Construction of subdirectly irreducible SQS-skeins of cardinality $n^2m$

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Abstract. We give a construction for subdirectly irreducible SQS-skeins of cardinality $n^2m$ having a monolith with a congruence class of cardinality $2^m$ for each integer $m \geq 2$. Moreover, the homomorphic image of the constructed SQS-skein modulo its atom is isomorphic to the initial SQS-skein. Consequently, we will construct an $SK(n^2m)$ with a derived $SL(n^2m)$ such that $SK(n^2m)$ and $SL(n^2m)$ are subdirectly irreducible and have the same congruence lattice. Also, we may construct an $SK(n^2m)$ with a derived $SL(n^2m)$ in which the congruence lattice of $SK(n^2m)$ is a proper sublattice of the congruence lattice of $SK(n^2m)$.

1. Introduction

A Steiner quadruple (triple) system is a pair $(S; B)$ where $S$ is a finite set and $B$ is a collection of 4-subsets (3-subsets) called blocks of $S$ such that every 3-subset (2-subset) of $S$ is contained in exactly one block of $B$ (see [8] and [11]). Let $SQS(m)$ denote a Steiner quadruple system (briefly quadruple system) of cardinality $m$ and $STS(n)$ denote Steiner triple system (briefly triple system) of cardinality $n$. It is well-known that $SQS(m)$ exists iff $m \equiv 2$ or $4 \pmod{6}$ and $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod{6}$ [8] and [11]. Let $(S; B)$ be an $SQS$. If one considers $S_a = S - \{a\}$ for any point $a \in S$ and deletes that point from all blocks which contain it then the resulting system $(S_a; B(a))$ is a triple system, where $B(a) = \{b - \{a\} \mid b \in B, a \in b\}$. Now, $(S_a; B(a))$ is called a derived triple system (or briefly DTS) of $(S; B)$ (cf. [8] and [11]).

A sloop (briefly SL) $L = (L; \cdot, 1)$ is a groupoid with a neutral element 1 satisfying the identities:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad x \cdot (x \cdot y) = y.$$ 

A sloop $L$ is called Boolean if it satisfies the associative law. The cardinality of the Boolean sloop is equal $2^m$. 

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There is one to one correspondence between STSs and Steiner loops (sloops) [8].

An SQS-skein (briefly an SK) \((Q; q)\) is an algebra with a unique ternary operation \(q\) satisfying:

\[
q(x, y, z) = q(x, z, y) = q(z, x, y), \quad q(x, x, y) = y, \quad q(x, y, q(x, y, z)) = z.
\]

An SQS-skein \((Q; q)\) is called Boolean if it satisfies in addition the identity:

\[
q(a, x, q(a, y, z)) = q(x, y, z).
\]

There is one to one correspondence between SQSs and SQS-skeins (cf. [8] and [11]).

The sloop associated with a derived triple system is also called derived. A derived sloop of an SQS-skein \((Q; q)\) with respect to \(a \in Q\) is the sloop \((Q_a; \cdot, a)\) with the binary operation \(\cdot\) defined by \(x \cdot y = q(a, x, y)\).

A subslop \(N\) of \(L\) (sub-SQS-skein of \(Q\)) is called normal if and only if \(N = [1] \theta\) \((N = [a] \theta)\) for a congruence \(\theta\) on \((L, Q)\) (cf. [8] and [12]). The associated congruence \(\theta\) with the normal subslop (sub-SQS-skein) \(N\) is given by:

\[
\theta = \{(x, y) : x \cdot y \ (or \ q(a, y, z)) \in N\}.
\]

Quickonbush in [12] and similarly Armanious in [1] have proved that the congruences of sloops (SQS-skeins) are permutable, regular and uniform. Also, we may say that the congruence lattice of each of sloops and SQS-skeins is modular. Moreover, they proved that a maximal subslop (sub-SQS-skein) has the same property as in groups.

