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A Zariski topology for k-semirings

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Abstract. The prime k-spectrum $\mathrm{Spec}_k(R)$ of a k-semiring R will be introduced. It will be proven that it is a topological space, and some properties of this space will be investigated. Connections between the topological properties of $\mathrm{Spec}_k(R)$ and possible algebraic properties of the k-semiring R will be established.

1. Introduction

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different disciplines of applied mathematics and information sciences because semirings provides an algebraic framework for modeling. Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason; their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Let R be a commutative ring with identity. The prime spectrum $\mathrm{Spec}(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of R play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $\mathrm{Spec}(M)$, the set of all prime submodules of a module M over R, are studied by many authors (for example see [11]). In this paper, we concentrate on Zariski topology of semirings and generalize the some well known results of Zariski topology on the sets of prime ideals of a commutative ring to prime ideals of a commutative semiring and investigate the basic properties of this topology. For example, we prove that if R is a k-semiring, then $\mathrm{Spec}_k(R)$ is a T_0-space and it is a compact space.

Throughout this paper R is a commutative semiring with identity. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [1, 6, 8, 10, 11]. All semiring in this paper are commutative with non-zero identity. Allen [1] has presented the notion of Q-ideal I in the semiring R and constructed the quotient semiring R/I (also see [3, 5, 7]). Let R be a semiring. A *subtractive ideal* (= *k-ideal*) I is a ideal of R such that if $x, x + y \in I$, then $y \in I$ (so $\{0_R\}$ is a k-ideal of R). A *prime ideal* of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. So P is prime if and only if whenever $IJ \subseteq P$ for some

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ideals I, J of R implies that $I \subseteq P$ or $J \subseteq P$. Furthermore, the collection of all prime k-ideals of R is called the *spectrum* of R and denoted by $\mathrm{Spec}_k(R)$. An ideal I of R is said to be *semiprime* if I is an intersection of prime k-ideals of R. If I is a proper ideal of R, then the *radical* $\mathrm{rad}(I)$ of I (in R) is the intersection of all prime k-ideals of R containing I (see [4]). Note that $I \subseteq \mathrm{rad}(I)$ and that $\mathrm{rad}(I)$ is a semiprime k-ideal of R. An ideal I of R is called *extraordinary* if whenever A and B are semiprime k-ideals of R with $A \cap B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. A semiring is called a *partitioning semiring*, if every proper principal ideal of R is a partitioning ideal (= a Q-ideal) (see [7]). A non-zero element a of a semiring R with identity is said to be a *semiunit* in R if $1 + ra = sa$ for some $r, s \in R$.

Lemma 1.1. *Let R be a semiring. If $\{I_i\}_{i\in\Lambda}$ is a collection of k-ideals of R, then $\sum_{i\in\Lambda} I_i$ and $\bigcap_{i\in\Lambda} I_i$ are k-ideals of R.* □

2. Properties of top semirings

Let R be a semiring with $1 \neq 0$. Then R has at least one maximal k-ideal and if I is a proper Q-ideal of R, then $I \subseteq P$ for some maximal k-ideal P of R (see [5]). Now by [3], R/P is a semifield and hence it is a semidomain. Thus P is prime and $\mathrm{Spec}_k(R) \neq \emptyset$ (see [3]). Then we have the following

Lemma 2.1. *If P is a maximal Q-ideal of a semiring R, then P is a prime k-ideal of R. In particular,* $\mathrm{Spec}_k(R) \neq \emptyset$. □

Let R be a semiring R with non-zero identity. For any k-ideal I of R by $V(I)$ we mean the set of all prime k-ideals of R containing I. Clearly, $V(R) = \emptyset$ and $V(\{0\}) = \mathrm{Spec}(R)$.

Definition 2.2. A semiring is called a *k-semiring*, if every ideal of R is a k-ideal.

Example 2.3. Assume that E_+ be the set of all non-negative integers and let $R = E_+ \cup \{\infty\}$. Define $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for all $a, b \in R$. Then R is a commutative semiring with $1_R = \infty$ and $0_R = 0$. An inspection will show that the list of ideals of R are: R, E_+ and for every non-negative integer n

$$I_n = \{0, 1, \ldots, n\}.$$

It is clear that every ideal of R is a k-ideal; so R is a k-semiring. Moreover, every proper ideal of R is a prime k-ideal; so $\mathrm{Spec}(R) = \{E_+, I_0, \ldots\}$. □

Lemma 2.4. *Let R be a k-semiring. Then the following statements hold:*

(*i*) *If S is a subset of R, then $V(S) = V(\langle S\rangle)$.*

(*ii*) *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$ for every k-ideals I and J of R.*

(*iii*) *If I is a k-ideal of R, then $V(I) = V(\mathrm{rad}(I))$.*

(*iv*) *If* $V(I) \subseteq V(J)$*, then* $J \subseteq \mathrm{rad}(I)$ *for every deals* I, J *of* R.

(*v*) $V(I){=}V(J)$ *if and only if* $\mathrm{rad}(I){=}\mathrm{rad}(J)$ *for every ideals* I, J *of* R.

