Some enumerational results relating the numbers of latin and frequency squares of order n

Francis N. Castro, Gary L. Mullen and Ivelisse Rubio

Abstract We discuss some enumerational results relating the numbers of $F(n;\lambda_1,...,\lambda_m)$ and $F(n;\lambda'_1,...,\lambda'_k)$ frequency squares of order n. In particular, for any frequency vector $(\lambda_1,...,\lambda_m)$ of n, we discuss some enumerational results relating the number of $F(n;\lambda_1,...,\lambda_m)$ frequency squares and the number of latin squares of order n. In Section 4 we also discuss some enumerational results for latin rectangles.

1. Introduction

A *latin square of order* n is an $n\times n$ array in which each of the numbers $1,2,\dots,n$ appears exactly once in each row and each column. By an $F(n;\lambda_1,\dots,\lambda_m)$ *frequency square* is meant an $n\times n$ array in which each of the numbers i with $1\leqslant i\leqslant m$ appears exactly λ_i times in each row and each column. Thus we have $n=\lambda_1+\cdots+\lambda_m$ and an $F(n;1,\dots,1)$ frequency square is a latin square of order n.

Let $\mathcal{F}(n;\lambda_1,\dots,\lambda_m)$ denote the total number of distinct $F(n;\lambda_1,\dots,\lambda_m)$ frequency squares and let $f(n;\lambda_1,\dots,\lambda_m)$ represent the number of reduced squares where a frequency square as above is reduced if the first row and first column are both in standard order with λ_1 1's, λ_2 2's, and continuing, λ_m m's.

It is known from [1] that

Theorem 1.1. *For any frequency vector* $(\lambda_1,\dots,\lambda_m)$ *of* n

$$\mathcal{F}(n;\lambda_1,\dots,\lambda_m)=\binom{n}{\lambda_1,\dots,\lambda_m}\binom{n-1}{\lambda_1-1,\dots,\lambda_m}f(n;\lambda_1,\dots,\lambda_m). \qquad \square$$

See [9] for some enumerational and classification results concerning latin squares. Let L_n denote the total number of latin squares of order n and let l_n denote the number of reduced latin squares of order n. It is known ([2], page 142) and easy to prove that

Corollary 1.2. *For* $n\geqslant 2$, $L_n=n!(n-1)!l_n$. $\square$

2010 Mathematics Subject Classification: 05B15.
Keywords: Latin square, frequency square, latin rectangle.

In this paper we prove several results relating the total number L_n of distinct latin squares of order n and the number of frequency squares with a fixed frequency vector. We also prove results relating the numbers of frequency squares of order n with two different frequency vectors.

It is known (see for example [8], Thm. 7.1) that a latin square of order n is equivalent to a 1-factorization of $K_{n,n}$, a bipartite graph in which each vertex of U is joined to each vertex of W, where U, W represent the rows and columns of a latin square of order n so that both U and W contain exactly n elements. If the symbol in position (i, j) is k, then we color the edge from i to j with color k. See page 107 of [8] for more details.

Now let $\vec{K_n}$ (see page 111 of [8]) be the complete directed graph with loops on n vertices. Then in Cor. 7.10 of [8] it is shown that the number of latin squares of order n with first row in standard order is the same as the number of 1-factorizations of $\vec{K_n}$. Also see [5] for connections between enumerating certain frequency squares and 1-factorizations of certain graphs.

Thus one can certainly show that counting latin squares can be done by counting 1-factorizations of an appropriate graph. In our paper we are not just counting or enumerating frequency squares, rather we are showing how to enumerate frequency squares with one frequency vector relative to the number of frequency squares with a different frequency vector. This is the main point of the current paper.

In [10] Wanless considers k-plexes for latin squares. Such objects are generalizations of transversals in latin squares. Many of our results could be stated using the terminlogy of k-plexes, but we prefer to use terminology involving i-transversals that is defined in the next section.

In [6] it was shown in Theorem 3.1 that one could relate the number of latin squares of order n to the number of 1-factorizations of frequency squares with frequency vector $\lambda_1, ..., \lambda_m$ via the use of isotopy classes. While the result in that paper is valid, the proof was incomplete in that it assumed (without proof) that each frequency square in an isotopy class had the same number of 1-factorizations. While this fact turns out to be true, it does require some proof. This proof is now given in Lemma 2.1 of the current paper.

In this paper we also extend the result from equation (2) in [6] dealing with latin and frequency squares, to the case where we relate the number of frequency squares with one frequency vector to the number of frequency squares with a different frequency vector.

2. Numbers of frequency and latin squares

Let $F(n; \lambda_1, \ldots, \lambda_m)$ be a frequency square of order n with frequency vector $(\lambda_1, \ldots, \lambda_m)$. For $i = 1, \ldots, m$, by an *i-transversal* is meant a set of n cells, one in each row and one in each column, each containing the symbol i. A set of n transversals containing λ_i, i-transversals for each $i = 1, \ldots, m$, forms a *partition* of the frequency square if for each i, the i-transversals disjointly partition the set of $n\lambda_i$ cells containing i. We define an *i-partition* to be the subset of a partition consisting of all i-transversals in the partition.