**Theorem 1.** (cf. [1] and [8]) Every subslop (sub-SQS-skein) of a finite sloop \((L; \cdot, 1)\) (SQS-skein \((Q; q)\)) with cardinality \(\frac{1}{2} |L| \ (\frac{1}{2} |Q|)\) is normal. □

A Boolean sloop is a Boolean group. If \((G; +)\) is a Boolean group, then \((G; q(x, y, z) = x + y + z)\) is a Boolean SQS-skein [1].

Guelzo [10] and Armanious [2], [3] gave generalized doubling constructions for nilpotent subdirectly irreducible SQS-skeins and sloops of cardinality \(2n\). In [6] the authors gave recursive construction theorems as \(n \rightarrow 2n\) for subdirectly irreducible sloops and SQS-skeins. All these constructions supplies us with subdirectly irreducible SQS-skeins having a monolith \(\theta\) satisfying \([x] \theta = 2\) (the minimal possible order of a proper normal SQS-skeins). Also in these constructions, the authors begin with a subdirectly irreducible SK\((n)\) to construct a subdirectly irreducible SK\((2n)\) satisfying the property that the cardinality of the congruence class of its monolith is equal 2. Armanious [5] has given another construction of a subdirectly irreducible SK\((2n)\). He begins with a finite simple SK\((n)\) to construct a subdirectly irreducible SK\((2n)\) having a monolith \(\theta\) with \([x] \theta = n\) (the maximal possible order of a proper normal sub-SQS-skein).

In [7] the authors begin with an arbitrary SL\((n)\) to construct subdirectly irreducible SL\((n2^m)\) for each possible integers \(n \geq 4\) and \(m \geq 2\).
In this article, we begin with an arbitrary SK (n) for each possible value \( n \geq 4 \) to construct subdirectly irreducible \( \text{SK}(n^{2^m}) \) for each integer \( m \geq 2 \). This construction enables us to construct subdirectly irreducible SQS-skeins having a monolithic \( \theta \) satisfying that its congruence class is \( \text{SK}(2^m) \). Moreover, its homomorphic image modulo \( \theta \) is isomorphic to \( Q \).

We will show that our construction supplies us with construction of an \( \text{SK}(n^{2^m}) \) with a derived \( \text{SL}(n^{2^m}) \) such that the congruence lattices of \( \text{SK}(n^{2^m}) \) and \( \text{SL}(n^{2^m}) \) are the same for each possible case. Moreover, we may construct an \( \text{SK}(n^{2^m}) \) with a derived \( \text{SL}(n^{2^m}) \) such that the congruence lattices of \( \text{SK}(n^{2^m}) \) is a proper sublattice of the congruence lattice of \( \text{SL}(n^{2^m}) \).

2. Subdirectly irreducible SQS-skeins \( Q \times_\alpha B \)

Let \( Q := (Q; q) \) be an \( \text{SK}(n) \) and \( B := (B; \cdot, 1) \) be a Boolean \( \text{SL}(2^m) \), where \( Q = \{x_0, x_1, x_2, \ldots, x_{n-1}\} \) and \( B = \{1, a_1, a_2, \ldots, a_{2^{m-1}}\} \). In this section we extend the SQS-skein \( Q \) to a subdirectly irreducible SQS-skein \( Q \times_\alpha B \) of cardinality \( n^{2^m} \) having \( Q \) as a homomorphic image.

