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Varieties of rectangular quasigroups

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Abstract. For the given variety $\mathbb{V}$ of quaisgroups, the class of all *rectangular* $\mathbb{V}$*–quasigroups* is defined as the class of all groupoids isomorphic to $L \times Q \times R$, where $Q \in \mathbb{V}$ and $L(R)$ is a left (right) zero semigroup. The identities axiomatizing the new class are given, proving that it is a variety in the language of the original variety.

1. Introduction

In the papers [6], [7] and [8], the so called rectangular loops and rectangular quasigroups were defined.

Definition 1.1. Groupoid is a *rectangular quasigroup* (*loop*) iff it is isomorphic to the direct product of a left zero semigroup, a quasigroup (loop) and a right zero semigroup.

Several different axiomatizations for both these structures were given and the problems of the axiomatization by independent systems of axioms were posed.

In their paper [5] M. Kinyon and J. D. Phillips solved these problems by giving the following axioms:

$(RQ1)$ $$x\backslash xx = x$$

$(RQ2)$ $$xx/x = x$$

$(RQ3)$ $$x(x\backslash y) = x\backslash xy$$

$(RQ4)$ $$(x/y)y = xy/y$$

$(RQ5)$ $$(x\backslash y)\backslash((x\backslash y) \cdot zu) = (x\backslash xz)u$$

$(RQ6)$ $$(xy \cdot (z/u))/(z/u) = x(yu/u)$$

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(*RL*) $$x\backslash x(y\backslash y) = (x/x)y/y$$

The system (RQ1)–(RQ6) axiomatizes rectangular quasigroups and, if we add (RL) to it, we get axioms for rectangular loops.

In this paper we give some new axiomatizations of rectangular loops. More importantly, if $\mathbb{V}$ is a quasigroup variety, we give an axiomatization of the variety of rectangular $\mathbb{V}$–quasigroups.

2. Axioms for rectangular $\mathbb{V}$–quasigroups

We need to adjust the types of (equational) quasigroups and left (right) zero semigroups. To achieve this we extend the language of groupoids with further operations.

Definition 2.1. Let $L = \{\cdot, \backslash, /\}$ be the language of quasigroups and M a further (possibly empty) set of operation symbols disjoint from L. The language $\hat{L} = L \cup M$ is *an extended language of quasigroups.*

The language $L_1 = \{\cdot, \backslash, /, e\}$, obtained from L by the addition of a single constant, is *the language of loops.*

Definition 2.2. A left (right) zero semigroup is an algebra in $\hat{L}$ satisfying identities $x\backslash y = x/y = xy$ and $xy = x(xy = y)$.

Definition 2.3. Let $\mathbb{V}$ be a variety of quasigroups in an extended language $\hat{L}$. An algebra in the language $\hat{L}$ is a *rectangular* $\mathbb{V}$*–quasigroup* if it is isomorphic to the direct product of a left zero semigroup, a quasigroup from the variety $\mathbb{V}$ and a right zero semigroup.

There are three exceptions to the definition above. In the Section 3 (4) we consider rectangular left (right) symmetric quasigroups which have only two binary operations. But in that case one of the division operations coincide with multiplication, so this algebra is equivalent to the (proper) rectangular left (right) symmetric quasigroup with three binary operations. Similarly, for TS–quasigroups in which both division operations are equal to multiplication, rectangular TS–quasigroups are just special groupoids.

Theorem 2.4. *Let $\mathbb{V}$ be a variety of quasigroups satisfying additional identities $s_i = t_i$ $(i \in I)$ in an extended language $\hat{L}$ and let x be a variable which does not occur in either s_i or t_i. Then the variety $\Box\mathbb{V}$ of rectangular $\mathbb{V}$–quasigroups can be axiomatized by (RQ1)–(RQ6) together with (for all $i \in I$):*

(V_i) $$x \cdot s_i x = x \cdot t_i x \ .$$

Proof. Left (right) zero semigroups as well as all $\mathbb{V}$–quasigroups satisfy (RQ1)–(RQ6) and all (V_i) $(i \in I)$. So do their direct products i.e. rectangular $\mathbb{V}$–quasigroups.

