \times B$ to $B$ defined as follows

$$\chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) =$$

$$\begin{cases} i_m \bullet i_n \bullet \alpha^{-1}(i_m \bullet i_n) & \text{for } \ y_m = y_n = x_0,\ y_k = x_1 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\ i_m \bullet i_n \bullet \alpha(i_m \bullet i_n) & \text{for } \ y_m = y_n = x_1,\ y_k = x_0 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $\chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) = 1$ in two cases:

($i$) $y_1 = y_2 = y_3 = x_0$ or $y_1 = y_2 = y_3 = x_1$.

($ii$) $y_1, y_2$ or $y_3 \in Q - \{x_0, x_1\}$.

For this characteristic function we obtain the following result:

**Lemma 2**. *The characteristic function $\chi$ satisfies the properties:*

($i$) $\chi((x, a), (y, b), (z, c)) = \chi((x, a), (z, c), (y, b)) = \chi((z, c), (x, a), (y, b))$;

($ii$) $\chi((x, a), (x, a), (y, b)) = 1$;

($iii$) $\chi((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c)))) = \chi((x, a), (y, b), (z, c))$.

*Proof.* According to the definition of $\chi$, we may deduce that ($i$) is valid.

In ($ii$), if $x = x_0$ and $y = x_1$ then $\chi((x_0, a), (x_0, a), (x_1, b)) = a \bullet a \bullet \alpha^{-1}(a \bullet a) = 1$. If $x = x_1$ and $y = x_0$, then $\chi((x_1, a), (x_1, a), (x_0, b)) = a \bullet a \bullet \alpha(a \bullet a) = 1$. Otherwise if $x$ or$y \neq x_0$ or $x_1$, then $\chi((x, a), (x, a), (y, b)) = 1$.

To prove the third property, we have only three essential cases:
(1) If $x = y = x_0$ and $z = x_1$ then

$$\begin{aligned}&\chi((x_0, a), (x_0, b), (q(x_0, x_0, x_1), a \bullet b \bullet c \bullet \chi((x_0, a), (x_0, b), (x_1, c))))\\&= \chi((x_0, a), (x_0, b), (x_1, c \bullet \alpha^{-1}(a \bullet b))) = a \bullet b \bullet \alpha^{-1}(a \bullet b)\\&= \chi((x_0, a), (x_0, b), (x_1, c)).\end{aligned}$$

(2) If $x = y = x_1$ and $z = x_0$ then

$$\begin{aligned}&\chi((x_1, a), (x_1, b), (q(x_1, x_1, x_0), a \bullet b \bullet c \bullet \chi((x_1, a), (x_1, b), (x_0, c))))\\&= \chi((x_1, a), (x_1, b), (x_0, c \bullet \alpha(a \bullet b))) = a \bullet b \bullet \alpha(a \bullet b)\\&= \chi((x_1, a), (x_1, b), (x_0, c)).\end{aligned}$$

Note that
$\chi((x_0, a), (x_0, b), (x_1, c)) = \chi((x_0, a), (x_1, c), (x_0, b)) = \chi((x_1, c), ((x_0, a), (x_0, b))$
and
$\chi((x_1, a), (x_1, b), (x_0, c)) = \chi((x_1, a), (x_0, c), (x_1, b)) = \chi((x_0, c), ((x_1, a), (x_1, b))$.

(3) Otherwise, i.e., when $i)$ $x = y = z = x_0$ or $x = y = z = x_1$
$ii)$ $x, y$ or $z \notin \{x_0, x_1\}$,
we have

$$\chi((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c)))) = \chi((x, a), (y, b), (z, c) = 1.$$

This completes the proof of the lemma. □

**Lemma 3.** *Let $(Q; q)$ be an arbitrary $\mathbf{SK}(n)$, and $(B; \bullet, 1)$ be a Boolean $\mathbf{SL}(2^m)$ for $m \geqslant 2$. Also let $q'$ be a ternary operation on the set $Q \times B$ defined by :*

$$q'((x, a), (y, b), (z, c)) := (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))).$$

*Then $Q \times_\alpha B = (Q \times B; q')$ is an $\mathbf{SK}(n2^m)$ for each possible number $n \geqslant 4$.*

