Essential operations of clones

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Abstract. Clones of algebras consist not only of essential operations but also of operations not depending on every variable. However, the sets of all essential operations of clones uniquely determine the clones. In this note we present a short precise proof of this fact and indicate these essential operations that are equal to inessential elements of clones.

1. Introduction

In the last century research in the theory of finite automata and deterministic operators led to problems concerning essential variables of functions. From that time the theory of essential variables of finite operations became a quite frequent research direction. The study of essential variables in functions defined on finite sets, initiated by A. Salomaa in [11], goes with multiple-valued logic and currently plays an important role in computer sciences. Essential variables of functions and essential term operations of algebras were widely studied under different aspects, see e.g. [1]–[6], [8],[9], [12],[13].

The clone of a given algebra consists of all its term operations – it contains both essentially n-ary term operations as well as term operations not depending on every variable. But the clone is uniquely determined by the set of all its constants and essential operations. This fact is sometimes assumed as intuitive, since every term operation not depending on every its variable can be obtained by adding inessential variables to an essential operation. However, this argumentation is imprecise and it cannot be regarded as sufficient, especially when the essential operation equal to a given inessential one has to be indicated, as e.g. in [10]. Therefore we decided to give in this note a short precise argument that clones of algebras are determined only by constants and essentially n-ary term operations. We indicate these essential elements of clones that are equal to the elements not depending on every variable.

By an *algebra* we mean a pair $\mathfrak{A} = (A; F^{\mathfrak{A}})$, where A is a nonempty set and $F^{\mathfrak{A}}$ is a family of mappings $f^{\mathfrak{A}}\colon A^n \to A$ called fundamental operations of $\mathfrak{A}$. The number n is called the *arity* of $f^{\mathfrak{A}}$. A *type* of algebras we define as a mapping $\tau\colon F \to \mathbb{N} \cup \{0\}$, where F is a nonempty set of fundamental operation symbols and $\mathbb{N}$ is the set of positive integers. An algebra is said to be of type τ if it is of

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the form $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$, where $F^{\mathfrak{A}} = \{f^{\mathfrak{A}}\colon\, f \in F\}$, and the arity of $f^{\mathfrak{A}}$ equals $\tau(f)$ for every $f \in F$.

Let an algebra $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$ of type τ be given. Recall that for every $1 \leqslant i \leqslant n$, the i-th n-ary *projection* is the mapping $(a_1, \ldots, a_n) \mapsto a_i$. It is usually denoted by $e_i^n(x_1, \ldots, x_n) = x_i$. The smallest set containing all projections and all elements of $F^{\mathfrak{A}}$ that is closed under superpositions is called the set of *term operations* of $\mathfrak{A}$, or the *clone* of $\mathfrak{A}$. We denote it by $Cl(\mathfrak{A})$. An n-ary term operation $f^{\mathfrak{A}} \in Cl(\mathfrak{A})$ *depends* on the variable x_i, if there exist elements $a_1, \ldots, a_n, b \in A$ such that

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The number of essential variables in $f^{\mathfrak{A}}$ is called the *essential arity* of $f^{\mathfrak{A}}$. If the term operation $f^{\mathfrak{A}}$ depends on every of its variable, then it is said to be *essentially* n-ary, or an *essential operation* of $\mathfrak{A}$. Otherwise $f^{\mathfrak{A}}$ is called *inessential*.

Following [6], for an algebra $\mathfrak{A}$ and every positive integer n, $\mathbb{P}_n(\mathfrak{A})$ denotes the set of all essentially n-ary term operations of $\mathfrak{A}$. $\mathbb{P}_0(\mathfrak{A})$ denotes the set of all constant non-nullary term operations of $\mathfrak{A}$ and all its nullary operations.

2. The result

Let an algebra $\mathfrak{A} = (A;\, F^{\mathfrak{A}})$ of type τ be given. For an n-ary term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n) \in Cl(\mathfrak{A})$ and a permutation σ of $1, \ldots, n$, define

$$f_\sigma^{\mathfrak{A}}(x_1, \ldots, x_n) = f^{\mathfrak{A}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Recall the following two simple observations. They are both easily provable by induction on the complexity of term operation, see also [7], § 8.