If an algebra satisfies (RQ1)–(RQ6) then it is a rectangular quasigroup. Since all (V_i) are satisfied, the quasigroup factor has to satisfy them too. But in quasigroups identities (V_i) are equivalent to $s_i = t_i$ and these define $\mathbb{V}$. □

Theorem 2.5. *Theorem* 2.4 *remains valid if we replace* (V_i) *by any of the following identities:*

$$x \circ (s_i \diamond x) = x \circ (t_i \diamond x)$$

$$(x \circ s_i) \diamond x = (x \circ t_i) \diamond x$$

$$x/(s_i \backslash x) = (x/t_i)\backslash x$$

$$x \circ (s_i \diamond y) = x \circ (t_i \diamond y)$$

$$(x \circ s_i) \diamond y = (x \circ t_i) \diamond y$$

where x, y *do not occur in* s_i, t_i *and* $\circ, \diamond \in \{\cdot, \backslash, /\}$.

Proof. In the proof of Theorem 2.4 we can replace any (V_i) by some of the above identities which are, in quasigroups, equivalent to $s_i = t_i$. The line of reasoning remains the same. □

Definition 2.6. $head(t)(tail(t))$ is the first (last) variable of the term t.

Theorem 2.7. *The equality* $u = v$ *is true in all rectangular* $\mathbb{V}$*–quasigroups iff* $head(u) = head(v)$, $tail(u) = tail(v)$ *and* $u = v$ *is true in all* $\mathbb{V}$*–quasigroups.*

Proof. In one direction the theorem is true because projections are epimorphisms and so preserve identities. The converse is true because direct products also preserve identities. □

Theorem 2.8. *Theorem* 2.4 *remains valid if we replace* (V_i) *by any of the following identities:*

$$s_i \circ x = t_i \circ x \quad (\textit{if } head(s_i) = head(t_i))$$

$$x \circ s_i = x \circ t_i \quad (\textit{if } tail(s_i) = tail(t_i))$$

$$s_i = t_i \quad (\textit{provided both } head(s_i) = head(t_i) \textit{ and } tail(s_i) = tail(t_i))$$

where x *does not occur in* s_i, t_i *and* $\circ \in \{\cdot, \backslash, /\}$. □

Example 2.9. Adding associativity $x \cdot yz = xy \cdot z$ to identities $(RQ1)$–$(RQ6)$ gives yet another axiomatization of *rectangular groups.* □

Example 2.10. Adding identity $x \cdot yx = x \cdot zx$ to (RQ1)–(RQ6) gives a (way too complicated) axiomatization of *rectangular bands.* □

Example 2.11. Rectangular commutative quasigroups have identities (RQ1) – (RQ6) and $x(yz \cdot x) = x(zy \cdot x)$ as axioms. □

However, note that commutative rectangular quasigroups are just commutative quasigroups.

Example 2.12. Rectangular medial quasigroups are axiomatized by $(RQ1)$–$(RQ6)$ and $xy \cdot uv = xu \cdot yv$. □

Example 2.13. Commutative medial quasigroups are characterized by the axiom $xy \cdot uv = uy \cdot xv$ (among others). *Rectangular commutative medial quasigroups* are rectangular quasigroups satisfying $x(yz \cdot uv) = x(uz \cdot yv)$. □

Example 2.14. Paramedial quasigroups are characterized by the identity $xy \cdot uv = vy \cdot ux$. *Rectangular paramedial quasigroups* are axiomatized by adding identity $x \cdot (yz \cdot uv)x = x \cdot (vz \cdot uy)x$ to (RQ1)–(RQ6). □

It is rather obvious that the following corollaries are true:

Corollary 2.15. *If the variety $\mathbb{V}$ of quasigroups is defined by the identities $s_i = t_i$ $(i \in I)$ such that $head(s_i) = head(t_i)\,, tail(s_i) = tail(t_i)$ for all $i \in I$, then the class of rectangular quasigroups satisfying all identities $s_i = t_i$ $(i \in I)$ is the class of all rectangular $\mathbb{V}$–quasigroups.* □

Corollary 2.16. *If the variety $\mathbb{V}$ of quasigroups is defined by the identities $s_i = t_i$ $(i \in I)$ such that $head(s_i) \neq head(t_i)$ and $tail(s_j) \neq tail(t_j)$ for some $i, j \in I$, then the class of rectangular quasigroups satisfying all identities $s_i = t_i$ $(i \in I)$ is just the class of all $\mathbb{V}$–quasigroups.* □