*Proof.* Let $Q = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ and $B = \{1, a_1, a_2, \dots, a_{2m_{-1}}\}$. For all $(x, a), (y, b), (z, c) \in Q \times B$, according to Lemma 2 $(i)$ and the properties of the operations "$q$" and "$\bullet$" we find that:

$$q'((x, a), (y, b), (z, c)) = q'((x, a), (z, c), (y, b)) = q'((z, c), (x, a), (y, b)).$$

By using Lemma 2 $(ii)$

$$q'((x, a), (x, a), (y, b) = (q(x, x, y), a \bullet a \bullet b \bullet \chi((x, a), (x, a), (y, b))) = (y, b).$$

Also, Lemma 2 $(iii)$ gives us that

$$\begin{aligned}&q'((x, a), (y, b), (q'((x, a), (y, b), (z, c)))\\&= q'((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c)) = (z, c).\end{aligned}$$

Hence $Q \times_\alpha B = (Q \times B; q')$ is an $\mathbf{SQS}$-skein. □

In the next theorem we prove that the constucted $Q\times_\alpha B$ is a subdirectly irreducible **SQS**-skein having a monolith $\theta_1$ satisfying that the cardinality of its congruence class equal $2^m$.

**Theorem 4.** *The constructed sloop $Q\times_\alpha B = (Q\times B; q')$ is a subdirectly irreducible* ***SQS****-skein.*

*Proof.* The projection $\Pi : (x, a) \to x$ from $Q \times B$ into $Q$ is an onto homomorphism and the congrurnce Ker $\Pi := \theta_1$ on $Q \times B$ is given by:

$$\theta_1 = \bigcup_{i=0}^{n-1} \left\{(x_i, 1), (x_i, a_1), \ldots, (x_i, a_{2m-1})\right\}^2,$$

so one can directly see that $[(x_0, 1)]\theta_1 = \left\{(x_0, 1), (x_0, \alpha_1), \ldots, (x_0, a_{2m-1})\right\}$.

Now $\mathbf{C}(Q) \cong \mathbf{C}((Q\times_\alpha B)/\theta_1) \cong [\theta_1 : 1]$. Our proof will now be complete if we show that $\theta_1$ is the unique atom of $\mathbf{C}(Q\times_\alpha B)$.

First, assume that $\theta_1$ is not an atom of $\mathbf{C}(Q\times_\alpha B)$, then we can find an atom $\gamma$ satisfying that: $\gamma \subset \theta_1$ and $|[(x_i, a_i)]\,\gamma| = r < |[(x_i, a_i)]\,\theta_1| = 2^m$. In the following we get a contradiction by proving $[(x_1, 1)]\,\gamma$ is not a normal sub-**SQS**-skein of $Q\times_\alpha B$.

Suppose $[(x_1, 1)]\gamma = \left\{(x_1, 1), (x_1, a_{s_1}), \ldots, (x_1, a_{s_{r-1}})\right\}$. If $\left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$ is an increasing successive subsequence of $\left\{a_1, a_2, \ldots, a_{2m-1}\right\}$ and if $\alpha(a_{s_i}) = a_{s_{i+1}}$ for all $i = 1, 2, \ldots, r-1$, then $\alpha(a_{s_{r-1}}) = a_{s_r} \notin \left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$. If $\left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$ is an increasing and not successive subsequence selected from $\left\{a_1, a_2, \ldots, a_{2m-1}\right\}$ then there exists an element $a_j \in \left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$ such that $\alpha(a_j) = a_{j+1} \notin \left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$. For both cases, we can always find an element $(x_1, a_k) \in [(x_1, 1)]\,\gamma$ such that $(x_1, \alpha(a_k)) \notin [(x_1, 1)]\,\gamma$ ($a_k = a_{s_{r-1}}$ for the first case, and $a_k = a_j$ for the second case).

We can determine the class containing $(x_0, 1)$ when we use the fact that $[(x_0, 1)]\,\gamma = q'\left([(x_1, 1)]\,\gamma, (x_1, 1), (x_0, 1)\right)$, hence we will find that

$$[(x_0, 1)]\,\gamma = \left\{(x_0, 1), (x_0, \alpha(a_{s_1})), (x_0, \alpha(a_{s_2})), \ldots, \left(x_0, \alpha\left(a_{s_{r-1}}\right)\right)\right\}.$$