We divide the set of elements of the direct product \( Q \times B \) into two subsets \( \{x_0, x_1\} \times B \) and \( \{x_2, \ldots, x_{n-1}\} \times B \). Consider the cyclic permutation \( \alpha = (a_1 a_2 \ldots a_{2^{m-1}}) \) on the set \( \{1, a_1, a_2, \ldots, a_{2^{m-1}}\} \) and the characteristic function \( \chi \) from the direct product \( Q \times B \) to \( B \) defined as follows

\[
\chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) = \begin{cases} 
  i_m \cdot i_n \cdot \alpha^{-1}(i_m \cdot i_n) & \text{for } y_m = y_n = x_0, \ y_k = x_1 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\
  i_m \cdot i_n \cdot \alpha(i_m \cdot i_n) & \text{for } y_m = y_n = x_1, \ y_k = x_0 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\
  1 & \text{otherwise.}
\end{cases}
\]

It is clear that \( \chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) = 1 \) in two cases:

\( (i) \) \( y_1 = y_2 = y_3 = x_0 \) or \( y_1 = y_2 = y_3 = x_1 \).

\( (ii) \) \( y_1, y_2 \) or \( y_3 \in Q - \{x_0, x_1\} \).

For this characteristic function we obtain the following result:

**Lemma 2.** The characteristic function \( \chi \) satisfies the properties:

\( (i) \) \( \chi((x, a), (y, b), (z, c)) = \chi((x, a), (z, c), (y, b)) = \chi((z, c), (x, a), (y, b)) \);

\( (ii) \) \( \chi((x, a), (x, a), (y, b)) = 1; \)

\( (iii) \) \( \chi((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))) = \chi((x, a), (y, b), (z, c)) \).

**Proof.** According to the definition of \( \chi \) we may deduce that (i) is valid.

In (ii), if \( x = x_0 \) and \( y = x_1 \) then \( \chi((x_0, a), (x_0, a), (x_1, b)) = a \bullet a \bullet a^{-1}(a \bullet a) = 1 \). If \( x = x_1 \) and \( y = x_0 \), then \( \chi((x_1, a), (x_1, a), (x_0, b)) = a \bullet a \bullet a(a \bullet a) = 1 \). Otherwise if \( x \neq x_0 \) or \( x_1 \), then \( \chi((x, a), (x, a), (y, b)) = 1 \).
To prove the third property, we have only three essential cases:

(1) If $x = y = x_0$ and $z = x_1$ then

\[
\chi((x_0, a), (x_0, b), (q(x_0, x_0, x_1), a \bullet b \bullet c \bullet \chi((x_0, a), (x_0, b), (x_1, c)))) = \chi((x_0, a), (x_0, b), (x_1, c \bullet c^{-1}(a \bullet b))) = a \bullet b \bullet a^{-1}(a \bullet b)
\]

= \chi((x_0, a), (x_0, b), (x_1, c)).

(2) If $x = y = x_1$ and $z = x_0$ then

\[
\chi((x_1, a), (x_1, b), (q(x_1, x_1, x_0), a \bullet b \bullet c \bullet \chi((x_1, a), (x_1, b), (x_0, c)))) = \chi((x_1, a), (x_1, b), (x_0, c \bullet a \bullet b)) = a \bullet b \bullet a \bullet b
\]

= \chi((x_1, a), (x_1, b), (x_0, c)).

Note that

\[
\chi((x_0, a), (x_0, b), (x_1, c)) = \chi((x_0, a), (x_1, c), (x_0, b)) = \chi((x_1, c), (x_0, a), (x_0, b))
\]

and

\[
\chi((x_1, a), (x_1, b), (x_0, c)) = \chi((x_1, a), (x_0, c), (x_1, b)) = \chi((x_0, c), (x_1, a), (x_1, b)).
\]

(3) Otherwise, i.e., when i) $x = y = z = x_0$ or $x = y = z = x_1$

ii) $x, y$ or $z \notin \{x_0, x_1\}$,

we have

\[
\chi((x, a), (y, b), q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))) = \chi((x, a), (y, b), (z, c)) = 1.
\]

This completes the proof of the lemma. 

\[\square\]

Lemma 3. Let $(Q; q)$ be an arbitrary $\mathbf{SK}(n)$, and $(B; \bullet, 1)$ be a Boolean $\mathbf{SL}(2^n)$ for $m \geq 2$. Also let $q'$ be a ternary operation on the set $Q \times B$ defined by :

\[
q'((x, a), (y, b), (z, c)) := (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))).
\]

Then $Q \times B = (Q \times B; q')$ is an $\mathbf{SK}(n2^n)$ for each possible $n \geq 4$.