Example 2.17. Moufang loops are defined as loops satisfying any of the four equivalent identities:

$$xy \cdot zx = (x \cdot yz)x$$

$$x(yz \cdot x) = xy \cdot zx$$

$$x(y \cdot xz) = (xy \cdot x)z$$

$$x(y \cdot zy) = (xy \cdot z)y.$$

K. Kunen recently proved in [9] that the existence of the neutral element follows from any of these identities. Therefore, *rectangular Moufang loops* are axiomatized by $(RQ1)$–$(RQ6)$ and for example $xy \cdot zx = (x \cdot yz)x$. □

Example 2.18. Let $(S; \cdot)$ and $(T; \circ)$ be groupoids and $f, g, h : S \longrightarrow T$ three bijections. If $f(xy) = g(x) \circ h(y)$ we say that $(T; \circ)$ is an *isotope* of $(S; \cdot)$. Isotopy is an important invariant of quasigroups which generalizes isomorphism. □

The result that every quasigroup is an isotope of some loop is a classical one in quasigroup theory. The class of all isotopes of groups is also significant and constitutes a variety of quasigroups as proved by V. D. Belousov in [1]. The defining identity of group isotopes is

$$x(y\backslash(z/u)v) = (x(y\backslash z)/u)v. \tag{2.1}$$

By the theorem 2.8 the axioms for the class of all *rectangular group isotopes* are $(RQ1)$–$(RQ6)$ and (2.1).

Note that the class of all *isotopes of rectangular groups* is strictly greater than the class of all rectangular group isotopes. Namely, if $S = \{0, 1\}, f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $xy = f(x)$, then $(S; \cdot)$ is an isotope of the left zero semigroup with two elements but is not a rectangular quasigroup. □

Example 2.19. The variety of *rectangular quasigroups with an idempotent* may be axiomatized by $(RQ1)$–$(RQ6)$ and $ee = e$. □

Example 2.20. The variety of *rectangular left loops* is axiomatized by $(RQ1)$–$(RQ6)$ and any of the following 37 identities:

$$x \circ ((y/y) \diamond x) = x \circ ((z/z) \diamond x)$$

$$(x \circ (y/y)) \diamond x = (x \circ (z/z)) \diamond x$$

$$x/((y/y)\backslash x) = (x/(z/z))\backslash x$$

$$x \circ ((y/y) \diamond u) = x \circ ((z/z) \diamond u)$$

$$(x \circ (y/y)) \diamond u = (x \circ (z/z)) \diamond u$$

where $\circ, \diamond \in \{\cdot, \backslash, /\}$. □

Example 2.21. If the variety of left loops is defined in the language of loops i.e. by the identity $ex = x$, then the variety of rectangular left loops is axiomatized by $(RQ1)$–$(RQ6)$ and

$$x \cdot ey = xy. \tag{2.2}$$

Example 2.22. The variety of rectangular loops is axiomatized by $(RQ1)$–$(RQ6)$ and any of the identities from the Example 2.20, together with the dual of one of them (to ensure the existence of a right neutral in quasigroup). However, we can apply the Theorem 2.5 to the single identity $y\backslash y = z/z$ which axiomatizes loops within quasigroups, and add any of the following identities to $(RQ1)$–$(RQ6)$ to obtain axioms for rectangular loops.

$$x \circ ((y\backslash y) \diamond x) = x \circ ((z/z) \diamond x)$$

$$(x \circ (y\backslash y)) \diamond x = (x \circ (z/z)) \diamond x$$

$$x/((y\backslash y)\backslash x) = (x/(z/z))\backslash x$$

$$x/((y/y)\backslash x) = (x/(z\backslash z))\backslash x$$

$$x \circ ((y\backslash y) \diamond u) = x \circ ((z/z) \diamond u)$$

$$(x \circ (y\backslash y)) \diamond u = (x \circ (z/z)) \diamond u$$

where $\circ, \diamond \in \{\cdot, \backslash, /\}$. This gives us a total of 1407 axiom systems for rectangular loops. □

Example 2.23. In the language of loops, the variety of rectangular loops can be axiomatized by $(RQ1)$–$(RQ6)$, (2.2) and

$$xe \cdot y = xy \tag{2.3}$$

The identity (2.2) may be replaced by any of identities from the Example 2.20. Likewise, the identity (2.3) may be replaced by the dual of some of these identities. This gives us 75 further axiomatizations of rectangular loops.