By the same way $[(x_2, 1)]\,\gamma = q'\left([(x_1, 1)]\,\gamma, (x_1, 1), (x_2, 1)\right)$, and this leads to

$$[(x_2, 1)]\,\gamma = \left\{(x_2, 1), (x_2, a_{s_1}), (x_2, a_{s_2}), \ldots, \left(x_2, a_{s_{r-1}}\right)\right\}.$$

From the other side $[(x_2, 1)]\,\gamma = q'\left([(x_0, 1)]\,\gamma, (x_0, 1), (x_2, 1)\right)$, here we will find that

$$[(x_2, 1)]\,\gamma = \left\{(x_2, 1), (x_2, \alpha(a_{s_1})), (x_2, \alpha(a_{s_2})), \ldots, \left(x_2, \alpha\left(a_{s_{r-1}}\right)\right)\right\}.$$

This means that for each $a_k \in \left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$ $\alpha(a_k) \in \left\{a_{s_1}, a_{s_2}, \ldots, a_{s_{r-1}}\right\}$. This contradicts the assumption that $(x_1, \alpha(a_k)) \notin [(x_1, 1)]\,\gamma$. Hence, we may say that there is no atom $\gamma$ of $\mathbf{C}(Q\times_\alpha B)$ satisfying $\gamma \subset \theta_1$. Therefore, $\theta_1$ is an atom of the lattice $\mathbf{C}(Q\times_\alpha B)$.

Secondly, to prove that $\theta_1$ is the unique atom of $\mathbf{C}(Q \times_\alpha B)$. Assume that $\delta$ is another atom of $\mathbf{C}(Q \times_\alpha B)$, then $\theta_1 \cap \delta = 0$. Hence, one can easily see that there is only one element $(x, a_i) \in [(x, a_i)]\delta$ with the first component $x$ (note that $[(x, a_i)]\,\theta_1 = \big\{(x, 1), (x, a_1), \ldots, (x, a_i), \ldots, (x, a_{2m_{-1}})\big\}$). For this reason we may say that the class $[(x_0, 1)]\delta$ has at most one pair $(x_1, a_i)$ with first component $x_1$. So we have two possibilities; either

$(i)$ $[(x_0, 1)]\delta$ contains only one pair $(x_1, a_i)$ with first component $x_1$, or

$(ii)$ $[(x_0, 1)]\delta$ has not any pairs with first component $x_1$.

For the first case, let $((x, a), (x_1, a_s)) \in \delta$ such that $x_0 \neq x \neq x_1$, and $a_s \neq a_i$. Then

$$q'((x_0, 1), (x, a), (x_1, a_s)) \in [(x_0, 1)]\delta.$$

In this case $(x_1, a_i) \in [(x_0, 1)]\,\delta$. Thus

$$q'((x_0, 1), (x_1, a_i), q'((x_0, 1), (x, a), (x_1, a_s))) \in [(x_0, 1)]\delta.$$

Hence, $(x, a_i \bullet a \bullet a_s) \in [(x_0, 1)]\,\delta$.

By using the properties of congruences, $((x_0, 1), (x_1, a_i)), ((x_1, a_s), (x, a))$ and $((x_1, a_i), (x_1, a_i)) \in \delta$, we shall find that $(q'((x_0, 1), (x_1, a_s), (x_1, a_i)), (x, a)) \in \delta$. This means that

$$q'((x_0, 1), (x, a), q'((x_0, 1), (x_1, a_s), (x_1, a_i))) \in [(x_0, 1)]\delta.$$

So,

$$(x, a \bullet \alpha(a_i \bullet a_s)) \in [(x_0, 1)]\,\delta.$$

Since the class $[(x_0, 1)]\,\delta$ contains at most one element with a first component $x$, it follows that $\alpha(a_i \bullet a_s) = a_i \bullet a_s$ hence $a_i \bullet a_s = 1$, which contradicts the choice that $a_s \neq a_i$. This implies that $[(x_0, 1)]\,\delta$ is not a normal sub-**SQS**-skein of $Q \times_\alpha B$.