**Proof.** Let $Q = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$ and $B = \{1, a_1, a_2, \ldots, a_{2^{2^n}}\}$. For all $(x, a), (y, b), (z, c) \in Q \times B$, according to Lemma 2 (i) and the properties of the operations "$q$" and "$\bullet$" we find that:

\[
q'((x, a), (y, b), (z, c)) = q'((x, a), (z, c), (y, b)) = q'((z, c), (x, a), (y, b)).
\]

By using Lemma 2 (ii)

\[
q'((x, a), (x, a), (y, b)) = (q(x, x, y), a \bullet a \bullet b \bullet \chi((x, a), (x, a), (y, b))) = (y, b).
\]

Also, Lemma 2 (iii) gives us that

\[
q'((x, a), (y, b), (q'((x, a), (y, b), (z, c)))
\]

= $q'((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))) = (z, c)$.

Hence $Q \times B = (Q \times B; q')$ is an $\mathbf{SQS}$-skein. \[\square\]
In the next theorem we prove that the constructed \( Q \times_\alpha B \) is a subdirectly irreducible SQS-slein having a monolith \( \theta_1 \) satisfying that the cardinality of its congruence class equal \( 2^m \).

**Theorem 4.** The constructed loop \( Q \times_\alpha B = (Q \times B; q') \) is a subdirectly irreducible SQS-slein.

**Proof.** The projection \( \Pi : (x, a) \to x \) from \( Q \times B \) into \( Q \) is an onto homomorphism and the congruence \( \operatorname{Ker} \Pi := \theta_1 \) on \( Q \times B \) is given by:

\[
\theta_1 = \bigcup_{i=0}^{n-1} \{(x_i, 1), (x_i, a_1), \ldots, (x_i, a_{2^m-1})\}^2,
\]

so one can directly see that \( [(x_0, 1)]_{\theta_1} = \{(x_0, 1), (x_0, a_1), \ldots, (x_0, a_{2^m-1})\} \).

Now \( \operatorname{C}(Q) \cong \operatorname{C}((Q \times_\alpha B)/\theta_1) \cong [\theta_1 : 1] \). Our proof will now be complete if we show that \( \theta_1 \) is the unique atom of \( \operatorname{C}(Q \times_\alpha B) \).

First, assume that \( \theta_1 \) is not an atom of \( \operatorname{C}(Q \times_\alpha B) \), then we can find an atom \( \gamma \) satisfying that: \( \gamma \subset \theta_1 \) and \( |[(x_i, a_i)]_{\gamma}| = r < |[(x_i, a_i)]_{\theta_1}| = 2^m \). In the following we get a contradiction by proving \( [(x_1, 1)]_{\gamma} \) is not a normal sub-SQS-slein of \( Q \times_\alpha B \).

Suppose \( [(x_1, 1)]_{\gamma} = \{(x_1, 1), (x_1, a_{s_1}), \ldots, (x_1, a_{s_{s-1}})\} \). If \( \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \) is an increasing successive subsequence of \( \{a_1, a_2, \ldots, a_{2^m-1}\} \) and if \( \alpha(a_{s_i}) = a_{s_{i+1}} \) for all \( i = 1, 2, \ldots, r - 1 \), then \( \alpha(a_{s_{s-1}}) = a_s \notin \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \). If \( \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \) is an increasing and not successive subsequence selected from \( \{a_1, a_2, \ldots, a_{2^m-1}\} \) then there exists an element \( a_j \in \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \) such that \( \alpha(a_j) = a_{j+1} \notin \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \). For both cases, we can always find an element \( (x_1, a_k) \in [(x_1, 1)]_{\gamma} \) such that \( (x_1, a_k) \notin [(x_1, 1)]_{\gamma} (a_k = a_{s_{s-1}} \) for the first case, and \( a_k = a_j \) for the second case).