However, it should be admitted that the axiom system of Kinyon and Phillips is shorter (smaller language and/or less identities and/or less variables and/or less symbols) and more appealing then any of the above 1482 systems. The only exception is perhaps the system with identities (2.2) and (2.3). □

3. Rectangular left symmetric quasigroups

The important class of *left symmetric quasigroups* is characterized by the identity $x \cdot xy = y$. Just as in numerous examples in the previous section, we can axiomatize rectangular left symmetric quasigroups by identities $(RQ1)$–$(RQ6)$ and the identity

$$x(y \cdot yz) = xz \tag{LS}$$

as prescribed by the Theorem 2.8.

However, in this case we can do more. Note that by the Definition 2.2 $x\backslash y = xy$ in both left and right zero semigroups. In left symmetric quasigroups this is also true. Therefore, the identity $x\backslash y = xy$ is true in rectangular left symmetric quasigroups as well. But then the operation $\backslash$ can be eliminated from axioms and from the language itself. We have:

Theorem 3.1. *An algebra $(S; \cdot, /)$ is a rectangular left symmetric quasigroup iff it satisfies:*

$$x \cdot xx = x \tag{LS1}$$

$$xx/x = x \tag{LS2}$$

$$(x/y)y = xy/y \tag{LS3}$$

$$xy \cdot (xy \cdot uv) = (x \cdot xu)v \tag{LS4}$$

$$(xy \cdot (u/v))/(u/v) = x(yv/v). \tag{LS5}$$

Proof. Axiom $(RQ3)$ transforms into trivial identity and may be eliminated. Axioms $(RQ1)$ and $(RQ5)$ become axioms $(LS1)$ and $(LS4)$ respectively.

Only (LS) remains to be proved. We do it by the series of lemmas below. □

Lemma 3.2. $(x \cdot xy)z = x(x \cdot yz)$

Proof.
$$\begin{aligned} (x \cdot xy)z &= (x \cdot xx) \cdot ((x \cdot xx) \cdot yz) && \text{(by (LS4))} \\ &= x(x \cdot yz) && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.3. $xy \cdot (xy \cdot z) = x \cdot xz$

Proof.
$$\begin{aligned} xy \cdot (xy \cdot z) &= xy \cdot (xy \cdot (z \cdot zz)) && \text{(by (LS1))} \\ &= (x \cdot xz) \cdot zz && \text{(by (LS4))} \\ &= (x \cdot xx) \cdot ((x \cdot xx) \cdot (z \cdot zz)) && \text{(by (LS4))} \\ &= x \cdot xz && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.4. $x(x \cdot xy) = xy$

Proof.
$$\begin{aligned} x(x \cdot xy) &= (x \cdot xx)y && \text{(by Lemma 3.2)} \\ &= xy && \text{(by (LS1))} \end{aligned}$$
□

Lemma 3.5. $xy \cdot x(x \cdot zu) = xy \cdot zu$

Proof.
$$\begin{aligned} xy \cdot x(x \cdot zu) &= xy \cdot (x \cdot xz)u && \text{(by Lemma 3.2)} \\ &= xy \cdot (xy \cdot (xy \cdot zu)) && \text{(by (LS4))} \\ &= xy \cdot zu && \text{(by Lemma 3.4)} \end{aligned}$$
□

Lemma 3.6. $x \cdot x(y \cdot yz) = x \cdot xz$

Proof.
$$\begin{aligned} x \cdot x(y \cdot yz) &= (x \cdot xy) \cdot yz && \text{(by Lemma 3.2)} \\ &= (x \cdot xy) \cdot x(x \cdot yz) && \text{(by Lemma 3.5)} \\ &= (x \cdot xy) \cdot (x \cdot xy)z && \text{(by Lemma 3.2)} \\ &= x \cdot xz && \text{(by Lemma 3.3)} \end{aligned}$$
□

Lemma 3.7. $x(y \cdot yz) = xz$

Proof.
$$\begin{aligned} x(y \cdot yz) &= x(x \cdot x(y \cdot yz)) && \text{(by Lemma 3.4)} \\ &= x(x \cdot xz) && \text{(by Lemma 3.6)} \\ &= xz && \text{(by Lemma 3.4)} \end{aligned}$$
□

The proof above is an adaptation of the proof found by the automated reasoning program `Prover9`. `Prover9` is the first order logic theorem prover developed by W. W. McCune [11] which is capable of solving difficult mathematical problems. For instance, McCune in [10] solved the so called Robbins conjecture using `Otter` (an earlier version of `Prover9`). See [12] for the gentle introduction to `Otter` with the leaning to quasigroup theory.