(2.i) *Let $n > 1$. For every n-ary term operation $f^{\mathfrak{A}} \in Cl(\mathfrak{A})$, there exists an $(n-1)$-ary term operation $g^{\mathfrak{A}} \in Cl(\mathfrak{A})$ such that*

$$f^{\mathfrak{A}}(a_1, \ldots, a_{n-1}, a_{n-1}) = g^{\mathfrak{A}}(a_1, \ldots, a_{n-1})$$

for all $a_1, \ldots, a_{n-1} \in A$.

(2.ii) *If an n-ary term operation $f^{\mathfrak{A}} \in \mathbb{P}_n(\mathfrak{A})$, then also $f_\sigma^{\mathfrak{A}} \in \mathbb{P}_n(\mathfrak{A})$ for every permutation σ of $1, \ldots, n$.*

Then we have the following.

Lemma. *For a given algebra $\mathfrak{A}$, if a term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n)$ depends only on the variables $x_1, \ldots, x_k$ for some $k < n$, then there exists a term operation $(f^*)^{\mathfrak{A}}(x_1, \ldots, x_k) \in \mathbb{P}_k(\mathfrak{A})$ such that*

$$f^{\mathfrak{A}}(x_1, \ldots, x_n) = (f^*)^{\mathfrak{A}}\big(e_1^n(x_1, \ldots, x_n), \ldots, e_k^n(x_1, \ldots, x_n)\big),$$

where $e_i^n(x_1, \ldots, x_n) = x_i$ for every $i = 1, \ldots, k$.

Proof. Consider a term operation $f^{\mathfrak{A}}(x_1, \ldots, x_n) \in Cl(\mathfrak{A})$ that depends on $x_1, ..., x_k$ for some $k < n$. From (2.i), there exists a k-ary term operation $(f^*)^{\mathfrak{A}} \in Cl(\mathfrak{A})$ such that

$$(f^*)^{\mathfrak{A}}(a_1, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, \ldots, a_k)$$

for every $a_1, \ldots, a_k \in A$. We shall prove that $(f^*)^{\mathfrak{A}}$ is essentially k-ary. Indeed, since $f^{\mathfrak{A}}(x_1, \ldots, x_n)$ depends on x_i for every $i = 1, \ldots, k-1$, there exist elements $a_1, \ldots, a_i, \ldots, a_n, b_i \in A$ such that

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \quad \neq \quad f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n).$$

Since $f^{\mathfrak{A}}$ does not depend on x_j for $j > k$, so we have

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k, a_{k+1}, \ldots, a_n) = \\ f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k, a_k, \ldots, a_k)$$

and

$$f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k, a_{k+1}, \ldots, a_n) = \\ f^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k, a_k, \ldots, a_k),$$

and consequently

$$(f^*)^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \neq (f^*)^{\mathfrak{A}}(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k)$$

for every $i = 1, \ldots, k-1$. Therefore the term operation $(f^*)^{\mathfrak{A}}$ depends on x_i for every $i < k$. Moreover, since $f^{\mathfrak{A}}$ depends also on x_k, we have

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_n) \quad \neq \quad f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, c_{k+1}, \ldots, c_n)$$

for some elements $c_1, \ldots, c_n, d_k \in A$. But $f^{\mathfrak{A}}$ does not depend on x_j for every $j > k$, so we have

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_{k+1}, \ldots, c_n) \quad = \quad f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k, c_k, \ldots, c_k)$$

and

$$f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, c_{k+1}, \ldots, c_n) \quad = \quad f^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k, d_k, \ldots, d_k)$$

and consequently

$$(f^*)^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, c_k) \quad \neq \quad (f^*)^{\mathfrak{A}}(c_1, \ldots, c_{k-1}, d_k).$$

Thus $(f^*)^{\mathfrak{A}}(x_1, \ldots, x_k) \in \mathbb{P}_k(\mathfrak{A}) \subset Cl(\mathfrak{A})$. Finally, let $(f^{**})^{\mathfrak{A}}$ denote the term operation obtained from $(f^*)^{\mathfrak{A}}$ by substituting every its variable x_i for the n-ary projection $e_i^n(x_1, \ldots, x_n)$ for every $i = 1, \ldots, k$. We have

$$(f^{**})^{\mathfrak{A}}(x_1, \ldots, x_n) \quad = \quad (f^*)^{\mathfrak{A}}\big(e_1^n(x_1, \ldots, x_n), \ldots, e_k^n(x_1, \ldots, x_n)\big).$$