We can determine the class containing \( (x_0, 1) \) when we use the fact that \( [(x_0, 1)]_{\gamma} = q'([(x_1, 1)]_{\gamma}, (x_1, 1), (x_0, 1)) \), hence we will find that

\[
[(x_0, 1)]_{\gamma} = \{(x_0, 1), (x_0, a(a_{s_1})), (x_0, a(a_{s_2})), \ldots, (x_0, a(a_{s_{s-1}}))\}.
\]

By the same way \( [(x_2, 1)]_{\gamma} = q'([(x_1, 1)]_{\gamma}, (x_1, 1), (x_2, 1)) \), and this leads to

\[
[(x_2, 1)]_{\gamma} = \{(x_2, 1), (x_2, a(a_{s_1})), (x_2, a(a_{s_2})), \ldots, (x_2, a(a_{s_{s-1}}))\}.
\]

From the other side \( [(x_2, 1)]_{\gamma} = q'([(x_0, 1)]_{\gamma}, (x_0, 1), (x_2, 1)) \), here we will find that

\[
[(x_2, 1)]_{\gamma} = \{(x_2, 1), (x_2, a(a_{s_1})), (x_2, a(a_{s_2})), \ldots, (x_2, a(a_{s_{s-1}}))\}.
\]

This means that for each \( a_0 \in \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \) \( \alpha(a_0) \in \{a_{s_1}, a_{s_2}, \ldots, a_{s_{s-1}}\} \). This contradicts the assumption that \( (x_1, a(a_k)) \notin [(x_1, 1)]_{\gamma} \). Hence, we may say that there is no atom \( \gamma \) of \( \operatorname{C}(Q \times_\alpha B) \) satisfying \( \gamma \subset \theta_1 \). Therefore, \( \theta_1 \) is an atom of the lattice \( \operatorname{C}(Q \times_\alpha B) \).
Secondly, to prove that \( \theta_1 \) is the unique atom of \( C(Q \times_\alpha B) \). Assume that \( \delta \) is another atom of \( C(Q \times_\alpha B) \), then \( \theta_1 \cap \delta = 0 \). Hence, one can easily see that there is only one element \( (x, a_i) \in [(x, a_i)]\delta \) with the first component \( x \) (note that \([[(x, a_i)]\theta_1 = \{(x, 1), (x, a_1), \ldots, (x, a_i), \ldots, (x, a_{2m-1})\}\)). For this reason we may say that the class \( [(x_0, 1)]\delta \) has at most one pair \( (x_1, a_i) \) with first component \( x_1 \). So we have two possibilities; either

(i) \( [(x_0, 1)]\delta \) contains only one pair \( (x_1, a_i) \) with first component \( x_1 \), or

(ii) \( [(x_0, 1)]\delta \) has not any pairs with first component \( x_1 \).

For the first case, let \( ((x, a), (x_1, a_i)) \in \delta \) such that \( x_0 \neq x \neq x_1 \), and \( a_s \neq a_i \).

Then

\[ q'((x_0, 1), (x, a), (x_1, a_i)) \in [(x_0, 1)]\delta. \]

In this case \( (x_1, a_i) \in [(x_0, 1)]\delta \). Thus

\[ q'((x_0, 1), (x, a), q'((x_0, 1), (x, a), (x_1, a_i))) \in [(x_0, 1)]\delta. \]

Hence, \( (x, a \bullet a \bullet a_s) \in [(x_0, 1)]\delta \).