As in [1] two frequency squares F_1 and F_2 of the same order and frequency vector, are said to be *isotopic* if there exist permutations $\sigma_r, \sigma_c, \sigma_\#$ so that F_2 can be obtained from F_1 by applying σ_r to the rows of F_1, and then successively applying σ_c to the columns and $\sigma_\#$ to the numbers of each resulting square, respectively.

We now prove that frequency squares from the same isotopy class yield exactly the same number of partitions. This will greatly reduce our calculations which will of course be very helpful for larger values of n.

Lemma 2.1. *Assume that two frequency squares F_1 and F_2 (of the same order n and frequency vector) are isotopic. Then the number of partitions of F_1 is the same as the number of partitions of F_2.*

Proof. Let F_1 and F_2 be frequency squares of order n with the same frequency vector. Suppose that F_1 and F_2 are *isotopic*. Fix permutations σ_r, σ_c and $\sigma_\#$ and define a function from the set of partitions of F_1 to the set of partitions of F_2 by applying $\sigma_r, \sigma_c, \sigma_\#$ to the transversals of the partitions. Let F_1^r be the frequency square obtained after we apply σ_r to F_1. Given an i-transversal $\{(1, i_1), (2, i_2), \dots, (n, i_n)\}$ of F_1 and applying σ_r to the i-transversal we obtain

$$\{(\sigma_r(1), i_1), \dots, (\sigma_r(n), i_n)\},$$

an i-transversal of F_1^r. Let F_1^c be the frequency square obtained after we apply σ_c to F_1^r. Given an i-transversal $\{(1, i_1), (2, i_2), \dots, (n, i_n)\}$ of F_1^r and applying σ_c to the i-transversal, we obtain $\{(1, \sigma_c(i_1)), \dots, (n, \sigma_c(i_n))\}$, an i-transversal of F_1^c. Let $F_1^\#$ be the frequency square obtained after we apply $\sigma_\#$ to F_1^c. Note that $F_2 = F_1^\#$ for some $r, c, \#$. Given an i-transversal $\{(1, i_1), (2, i_2), \dots, (n, i_n)\}$ of F_1^c we obtain the $\sigma_\#(i)$-transversal $\{(1, i_1), \dots, (n, i_n)\}$ of F_2. Hence $\sigma_r, \sigma_c, \sigma_\#$ take a transversal of F_1 to a transversal of F_2.

Let $A = \{(1, i_1), \dots, (n, i_n)\} \neq B = \{(1, j_1), \dots, (n, j_n)\}$ be two distinct i-transversals of F_1. We claim that applying σ_r, σ_c, or $\sigma_\#$ to A and B we obtain distinct transversals. Suppose that $\sigma_c(A) = \{(1, \sigma_c(i_1)), \dots, \ (n, \sigma_c(i_n))\} = \sigma_c(B) = \{(1, \sigma_c(j_1)), \cdots, (n, \sigma_c(j_n))\}$. Then $\sigma_c(i_k) = \sigma_c(j_k)$ for $k = 1, \dots, n$. This implies that $i_k = j_k$ for $k = 1, \dots, n$, contradicting the fact that $A \neq B$. The same can be proved for σ_r and $\sigma_\#$. We also claim that if $A \cap B = \emptyset$, then $\sigma_c(A) \cap \sigma_c(B) = \emptyset$. Suppose not. Then $(k, \sigma_c(i_k)) = (k, \sigma_c(j_k))$ for some $k = 1, \dots, n$. Then $i_k = j_k$, contradicting that $A \cap B = \emptyset$. The same can be proved for σ_r and $\sigma_\#$. Hence, applying $\sigma_r, \sigma_c, \sigma_\#$ to a partition of F_1 we obtain a partition of F_2.

The above shows that $\sigma_\# \circ \sigma_c \circ \sigma_r$ is a well defined function between the sets of partitions of F_1 and F_2. This implies that the number of partitions of F_1 is less than or equal to the number of partitions of F_2. But we can repeat the same process starting with F_2 and we obtain that the number of partitions of F_2 is less than or equal to the number of partitions of F_1. Therefore, the number of partitions of F_1 and F_2 are equal. $\square$

It is clear from the previous proof that permutations of rows and columns take an i-transversal to another i-transversal. These permutations also take different i-transversals into different i-transversals; hence the number of i-transversals is preserved by permutations of rows and columns as the next lemma states.

Lemma 2.2. *Let F_1 and F_2 be frequency squares of the same order and frequency vector. Suppose that F_2 can be obtained from F_1 by successively applying permutations of rows and columns. Then, F_1 and F_2 have the same number of i-transversals.* □

Remark 1. Note that permutations $\sigma_{\#}$ of symbols of a frequency square take i-transversals to $\sigma_{\#}(i)$-transversals and therefore it is false in general that the number of i-transversals of frequency squares belonging to the same isotopy class is fixed, as it is shown in the next example.