McCune also wrote the model builder program `Mace4` [11], which is used in the following examples to verify the independence of the axioms $(LS1)$–$(LS5)$.

Example 3.8. Table 1 is the smallest model that satisfies (LS2), (LS3), (LS4), and (LS5), but not (LS1).

•	0	1
0	1	1
1	0	0

\	0	1
0	1	1
1	0	0

Table 1. (LS2), (LS3), (LS4) and (LS5), but not (LS1). □

Example 3.9. Table 2 is the smallest model that satisfies (LS1), (LS3), (LS4), and (LS5), but not (LS2).

$\bullet$	0	1
0	0	1
1	1	0

$\backslash$	0	1
0	1	0
1	0	1

Table 2. (LS1), (LS3), (LS4) and (LS5), but not (LS2). □

Example 3.10. Table 3 is the smallest model that satisfies (LS1), (LS2), (LS4), and (LS5), but not (LS3).

$\bullet$	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

$\backslash$	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

Table 3. (LS1), (LS2), (LS4) and (LS5), but not (LS3). □

Example 3.11. Table 4 is the smallest model that satisfies (LS1), (LS2), (LS3), and (LS5), but not (LS4).

$\bullet$	0	1	2
0	0	1	0
1	1	2	1
2	2	0	2

$\backslash$	0	1	2
0	0	2	0
1	1	0	1
2	2	1	2

Table 4. (LS1), (LS2), (LS3) and (LS5), but not (LS4). □

Example 3.12. Table 5 is the smallest model that satisfies (LS1), (LS2), (LS3), and (LS4), but not (LS5).

$\bullet$	0	1
0	0	0
1	1	1

$\backslash$	0	1
0	0	0
1	0	1

Table 5. (LS1), (LS2), (LS3) and (LS4), but not (LS5). □

4. Right symmetric quasigroups

Right symmetric quasigroups are defined by the identity $xy \cdot y = x$. From the Theorem 3.1 it follows, by the duality principle for groupoids (see [2]), that the class of all *rectangular right symmetric quasigroups* can be axiomatized by the identities:

$$x\backslash xx = x \tag{RS1}$$

$$xx \cdot x = x \tag{RS2}$$

$$x(x\backslash y) = x\backslash xy \tag{RS3}$$

$$(x\backslash y)\backslash((x\backslash y)\cdot uv) = (x\backslash xu)v \tag{RS4}$$

$$(xy\cdot uv)\cdot uv = x(yv\cdot v) \tag{RS5}$$

in the language $\{\cdot, \backslash\}$. Moreover, the axioms are mutually independent.

If a quasigroup satisfies both left and right symmetry identities, i.e. if both $x\cdot xy = y$ and $xy\cdot y = x$ are true, then such a quasigroup is called a *totally symmetric* or TS*–quasigroup*. TS–quasigroups are commutative and both division operations in them coincide with multiplication. Applying Theorem 3.1 and its dual we get:

Theorem 4.1. *A groupoid* $(S;\cdot)$ *is a rectangular* TS*–quasigroup iff*

$$x\cdot xx = x \tag{TS1}$$

$$xx\cdot x = x \tag{TS2}$$

$$xy\cdot(xy\cdot uv) = (x\cdot xu)v \tag{TS3}$$

$$(xy\cdot uv)\cdot uv = x(yv\cdot v). \tag{TS4}$$

Example 4.2. Table 6 is the smallest model that satisfies (TS2), (TS3) and (TS4), but not (TS1).

•	0	1
0	1	1
1	0	0

Table 6. (TS2), (TS3) and (TS4), but not (TS1).

•	0	1	2
0	0	2	0
1	1	1	1
2	2	0	2

Table 7. (TS1), (TS2) and (TS4), but not (TS3).

Example 4.3. Table 7 is the smallest model that satisfies (TS1), (TS2), and (TS4), but not (TS3). □

Independence of (TS2) and (TS4) is proved by models dual to those in Examples 4.2 and 4.3 respectively.

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