By using the properties of congruences, \( ((x_0, 1), (x_1, a_i)), (x_1, a_s), (x, a) \) and \( ((x_1, a_i), (x_1, a_i)) \in \delta \), we shall find that \( (q'((x_0, 1), (x_1, a_i), (x_1, a_i)), (x, a)) \in \delta \). This means that

\[ q'((x_0, 1), (x, a), q'((x_0, 1), (x_1, a_s), (x_1, a_i))) \in [(x_0, 1)]\delta. \]

So,

\[ (x, a \bullet a(a_i \bullet a_s)) \in [(x_0, 1)]\delta. \]

Since the class \( [(x_0, 1)]\delta \) contains at most one element with a first component \( x \), it follows that \( a(a_i \bullet a_s) = a_i \bullet a_s \) hence \( a_s \bullet a_s = 1 \), which contradicts the choice that \( a_s \neq a_i \). This implies that \( [(x_0, 1)]\delta \) is not a normal sub-SQS-skein of \( Q \times_\alpha B \).

For the second case \( (ii) \) when \( [(x_0, 1)]\delta \) has not any pair with first component \( x_1 \). Let \( (x, a) \in [(x_0, 1)]\delta \) such that \( x_0 \neq x \neq x_1 \), and let \( (x, b) \) and \( (x, c) \) are two elements in \( Q \times B \) such that \( a \neq b \). Then

\[ q'((x_0, 1), (x, a), q'((x_0, 1), (x_1, c), (x, b))) \in [q'((x_0, 1), (x_1, c), (x, b))]\delta. \]

This means that \( (x_1, c \bullet a \bullet b) \in [q'((x_0, 1), (x_1, c), (x, b))]\delta \). Also,

\[ q'((x_0, 1), (x, c), q'((x_0, 1), (x, a), (x, b))) \in q'((x_0, 1), (x_1, c), [(x, b)]\delta) = [q'((x_0, 1), (x_1, c), (x, b))]\delta. \]

Therefore \( (x_1, c \bullet a^{-1}(a \bullet b)) \in [q'((x_0, 1), (x_1, c), (x, b))]\delta \).

By using the fact that the class \( [q'((x_0, 1), (x_1, c), (x, b))]\delta \) contains only one element with the first component \( x_1 \), we may say that \( a^{-1}(a \bullet b) = a \bullet b \) hence \( a \bullet b = 1 \), which contradicts that \( a \neq b \). Thus \( [(x_0, 1)]\delta \) is not a normal sub-SQS-skein of \( Q \times_\alpha B \). This means that there is no another atom \( \delta \), and \( \theta_1 \) is the unique atom of \( C(Q \times_\alpha B) \). Therefore, \( Q \times_\alpha B \) is a subdirectly irreducible SQS-skein. \( \square \)
Note that in the constructed SQS-skein $Q \times_\alpha B$, we may choose $B$ a Boolean $\text{SL}(2^m)$ for each $m \geq 2$. Therefore, as a consequence of the proof of Theorem 3, we obtain

**Corollary 5.** Let $B$ be a Boolean $\text{SL}(2^m)$ for an integer $m \geq 2$. Then the congruence class $[(x_0,1)]\theta_1$ of the monolith $\theta_1$ of the constructed subdirectly irreducible $\text{SQS}$-skein $Q \times_\alpha B$ is a Boolean $\text{SK}(2^m)$. \hfill $\square$

Also, Theorem 3 enable us to construct a subdirectly irreducible $\text{SQS}$-skein $Q \times_\alpha B$ having a monolith $\theta_1$ satisfying that $(Q \times_\alpha B)/\theta_1 \cong Q$.

**Corollary 6.** Every $\text{SQS}$-skein $Q$ is isomorphic to the homomorphic image of the subdirectly irreducible $\text{SQS}$-skein $Q \times_\alpha B$ over its monolith, for each Boolean slope $B$. \hfill $\square$

Remark: The $\text{SQS}$-skein $Q \times_\alpha B$ having $L \times_\alpha B$ as a derived slope.

Let $(Q; q)$ be an $\text{SK}(n)$ and $(L; \ast, x_0)$ be a derived $\text{SL}(n)$ of $Q$ with respect to the element $x_0$ with the same congruence lattice. This means that for $L = Q = \{x_0, x_1, \ldots, x_{n-1}\}$, the binary operation "\ast" is defined by $x \ast y = q(x_0, x, y)$.