For the second case $(ii)$ when $[(x_0, 1)]\delta$ has not any pair with first component $x_1$. Let $(x, a) \in [(x_0, 1)]\delta$ such that $x_0 \neq x \neq x_1$, and let $(x, b)$ and $(x, c)$ are two elements in $Q \times B$ such that $a \neq b$. Then

$$q'((x_0, 1), (x, a), q'((x_0, 1), (x_1, c), (x, b))) \in [q'((x_0, 1), (x_1, c), (x, b))]\,\delta.$$

This means that $(x_1, c \bullet a \bullet b) \in [q'((x_0, 1), (x_1, c), (x, b))]\,\delta$. Also,

$$\begin{aligned} q'((x_0, 1), (x_1, c), q'((x_0, 1), (x, a), (x, b))) &\in q'((x_0, 1), (x_1, c), [(x, b)]\delta) \\ &= [q'((x_0, 1), (x_1, c), (x, b))]\,\delta. \end{aligned}$$

Therefore $(x_1, c \bullet \alpha^{-1}(a \bullet b)) \in [q'((x_0, 1), (x_1, c), (x, b))]\,\delta$.

By using the fact that the class $[q'((x_0, 1), (x_1, c), (x, b))]\,\delta$ contains only one element with the first component $x_1$, we may say that $\alpha^{-1}(a \bullet b) = a \bullet b$, hence $a \bullet b = 1$, which contradicts that $a \neq b$.Thus $[(x_0, 1)]\delta$ is not a normal sub-**SQS**-skein of $Q \times_\alpha B$.This mean that there is no another atom $\delta$, and $\theta_1$ is the unique atom of $\mathbf{C}(Q \times_\alpha B)$. Therefore, $Q \times_\alpha B$ is a subdirectly irreducible **SQS**-skein. □

Note that in the constructed **SQS**-skein $Q \times_\alpha B$, we may choose $B$ a Boolean $\mathbf{SL}(2^m)$ for each $m \geqslant 2$. Therefore, as a consequence of the proof of Theorem 3, we obtain

**Corollary 5**. *Let $B$ be a Boolean $\mathbf{SL}(2^m)$ for an integer $m \geqslant 2$. Then the congruence class $[(x_0, 1)]\theta_1$ of the monolith $\theta_1$ of the constucted subdirectly irreducible **SQS**-skein $Q \times_\alpha B$ is a Boolean $\mathbf{SK}(2^m)$.* □

Also, Theorem 3 enable us to construct a subdirectly irreducible **SQS**-skein $Q \times_\alpha B$ having a monolith $\theta_1$ satisfying that $(Q \times_\alpha B)/\ \theta_1 \cong Q$.

**Corollary 6**. *Every **SQS**-skein $Q$ is isomorphic to the homomorphic image of the subdirectly irreducible **SQS**-skein $Q \times_\alpha B$ over its monolith, for each Boolean sloop $B$.* □

Remark: The **SQS**-skein $Q \times_\alpha B$ having $L \times_\alpha B$ as a derived sloop.

Let $(Q; q)$ be an $\mathbf{SK}(n)$ and $(L; *, x_0)$ be a derived $\mathbf{SL}(n)$ of $Q$ with respect to the element $x_0$ with the same congruence lattice. This means that for $L = Q = \{x_0, x_1, \dots, x_{n-1}\}$, the binary operation " $*$ " is defined by $x * y = q(x_0, x, y)$.

By using the construction in [7], we construct subdirectly irreducible $\mathbf{SL}(n2^m)$. This means that if we begin with our derived sloop $L := (L; *, x_0)$ of cardinality $n$ and the Boolean sloop $B := (B; \bullet, 1)$ of cardinality $2^m$, we get subdirectly irreducible sloop $L \times_\alpha B = (L \times B; \circ, (x_0, 1))$, where

$$(x, a) \circ (y, b) := (x * y, a \bullet b \bullet \chi((x, a), (y, b)))$$

and

$$\chi((x, a), (y, b))_L = \begin{cases} a \bullet \alpha^{-1}(a) & \text{for } \ x = x_0,\ y = x_1, \\ b \bullet \alpha^{-1}(b) & \text{for } \ x = x_1,\ y = 1, \\ c \bullet \alpha(c) & \text{for } \ x = x_1 = y \text{ and } a \bullet b = c, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $\chi((x, a), (y, b))_L = \chi((x_0, 1), (x, a), (y, b))$ (the characteristic function of our construction) for all $x, y \in L = Q$. Hence $(x, a) \circ (y, b) = q'((x_0, 1), (x, a), (y, b))$ for all $(x, a)\,, (y, b) \in L \times B = Q \times B$, this means directly that the constructed sloop $L \times_\alpha B$ is a derived sloop of the constructed **SQS**-skein $Q \times_\alpha B$. Therefore, we have the following result:

**Corollary 7.** *Let $L$ be a derived sloop of the **SQS**-skein $Q$ with respect to the element $x_0$, then the sloop $L \times_\alpha B$ is a derived sloop of the **SQS**-skein $Q \times_\alpha B$ with respect to $(x_0, 1)$.* □

Note that $Q$ is isomorphic to the homomorphic image of $Q \times_\alpha B$ over its monolith (Corollary 5) and also $L$ is isomorphic to the homomorphic image of $L \times_\alpha B$ over its monolith [7]. Hence according to [7], Theorem 4 and Corollary 6, we may say that:

*There is always an SQS-skein $Q \times_\alpha B$ with a derived sloop $L \times_\alpha B$, in which both $Q \times_\alpha B$ and $L \times_\alpha B$ are subdirectly irreducible of cardinality $n2^m$ having the same congruence lattice for each possible integers $n \geqslant 4$ and $m \geqslant 2$.*

The construction of a semi-Boolean **SQS**-skein (each derived sloop $L$ of $Q$ is Boolean) given in [9] satisfies that $C(Q)$ is a proper sublattice of the congruence lattice of its derived sloop $C(L)$. This means that we may begin with **SQS**-skein $Q$ with a derived sloop $L$ in which the congruence lattice of $Q$ is a proper sublattice of the congruence lattice of $L$, this leads to $C(L \times_\alpha B)$ is a proper sublattice of $C(Q \times_\alpha B)$.

Consequently, *we may construct SQS-skein $Q \times_\alpha B$ with a derived sloop $L \times_\alpha B$ such that $Q \times_\alpha B$ and $L \times_\alpha B$ are subdirectly irreducible of cardinality $n2^m$ and have the same congruence lattice, if we begin with $L$ derived sloop of $Q$ with the same congruence lattice. Also, we may construct SQS-skein $Q \times_\alpha B$ with a derived sloop $L \times_\alpha B$ in which the congruence lattice of $Q \times_\alpha B$ is a proper sublattice of the congruence lattice of $L \times_\alpha B$, if we begin with $L$ derived sloop of $Q$ such that the congruence lattice of $Q$ is a proper sublattice of the congruence lattice of $L$.*

# References

[1] **M.H. Armanious**, *Algebraische Theorie der Quadrupelsysteme*, Ph.D. Thesis, Fachbereich Mathematik der Technischen Hochschule Darmstadt, 1980.

[2] **M.H. Armanious**, *Construction of nilpotent sloops of class n*, Discrete Math. **171** (1997), 17 – 25.

[3] **M.H. Armanious**, *Nilpotent SQS-skeins with nilpotent derived sloops*, Ars Combin. **56** (2000), 193 – 200.

[4] **M.H. Armanious**, *On subdirectly irreducible Steiner loops of cardinality 2n*, Beitrage zur Algebra und Geometrie **43** (2002), 325 – 331.

[5] **M.H. Armanious**, *On subdirectly irreducible SQS-skeins*, J. Combin. Math. Combin. Comput. **52** (2005), 117 – 130.

[6] **M.H. Armanious and E.M. Elzayat**, *Subdirectly irreducible sloops and SQS-skeins*, Quasigroups and Reelated Systems **15** (2007), 233 – 250.

[7] **E.M. Elzayat and M.H. Armanious**, *Construction for subdirectly irreducible sloops of cardinality $n2^m$*, Quasigroups and Reelated Systems **17** (2009), 151 – 158.

[8] **B. Ganter and H. Werner**, *Co-ordinatizing Steiner systems*, Ann. Discrete Math. **7** (1980), 3 – 24.

[9] **A.J. Guelzow**, *Semi-boolean SQS-skeins*, J. Alg. Comb. **2** (1993), 147 – 153.

[10] **A.J. Guelzow**, *The structure of nilpotent Steiner quadruple systems*, J. Comb. Designs **1** (1993), 301 – 321.

[11] **C.C. Lindner and A. Rosa**, *Steiner quadruple systems*, Discrete Math. **21** (1978), 147 – 181.

[12] **R.W. Quackenbush**, *Varieties of Steiner loops and Steiner quasigroups*, Canadian J. Math. **28** (1978), 1187 – 1198.

Received August 25, 2012

Department of Mathematics, Faculty of Science and Arts, Khulais, King Abdulaziz University, Kingdom of Saudi Arabia
E-mail: enaselzyat@yahoo.com