Note that for every $a_1, \ldots, a_n \in A$ we have

$$(f^{**})^{\mathfrak{A}}(a_1, \ldots, a_n) = (f^*)^{\mathfrak{A}}\big(e_1^n(a_1, \ldots, a_n), \ldots, e_k^n(a_1, \ldots, a_n)\big) = (f^*)^{\mathfrak{A}}(a_1, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, a_k, \ldots, a_k)$$

and since $f^{\mathfrak{A}}$ does not depend on x_j for any $j > k$, we obtain

$$f^{\mathfrak{A}}(a_1, \ldots, a_k, a_k, \ldots, a_k) = f^{\mathfrak{A}}(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$$

and consequently

$$(f^{**})^{\mathfrak{A}}(x_1, \ldots, x_n) = f^{\mathfrak{A}}(x_1, \ldots, x_n),$$

completing the proof. □

Theorem. *Let* $\mathfrak{A}_1 = (A; F^{\mathfrak{A}_1})$ *and* $\mathfrak{A}_2 = (A; G^{\mathfrak{A}_2})$ *be algebras of types* τ_1 *and* τ_2*, respectively. Then* $Cl(\mathfrak{A}_1) = Cl(\mathfrak{A}_2)$ *if and only if* $\mathbb{P}_n(\mathfrak{A}_1) = \mathbb{P}_n(\mathfrak{A}_2)$ *for every* $n \in \mathbb{N} \cup \{0\}$.

In another words, the clone $Cl(\mathfrak{A})$ of a given algebra $\mathfrak{A}$ is uniquely determined by the subset of $Cl(\mathfrak{A})$ consisting of all term operations depending on every variable and all constant operations.

Proof. The necessity of the theorem is obvious. For the proof of sufficiency assume that $\mathbb{P}_n(\mathfrak{A}_1) = \mathbb{P}_n(\mathfrak{A}_2)$ for every nonnegative integer n. Let a mapping f be a nullary, constant non-nullary or essentially n-ary term operation of $\mathfrak{A}_1$. Then, by the assumption, $f \in \mathbb{P}_n(\mathfrak{A}_1)$ if and only if $f \in \mathbb{P}_n(\mathfrak{A}_2)$ for some $n \in \mathbb{N} \cup \{0\}$. Let $f^{\mathfrak{A}_1}(x_1, \ldots, x_n) \in Cl(\mathfrak{A}_1)$ be a term operation depending only on k, $k < n$, its variables. Consider a term operation $f_\sigma^{\mathfrak{A}_1}(x_1, \ldots, x_n) = f^{\mathfrak{A}_1}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for a permutation $\sigma \in \mathbb{S}_n$ such that $f_\sigma^{\mathfrak{A}_1}$ depends on $x_1, \ldots, x_k$. From (2.ii), $f^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_1)$ implies that $f_\sigma^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_1)$. Then, from Lemma, there exists a term operation $(f_\sigma^*)^{\mathfrak{A}_1} \in \mathbb{P}_k(\mathfrak{A}_1)$ such that

$$(f_\sigma^*)^{\mathfrak{A}_1}(a_1, \ldots, a_k) = f_\sigma^{\mathfrak{A}_1}(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$$

for every $a_1, \ldots, a_n \in A$. But since $(f_\sigma^*)^{\mathfrak{A}_1}$ is essentially k-ary, so – by the assumption – $(f_\sigma^*)^{\mathfrak{A}_1}$ belongs also to the set $\mathbb{P}_k(\mathfrak{A}_2) \subseteq Cl(\mathfrak{A}_2)$ and hence $f_\sigma^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_2)$. Now, from (2.ii) again, $f^{\mathfrak{A}_1} \in Cl(\mathfrak{A}_2)$ and consequently the inclusion $Cl(\mathfrak{A}_1) \subseteq Cl(\mathfrak{A}_2)$ holds. The proof of the opposite inclusion is analogous. So, $Cl(\mathfrak{A}_1) = Cl(\mathfrak{A}_2)$, completing the proof. □

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