Example 2.3. Considere the following reduced frequency squares with vector $(5; 2, 2, 1)$:

$$F_1 = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 3 & 1 & 1 \\ 2 & 3 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 \end{pmatrix}, \quad F_1' = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 3 & 1 & 2 & 2 \\ 2 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

The square F_1' can be obtained from square F_1 by interchanging entries $1 \leftrightarrow 2$ and permuting the rows and columns to convert it into a reduced square and hence the two squares are isotopic. It can be checked that F_1 has 2, 1-transversals and 4, 2-transversals, and F_1' has 4, 1-transversals and 2, 2-transversals. Note that $\sigma_{\#}(1) = 2$ and the number of 1-transversals of F_1 is the number of 2-transversals of F_1'. □

Let $\Lambda(n; \lambda_1, \ldots, \lambda_m)$ denote the number of distinct isotopy classes of frequency squares $F(n; \lambda_1, \ldots, \lambda_m)$. For a fixed frequency vector, from Theorem 1.1, we know that the number of isotopy classes of frequency squares is the same as the number of isotopy classes of reduced frequency squares. Assume that the j-th class contains n_j reduced squares so that

$$\sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j = f(n; \lambda_1, \ldots, \lambda_m). \tag{1}$$

We now prove

Theorem 2.4. *For any frequency vector $(\lambda_1, \ldots, \lambda_m)$ of n*

$$\binom{n}{\lambda_1, \ldots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} \lambda_1! \cdots \lambda_m! \tag{2}$$

$$= n!(n-1)!l_n = L_n,$$

where $\delta^{(j)}$ denotes the number of distinct partitions of any reduced frequency square $F(n; \lambda_1, \ldots, \lambda_m)$ in the j-th isotopy class of reduced squares which contains n_j reduced squares.

Proof. How many distinct latin squares of order n does the left hand side of (2) generate? Consider the j-th isotopy class. By Lemma 2.1 each frequency square in this class has the same number $\delta^{(j)}$ of partitions so consider a fixed reduced frequency square $F = F(n; \lambda_1, \ldots, \lambda_m)$ in this class. Using this reduced frequency square one can construct different latin squares in the following way.

Fix a partition P of F. For each 1-transversal in P, replace each value 1 in the cells given by the 1-transversal by a number k, $k = 1, \cdots, \lambda_1$, one number for each of the λ_1 1-transversals. Since the 1-transversals are disjoint, this gives $\lambda_1!$ different latin squares of order n. Similarly, for each 2-transversal of F, replace the number 2 by $\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2$. Doing the same for each $i = 1, \cdots, m$, the partition P generates $\lambda_1! \times \cdots \times \lambda_m!$ distinct latin squares of order n. Each of the $\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m}$ distinct frequency squares obtained by permuting rows and columns of F will also produce $\lambda_1! \times \cdots \times \lambda_m!$ latin squares.

Continuing, this can be repeated for each of the n_j reduced squares in the j-th isotopy class. Finally, we doing this for each class we get that the number of latin squares of order n generated from the left hand side will be at most L_n.

Conversely, given a latin square L_1 of order n, construct a frequency square $FS_1 = F_1(n; \lambda_1, \ldots, \lambda_m)$ in the following way: replace the numbers $1, 2, \ldots, \lambda_1$ in the latin square by 1, the numbers $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ by 2 and continuing, until the numbers $\lambda_1 + \cdots + \lambda_{m-1} + 1, \ldots, n$ by m.

Consider the $a_1, \ldots, a_{\lambda_1}$, 1-transversals forming a 1-partition of FS_1. Note that any latin square with the numbers $\lambda_1 + 1, \ldots, n$ in the same positions as L_1 and with a value i_1, $1 \leqslant i_1 \leqslant \lambda_1$ in the positions of a_1, a value $i_2 \neq i_1$, $1 \leqslant i_2 \leqslant \lambda_1$ in the positions of a_2 and so on gives FS_1 if we apply the above construction. There are $\delta_1(FS_1)\lambda_1!$ latin squares that give FS_1 under this construction, where $\delta_1(FS_1)$ is the number of 1-partitions of FS_1 and there are no other latin squares that give FS_1 under this construction. Something similar happens for all the other i-partitions. Let C_1 be the set of all these latin squares; this is, C_1 is the set of all the latin squares that give FS_1 under this construction. There are exactly $\delta_1(FS_1) \cdots \delta_m(FS_1)\lambda_1! \cdots \lambda_m!$ different latin squares in C_1, where $\delta_i(FS_1)$ is the number of i-partitions of FS_1.

Take another latin square of order n that it is not in C_1 and construct a frequency square FS_2 with the above construction. This gives another set C_2 of latin squares associated to FS_2. Repeat until we have a set $\{C_1, \cdots, C_k\}$ such that any latin square of order n belongs to a C_s and each C_s corresponds to a unique FS_s. We then have that

$$L_n = \sum_{s=1}^{k} |C_s| = \sum_{s=1}^{k} \delta^{(s)} \lambda_1! \cdots \lambda_m!$$

$$\leqslant \sum_{s=1}^{\mathcal{F}} \delta^{(s)} \lambda_1! \cdots \lambda_m! = \binom{n}{\lambda_1, \ldots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{s=1}^{f} \delta^{(s)} \lambda_1! \cdots \lambda_m!,$$

where $\mathcal{F}$ is the total number of frequency squares $F(n; \lambda_1, \ldots, \lambda_m)$, f is the total number of reduced frequency squares with the same frequency vector and $\delta^{(s)} = \delta_1(FS_s) \cdots \delta_m(FS_s)$ is the number of partitions of the frequency square FS_s.