By using the construction in [7], we construct subdirectly irreducible $\text{SL}(n2^m)$. This means that if we begin with our derived slope $L := (L; \ast, x_0)$ of cardinality $n$ and the Boolean slope $B := (B; \bullet, 1)$ of cardinality $2^m$, we get subdirectly irreducible slope $L \times_\alpha B = (L \times B; \circ, (x_0, 1))$, where

$$(x, a) \circ (y, b) := (x \ast y, a \bullet b \bullet \chi((x, a), (y, b)))$$

and

$$
\chi((x, a), (y, b))_{L} = \begin{cases} 
    a \bullet a^{-1}(a) & \text{for } x = x_0, y = x_1, \\
    b \bullet a^{-1}(b) & \text{for } x = x_1, y = 1, \\
    c \bullet a(c) & \text{for } x = x_1 = y \text{ and } a \bullet b = c, \\
    1 & \text{otherwise.}
\end{cases}
$$

It is easy to see that $\chi((x, a), (y, b))_{L} = \chi((x_0, 1), (x, a), (y, b))$ (the characteristic function of our construction) for all $x, y \in L = Q$. Hence $(x, a) \circ (y, b) = q'((x_0, 1), (x, a), (y, b))$ for all $(x, a), (y, b) \in L \times B = Q \times B$, this means directly that the constructed slope $L \times_\alpha B$ is a derived slope of the constructed $\text{SQS}$-skein $Q \times_\alpha B$. Therefore, we have the following result:

**Corollary 7.** Let $L$ be a derived slope of the $\text{SQS}$-skein $Q$ with respect to the element $x_0$, then the slope $L \times_\alpha B$ is a derived slope of the $\text{SQS}$-skein $Q \times_\alpha B$ with respect to $(x_0, 1)$. \hfill $\square$

Note that $Q$ is isomorphic to the homomorphic image of $Q \times_\alpha B$ over its monolith (Corollary 5) and also $L$ is isomorphic to the homomorphic image of $L \times_\alpha B$ over its monolith [7]. Hence according to [7], Theorem 4 and Corollary 6, we may say that:

There is always an $\text{SQS}$-skein $Q \times_\alpha B$ with a derived slope $L \times_\alpha B$, in which both $Q \times_\alpha B$ and $L \times_\alpha B$ are subdirectly irreducible of cardinality $n2^m$ having the same congruence lattice for each possible integers $n \geq 4$ and $m \geq 2$. 

The construction of a semi-Boolean SQS-skein (each derived sloo\(L\) of \(Q\) is Boolean) given in [9] satisfies that \(C(Q)\) is a proper sublattice of the congruence lattice of its derived sloo\(L\). This means that we may begin with SQS-skein \(Q\) with a derived sloo\(L\) in which the congruence lattice of \(Q\) is a proper sublattice of the congruence lattice of \(L\), this leads to \(C(L \times_\alpha B)\) is a proper sublattice of \(C(Q \times_\alpha B)\).

Consequently, we may construct SQS-skein \(Q \times_\alpha B\) with a derived sloo\(L \times_\alpha B\) such that \(Q \times_\alpha B\) and \(L \times_\alpha B\) are subdirectly irreducible of cardinality \(n2^m\) and have the same congruence lattice, if we begin with \(L\) derived sloo of \(Q\) with the same congruence lattice. Also, we may construct SQS-skein \(Q \times_\alpha B\) with a derived sloo\(L \times_\alpha B\) in which the congruence lattice of \(Q \times_\alpha B\) is a proper sublattice of the congruence lattice of \(L \times_\alpha B\), if we begin with \(L\) derived sloo of \(Q\) such that the congruence lattice of \(Q\) is a proper sublattice of the congruence lattice of \(L\).

References


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