Using (1) one can now sum over the isotopy classes of reduced frequency squares to see that $\delta^{(s)}$ coincides with $\delta^{(j)}$ in equation (2) and get that

$$L_n \leqslant \binom{n}{\lambda_1, \ldots, \lambda_m} \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} \lambda_1! \cdots \lambda_m!. \qquad \square$$

One can easily simplify the result of the theorem to obtain

Corollary 2.5. *For any frequency vector $(\lambda_1, \ldots, \lambda_m)$ of n*

$$n! \binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m} \sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j \delta^{(j)} = n!(n-1)! l_n = L_n,$$

where $\delta^{(j)}$ denotes the number of distinct partitions of any reduced frequency square $F(n; \lambda_1, \ldots, \lambda_m)$ in the j-th isotopy class which contains n_j reduced squares. $\square$

We note that results for the number of isotopy classes of frequency squares of order $n \leqslant 6$ can be found in [1] while results for orders 7 and 8 can be found in [7].

Example 2.6. For $n = 4$, from [1] there are five reduced $F(4; 2, 2)$ frequency squares and these are given by

$$F_1 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{matrix}, \quad F_2 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{matrix}, \quad F_3 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{matrix}$$

$$F_4 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{matrix}, \quad F_5 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{matrix}$$

Square	$\#1 - trans.$	$\#2 - trans.$	δ_j
F_1	4	4	4
F_2	2	2	1
F_3	2	2	1
F_4	2	2	1
F_5	2	2	1

Note that from [1], there are just two distinct isotopy classes; the first containing just the square F_1 while the second class contains the four squares $F_2, \ldots, F_5$. Hence our theorem yields

$$\binom{4}{2,2}\binom{3}{2,1}[4(2!)(2!)+4(2!)(2!)] = 6(3)(16+16) = 576 = 4!3!(4) = L_4. \qquad \square$$

Remark 2. The above results simplify considerably when there is only one isotopy class. This is the case for frequency squares $F(n; n-1, 1)$.

The next argument shows that there is only one isotopy class for $F(n; n-1, 1)$ frequency squares. Since each row and column contains only one 2 and the rest 1's, we can easily interchange rows and columns to show that every $F(n; n-1, 1)$ frequency square is isotopic to the square

$$\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 2 \\ 1 & 1 & \cdots & 2 & 1 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 2 & 1 & \cdots & 1 & 1 \end{array}$$

which has 2's on the back diagonal. It is easy to see that there are $(n-2)!$ reduced frequency squares of this type.

3. Enumerating frequency squares

In this section we enumerate frequency squares of certain frequency vectors using the number of i-transversals of frequency squares of a related frequency vector. We also give a formula to compute the number of 1-transversals of frequency squares $F(n; n-1, 1)$. As a consequence we can compute the number of frequency squares $F(n; n-2, 1, 1)$ for any $n \geqslant 3$. Let $F(n)$ be a frequency square of order n and let $T_i(F(n))$ be the number of i-transversals of $F(n)$.

Lemma 3.1. *Let $(\lambda_1, \ldots, \lambda_m, \underbrace{1, \ldots, 1}_{s})$ be a frequency vector of n where $\lambda_m \neq \lambda_j$ for all $j \neq m$, and let $\Lambda = \Lambda(n; \lambda_1, \ldots, \lambda_m, \underbrace{1, \ldots, 1}_{s})$ be the number of distinct isotopy classes of frequency squares associated to it. Then*

$$\binom{n}{\lambda_1, \ldots, \lambda_m}\binom{n-1}{\lambda_1 - 1, \ldots, \lambda_m}\sum_{j=1}^{\Lambda} n_j T_m(F_j(n)) \tag{3}$$
$$= \mathcal{F}(n; \lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1, \underbrace{1, \ldots, 1}_{s+1})$$

where $\lambda_m \geqslant 2$, $s \geqslant 0$, and $T_m(F_j(n))$ denotes the number of distinct m-transversals of any reduced frequency square $F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ in the j-th isotopy class of reduced frequency squares which contains n_j reduced squares.

Proof. Assume that $\lambda_m \neq \lambda_j$ for all $j \neq m$. This implies that the permutations used to construct the isotopy classes of the frequency vector $(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ do not include permutations $\sigma_\#$ of the symbol m because, if one apply the permutation $\sigma_\#(m)$, the resulting frequency square will have a different frequency vector and all the vectors in the isotopy class must have the same frequency vector. Hence, by Lemma 2.2 the number of m-transversals within an isotopy class is fixed.

Given a frequency square $FS^m = F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ we construct another frequency square $FS^{m-1} = F(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,1,1,\ldots,1)$ in the following way: consider an m-transversal of FS^m and replace the m's in the entries given by the m-transversal by the number $l = m+s+1$. Each of the $T_m(FS^m)$ different m-transversals of FS^m gives a different frequency square FS^{m-1}. The same can be done with each of the $T_m(F_j(n))$ m-transversals of the $\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}$ different frequency squares FS^m given by each of the n_j reduced frequency squares in the j-th isotopy class of FS^m. Hence,

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda} n_j T_m(F_j(n))$$

$$\leqslant \mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1})$$

Conversely, given a frequency square FS_1^{m-1} construct a frequency square FS_1^m by replacing the number $l = m+s+1$ by the number m. Any frequency square with the number i in the λ_i positions of FS_1^{m-1} for $i \neq m, l$ will produce the same frequency square FS_1^m. Let C_1 be the set of all the frequency squares FS^{m-1} that produce FS_1^m under the above construction. The number of squares FS^{m-1} in C_1 is the number of m-transversals of FS_1^m. Take another frequency square FS_2^{m-1} that it is not in C_1 and construct FS_2^m. This gives another set C_2, and, repeating the construction, we get a set $\{C_1,\cdots,C_k\}$, where each frequency square FS^{m-1} belongs to a C_i and each C_s corresponds to a unique FS^m. This gives

$$\mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1}) = \sum_{i=1}^{k}|C_i|$$

$$= \sum_{i=1}^{k} T_m(FS_i^m) \leqslant \sum_{i=1}^{\mathcal{F}} T_m(FS_i^m),$$

where $\mathcal{F}$ is the total number of frequency squares $F(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$. Since the number of m-transversals do not change with row and column permutations and the number of m-transversals does not change within the isotopy classes we have that

$$\begin{aligned}
&\mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1})\\
&\leqslant \binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{f} T_m(F_j(n))\\
&= \binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda} n_j T_m(F_j(n)),
\end{aligned}$$

where f is the number of reduced frequency squares with frequency vector of the form $(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ and n_j is the number of reduced squares in the j-th isotopy class. □

Example 3.2. The above lemma gives a way to compute $\mathcal{F}(8;6,1,1)$ using reduced frequency squares with frequency vector $(7,1)$. Namely, it is known that $f(n;n-1,1)=(n-2)!$ and, by Remark 2, there is only one isotopy class of frequency squares with frequency vector $(n-1,1)$. Hence

$$\mathcal{F}(8;6,1,1) = 8\times 7\times 6!\times T_1(8;7,1) = 598,066,560,$$

as reported in [7]. □

Example 3.3. In general, to compute $\mathcal{F}(n;n-2,1,1)$ using reduced frequency squares with frequency vector $(n-1,1)$, we need to compute $T_1(F(n;n-1,1))$, and then

$$\mathcal{F}(n;n-2,1,1) = n!\times T_1(F(n;n-1,1)).$$

Theorem 3.8 gives a formula to compute $\mathcal{F}(n;n-2,1,1)$ for any n. □

Remark 3. If $\lambda_m=\lambda_i$ for some i, then Lemma 3.1 is false. The reason is that one can interchange the numbers m and i in a frequency square to obtain another frequency square in the same isotopy class but both having different numbers of m-transversals. In fact, two reduced frequency squares in the same isotopy class can have have different m-transversals as we saw in Example 2.3. Therefore, in this case one cannot group the reduced squares in the isotopy class to get n_j in equation (3). However, if instead of summing over the isotopy classes, one sums over all the reduced frequency squares, one obtains a formula that works for any frequency vector as we see in Lemma 3.5.

Remark 4. Note that, since one can relabel $i\leftrightarrow m$, and interchange the positions of λ_m,λ_i, it is enough to have any λ_i be such that $\lambda_i\neq\lambda_j$ for all $j\neq i$.

Lemma 3.1 can be applied successively to obtain the following result.

Theorem 3.4. *Let* $(\lambda_1,\ldots,\lambda_l,\cdots,\lambda_m,\underbrace{1,\ldots,1}_{s})$ *be a frequency vector of* n *where* $\lambda_i \neq \lambda_j$ *for* $i = l,\cdots,m,\ j = 1,\cdots,m,$ *and let* Λ *be the number of distinct isotopy classes of reduced frequency squares associated to it. Then*

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda} n_j T_{l+1}(F_j(n))\cdots T_m(F_j(n))$$
$$= \mathcal{F}(n;\lambda_1,\ldots,\lambda_l,\lambda_{l+1}-1,\ldots,\lambda_{m-1}-1,\lambda_m-1,\underbrace{1,\ldots,1}_{s+m-l+1}),$$

where $\lambda_l \geqslant 2,\ldots,\lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_l(F_j(n))$ *denote the number of distinct* l*-transversals of any reduced frequency square* $F_j(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ *in the* j*-th isotopy class of reduced squares which contains* n_j *reduced squares.* □

Note that Lemma 3.1 requires $\lambda_m \neq \lambda_i$ for all $i \neq m$. Alternatively, one can sum over all the reduced frequency squares and then this assumption is not needed:

Lemma 3.5. *For any frequency vector* $(\lambda_1,\ldots,\lambda_m,\underbrace{1,\ldots,1}_{s})$ *of* n, *let* f *be the number of distinct reduced frequency squares with this frequency vector. Then*

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{f} T_m(F_j(n))$$
$$= \mathcal{F}(n;\lambda_1,\ldots,\lambda_{m-1},\lambda_m-1,\underbrace{1,\ldots,1}_{s+1})$$

where $\lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_m(F_j(n))$ *denotes the number of distinct* m*-transversals of the reduced frequency square* $F_j(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ *and the sum is over the* f *different reduced frequency squares.* □

Theorem 3.6. *For any frequency vector* $(\lambda_1,\ldots,\lambda_m,\underbrace{1,\ldots,1}_{s})$ *of* n, *let* f *be the number of distinct reduced frequency squares with this frequency vector. Then*

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{f} T_{l+1}(F_j(n))\cdots T_m(F_j(n))$$
$$= \mathcal{F}(n;\lambda_1,\ldots,\lambda_l,\lambda_{l+1}-1,\ldots,\lambda_{m-1}-1,\lambda_m-1,\underbrace{1,\ldots,1}_{s+m-l+1}),$$

where $\lambda_l \geqslant 2,\ldots,\lambda_m \geqslant 2,\ s \geqslant 0,$ *and* $T_l(F_j(n))$ *denote the number of distinct* l*-transversals of the reduced frequency square* $F_j(n;\lambda_1,\ldots,\lambda_m,1,\ldots,1)$ *and the sum is over the* f *different reduced frequency squares.* □

The following is a well known result for derangements. When it is reinterpreted for frequency squares, it gives a formula to compute the number of 1-transversals of a frequency square with frequency vector $(n-1,1)$.

Lemma 3.7. *Let $T_1(F(n;n-1,1))$ be the number of 1-transversals of an $F(n;n-1,1)$ frequency square. Then*

$$T_1(F(n;n-1,1)) = (n-1)\left(T_1(F(n-1;n-2,1)) + T_1(F(n-2;n-3,1))\right)$$

$$= n!\sum_{i=2}^{n}\frac{(-1)^i}{i!}. \qquad \square$$

Note that this is the number of derangements of n symbols. The above result, together with Lemma 3.1, and the fact that there is only one isotopy class for frequency squares $F(n;n-1,1)$ with $(n-2)!$ reduced frequency squares is used to obtain a formula for the number of frequency squares $\mathcal{F}(n;n-2,1,1)$ for any $n \geqslant 3$.

Theorem 3.8. *Let $\mathcal{F}(n;n-2,1,1)$ be the number of frequency squares with frequency vector $(n-2,1,1)$. Then,*

$$\mathcal{F}(n;n-2,1,1) = n!n!\sum_{i=2}^{n}\frac{(-1)^i}{i!}. \qquad \square$$

The number of reduced frequency squares $f(n;n-2,1,1)$ for $n \leqslant 8$ where given in [1] and [7]. Theorem 3.8 gives a formula for the value of $f(n;n-2,1,1)$ for any $n \geqslant 3$.

Corollary 3.9. *Let $f(n;n-2,1,1)$ be the number of reduced frequency squares with frequency vector $(n-2,1,1)$. Then,*

$$f(n;n-2,1,1) = (n-3)!(n-2)!n\sum_{i=2}^{n}\frac{(-1)^i}{i!}.$$

n	$f(n,n-2,1,1)$
7	7416
8	254280
9	12014640
10	747578160
11	59329146240
12	5814256049280

4. Transversals and latin rectangles

Let $T_1(n; n-1, 1)$ be the number of 1-transversals of an $F(n; n-1, 1)$ frequency square. Consider the two line latin rectangles with first row 1,2,3:

$$R_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

We can associate 1-transversals to the above two line latin rectangles as follows. Consider the frequency square

$$F_d(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

with 2's on the main diagonal. The 1-transversal of $F_d(3)$ associated to R_1 is

$$\{(1,2), (2,3), (3,1)\},$$

and the 1-transversal associated to R_2 is

$$\{(1,3), (2,1), (3,2)\}.$$

Note that there are correspondences $\{(1,2), (2,3), (3,1)\} \mapsto (2\ 3\ 1)$ and $\{(1,3), (2,1), (3,2)\} \mapsto (3\ 1\ 2)$.

We can generalize this construction for any n since no 1-transversal of the frequency square $F_d(n)$ with 2's in the diagonal will contain the pair (i,i) for $i = 1, \ldots, n$. In general, consider the "diagonal" frequency square of order n

$$F_d(n) = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ & \vdots & & \\ 1 & 1 & \cdots & 2 \end{pmatrix}. \tag{4}$$

Note that the set of 1-transversals of $F_d(n)$ is

$$A = \{\{(1,i_1), (2,i_2), \cdots, (n,i_n)\} \mid i_l \neq l, i_k \neq i_l \text{ for } k \neq l\},$$

and

$$\{(1,i_1), (2,i_2), \cdots, (n,i_n)\} \mapsto (i_1\ i_2\ \cdots\ i_n)$$

defines a 1-1 correspondence between the set of 1-transversals A and the set of two line latin rectangles whose first row is in the natural order $1, 2, \ldots, n$ and second row is $(i_1\ i_2\ \cdots\ i_n)$.

For $m \leqslant n$, let $R(m,n)$ be the number of m line latin rectangles of order n whose first row is in standard order $1, 2, \ldots, n$.

Corollary 4.1. *For each $n \geqslant 2$, $R(2,n) = T_1(n; n-1, 1)$.* □

The correspondence of pairs of disjoint 1-transversals of $F_d(n)$ and 3 line latin rectangles is similar. Consider the diagonal frequency square (4) and note that the set of pairs of disjoint 1-transversals of this square is

$$A = \{\{\{(1,i_1),(2,i_2),\cdots,(n,i_n)\},\{(1,j_1),(2,j_2),\cdots,(n,j_n)\}\} \mid$$
$$i_l, j_l \neq l, i_k \neq i_l \text{ and } j_k \neq j_l \text{ for } k \neq l, \text{ and } i_k \neq j_k\}.$$

Now each element in A (a pair) defines the last two rows

$$(i_1\ i_2\ \cdots\ i_n),(j_1\ j_2\ \cdots\ j_n)$$

of a three line latin rectangle with first row in the natural order. Since we can interchange the order of the last 2 rows, we have 2 different three line latin rectangles with first row in the natural order for each element in A. Let $T_1^{(m)}(n;n-1,1)$ be the number of sets of m disjoint 1-transversals of the frequency square (2). Hence $T_1^{(1)}(n;n-1,1) = T_1(n;n-1,1)$.

Corollary 4.2. *For each $n \geqslant 3$, $R(3,n) = 2T_1^{(2)}(n;n-1,1)$.* □

The construction for m line latin rectangles is similar: the set A is the set of all sets of $m-1$ disjoint 1-transversals of (4). Each element in A gives $m-1$ rows of the m line latin rectangle. There are $(m-1)!$, m line latin rectangles for each element in A.

Corollary 4.3. *For $1 \leqslant m \leqslant n$, $R(m,n) = (m-1)!T_1^{(m-1)}(n;n-1,1)$.* □

See page 142 of [2] for the number of m line latin rectangles of order $n \leqslant 11$.

Corollary 4.4. *For each $n \geqslant 2$, $T_1^{(n-1)}(n;n-1,1) = l_n$, the number of reduced latin squares of order n.* □

5. Relating the numbers of frequency squares with two different frequency vectors

In this section we extend our results from Section 2 in order to be able to go from one frequency vector to another, not just from a given frequency vector to the vector $(1,\ldots,1)$ involving latin squares.

Let $\lambda_1 + \cdots + \lambda_m$ be a partition of n. Another partition

$$\lambda'_{11} + \cdots + \lambda'_{1e_1} + \cdots + \lambda'_{m1} + \cdots + \lambda'_{me_m}$$

of n is a *refinement*, if for each $i = 1,\ldots,m$, $\lambda_i = \lambda'_{i1} + \cdots + \lambda'_{ie_i}$. In this case, will call $(\lambda'_{11},\ldots,\lambda'_{me_m})$ a *refinement vector* of $(\lambda_1,\ldots,\lambda_m)$

For each $i = 1,\ldots,m$, we have $\lambda_i n$ cells (λ_i in each row and column) in the $F(n;\lambda_1,\ldots,\lambda_m)$ frequency square containing the symbol i. For each $i = 1,\ldots,m$, we now form an $(\lambda'_{i1},\ldots,\lambda'_{ie_i})$-array containing e_i disjoint blocks. The first block has $\lambda'_{i1}n$ cells with λ'_{i1} cells in each row and column. Continuing, the e_i-th block has $\lambda'_{ie_i}n$ cells with λ'_{ie_i} cells occurring in each row and column.

In Section 2, to construct latin squares from frequency squares, we replaced the values of the cells given by each of the i-transversals of an i-partition by a symbol, one symbol for each transversal, hence λ_i symbols for each i-partition. Now, to construct frequency squares with frequency vector $(n;\lambda'_{11},\ldots,\lambda'_{me_m})$, we will replace the values of the cells given in each block of a $(\lambda'_{i1},\ldots,\lambda'_{ie_i})$-array by a symbol, one symbol for each block, hence e_i symbols for each $(\lambda'_{i1},\ldots,\lambda'_{ie_i})$-array.

Let $\delta_i(F)$ be the number of such arrays arising from the symbol i which occurs in the reduced frequency square $F = F(n;\lambda_1,\ldots,\lambda_m)$. Following the proof of Lemma 2.1, one can prove that the product $\delta = \delta_1(F)\cdots\delta_m(F)$ is invariant in an isotopy class:

Lemma 5.1. *Assume that two frequency squares F_1 and F_2 (of the same order n and frequency vector) are isotopic. Then the number of arrays from F_1 is the same as the number of arrays from F_2; that is $\delta_1(F_1)\cdots\delta_m(F_1) = \delta_1(F_2)\cdots\delta_m(F_2)$.* □

Remark 5. As in Example 2.3, for a fixed i, $\delta_i(F_1)$ might not be equal to $\delta_i(F_2)$, but, since we are considering all the symbols in the product, we get that we have $\delta_1(F_1)\cdots\delta_m(F_1) = \delta_1(F_2)\cdots\delta_m(F_2)$.

We now obtain a theorem that extends the result in Theorem 2.4:

Theorem 5.2. *If $\lambda = (\lambda_1,\ldots,\lambda_m)$ is a frequency vector of n and $(\lambda'_{11},\ldots,\lambda'_{me_m})$ is a fixed refinement vector of λ, then*

$$\binom{n}{\lambda_1,\ldots,\lambda_m}\binom{n-1}{\lambda_1-1,\ldots,\lambda_m}\sum_{j=1}^{\Lambda(n;\lambda_1,\ldots,\lambda_m)} n_j\delta^{(j)}e_1!\cdots e_m!$$

$$=\binom{n}{\lambda'_{11},\ldots,\lambda'_{me_m}}\binom{n-1}{\lambda'_{11}-1,\ldots,\lambda'_{me_m}}f(n;\lambda'_{11},\ldots,\lambda'_{me_m})$$

$$=\mathcal{F}(n;\lambda'_{11},\ldots,\lambda'_{me_m})$$

where $\delta^{(j)}$ denotes the number of distinct arrays (as defined above) of any reduced frequency square $F(n;\lambda_1,\ldots,\lambda_m)$ in the j-th isotopy class of reduced squares which contains n_j reduced squares. □

As the proof of this theorem is similar to the proof of Theorem 2.4 in Section 2 for determining the total number of latin squares from reduced $F(n;\lambda_1,\ldots,\lambda_m)$ frequency squares, we omit the proof and instead, provide the reader with the following illustrative example.

We start with reduced $F(5;4,1)$ frequency squares and determine the total number of $F(5;2,2,1)$ frequency squares. There is only one isotopy class and

$(5-2)!$ reduced frequency squares with the frequency vector $(4,1)$. Consider

$$F = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{array} .$$

There are (4)(5)=20 cells containing the symbol 1. Form a (2,2)-array containing 2 blocks with 10 cells each, 2 per row and column. This is the same as considering a partition and selecting 2, 1-transversals to construct one block and 2 other 1-transversals to construct the other block. For example, from the partition

$$P = \{\{(1,1),(2,2),(3,4),(4,3),(5,5)\},\{(1,2),(2,3),(3,5),(4,1),(5,4)\},$$

$$\{(1,3),(2,5),(3,1),(4,4),(5,2)\},\{(1,4),(2,1),(3,2),(4,5),(5,3)\},$$

$$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}\},$$

one can form an array $\{B_1, B_2\}$ with the two blocks

$$B_1 = \{(1,1),(2,2),(3,4),(4,3),(5,5),(1,2),(2,3),(3,5),(4,1),(5,4)\},$$

$$B_2 = \{(1,3),(2,5),(3,1),(4,4),(5,2),(1,4),(2,1),(3,2),(4,5),(5,3)\}.$$

The 1's in B_1 can be changed to 3's to obtain

$$F' = \begin{array}{ccccc} 3 & 3 & 1 & 1 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 3 & 3 \end{array} .$$

Note that there are $e_1! = 2!$ ways to replace the symbol 1 using this array. There are a total of $\delta_1 = 108$ distinct arrays containing the symbol 1. Theorem 5.2 implies that there are 72 reduced frequency squares $F(5;2,2,1)$, which agrees with the results from [1].

Acknowledgements: The authors appreciate the careful review and comments made by the referees which helped to correct and improve this final version. The second author would like to sincerely thank the Departments of Computer Science and Mathematics at the University of Puerto Rico, Río Piedras for their hospitality during his two visits in 2008 and 2009 when most of this work was completed. The third author appreciates the hospitality of the Department of Mathematics of the Pennsylvania State University during her visit in 2011.

References

[1] **L.J. Brant and G.L. Mullen**, *Some results on enumeration and isotopic classification of frequency squares*, Utilitas Math. **29**(1986), 231 – 244.

[2] **C.J. Colbourn and J.H. Dinitz, Editors**, *Handbook of Combinatorial Designs*, Sec. Ed., Chapman and Hall/CRC, Boca Raton, FL, 2007.

[3] **J. Dénes and A.D. Keedwell**, *Latin Squares and their Applications*, Academic Press, New York, 1974.

[4] **J. Dénes and A.D. Keedwell**, *Latin Squares*, Annals of Disc. Math., Vol. **46** (1991), North-Holland, Amsterdam.

[5] **J. Dénes, A.D. Keedwell, and G.L. Mullen**, *Connections between r-factors of labelled graphs and frequency squares*, Ars Combinatoria **28**(1989), 201 – 202.

[6] **J. Dénes and G.L. Mullen**, *Enumeration formulas for latin and frequency squares*, Discrete Math. **111**(1993), 157 – 163.

[7] **V. Krčadinac**, *Frequency squares of orders 7 and 8*, Utilitas Math. **72**(2007), 89 – 95.

[8] **C.F. Laywine and G.L. Mullen**, *Discrete Mathematics Using Latin Squares*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1998.

[9] **B.D. McKay and I.M. Wanless**, On the number of latin squares, Ann. Comb. **9**(2005), 335 – 344.

[10] **I.M. Wanless**, A generalization of transversals for latin squares, Electron J. Combinatorics **9**(2002), #R12.

Received December 10, 2010
Revised January 10, 2012

F. N. Castro
Department of Mathematics, University of Puerto Rico, Río Piedras. PO Box 70377, San Juan, PR 00936-8377
E-mail: franciscastr@gmail.com

G. L. Mullen
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail: mullen@math.psu.edu

I. Rubio
Department of Computer Science, University of Puerto Rico, Río Piedras, PO Box 70377, San Juan, PR 00936-8377
E-mail: iverubio@gmail.com