(*vi*) *If* $\{I_i\}_{i\in\Lambda}$ *is a family of ideals of* R*, then* $V(\sum_{i\in\Lambda} I_i) = \bigcap_{i\in\Lambda} V(I_i)$.

Proof. (*i*) and (*iv*) are obvious.
(*ii*) It is clear that $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. Let $P \in V(IJ)$. Then $IJ \subseteq P$, and hence $I \subseteq P$ or $J \subseteq P$. Thus $P \in V(I)$ or $P \in V(J)$, i.e., $P \in V(I) \cup V(J)$. Hence $V(IJ) \subseteq V(I) \cup V(J)$.

(*iii*) Since $I \subseteq \mathrm{rad}(I)$, we have $V(\mathrm{rad}(I)) \subseteq V(I)$. For the reverse inclusion, assume that $P \in V(I)$. Then $I \subseteq P$. Hence $\mathrm{rad}(I) \subseteq P$, and so we have the equality.

(*v*) Let $V(I) = V(J)$. By (*iii*), we have $V(I) \subseteq V(\mathrm{rad}(J)$; hence $\mathrm{rad}(J) \subseteq \mathrm{rad}(I)$ by (*iv*). Similarly, $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$, and so we have the equality. The other implication is similar.

(*vi*) Let $P \in \bigcap_{i\in\Lambda} V(I_i)$. Then $I_i \subseteq P$ for every $i \in \Lambda$, so $\sum_{i\in\Lambda} I_i \subseteq P$, which implies that $\bigcap_{i\in\Lambda} V(I_i) \subseteq V(\sum_{i\in\Lambda} I_i)$. The reverse inclusion is similar. □

Let R be a k-semiring. If $\zeta(R)$ denotes the collection of all subsets $V(I)$ of $\mathrm{Spec}_k(R)$, then $\zeta(R)$ contains the empty set and $\mathrm{Spec}(R) = X$ and is closed under arbitrary intersection by Lemma 2.4 (*vi*). If also $\zeta(R)$ is closed under finite union, that is, for every ideals I and J of R such that $V(I)\cup V(J) = V(L)$ for some ideal L of R, for in this case $\zeta(R)$ satisfies the axioms of closed subsetes of a topological spaces, which is called *Zariski topology*. The following definition is the same as that introduced by MacCasland, Moore, and Smith in [11].

Definition 2.5. Let R be a k-semiring. An R-semimodule M equipped with Zariski topology is called *top semimodule*. A k-semiring R which is a top semimodule as an R-semimodule is called a *top semiring*.

Proposition 2.6. *Every* k*-semiring with a non-zero identity is a top semiring.*

Proof. Apply Lemma 2.4. □

Theorem 2.7. *Every ideal of a* k*-semiring with a non-zero identity is extraordinary.*

Proof. Note that $\mathrm{Spec}_k(R) \neq \emptyset$ by Lemma 2.1. Let P be any ideal of R and let I and J be semiprime ideals of R such that $I \cap J \subseteq P$. By Proposition 2.6, there exists an ideal U of R such that $V(I) \cup V(J) = V(U)$. Since $I = \bigcap_{i\in\Lambda} P_i$, where P_i are prime k-ideals of R ($i \in \Lambda$), for each $i \in \Lambda$, $P_i \in V(I) \subseteq V(U)$, so that $U \subseteq P_i$. Thus $U \subseteq I$. Similarly, $U \subseteq J$. Thus $U \subseteq I \cap J$. Now we have $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(U) = V(I) \cup V(J)$, that is, $V(I) \cup V(J) = V(I \cap J)$. Hence $P \in V(I \cap J)$ gives $I \subseteq P$ or $J \subseteq P$. □

Definition 2.8. A semiring is called a *strong partitioning semiring*, if every proper finitely generated ideal of R is a partitioning ideal (= a Q-ideal).

Proposition 2.9. *Assume that R is a strong partitioning semiring and let I be the proper ideal of R generated by a family $\{a_t\}_{t\in\Lambda}$ of elements R. Then I is a Q-ideal of R.*

Proof. Since $R = \bigcup\{q + Ra_t : q \in Q\}$ for some $t \in \Lambda$, we must have $R = \bigcup\{q + I : q \in Q\}$. Let $X \in (q_1 + I) \cap (q_2 + I) \neq \emptyset$. Then $X = q_1 + r_{i_1}a_{i_1} + \ldots + r_{i_n}a_{i_n} = q_2 + s_{j_1}a_{j_1} + \ldots + s_{j_m}a_{j_m}$ for some $a_{j_k}, a_{i_t} \in I$ and $r_{i_t}, s_{j_k} \in R$ ($1 \leqslant t \leqslant n$, $1 \leqslant k \leqslant m$). Let J be the ideal of R generated by $r_{i_1}a_{i_1}, ..., r_{i_n}a_{i_n}, s_{j_1}a_{j_1}, \ldots, s_{j_m}a_{j_m}$. By assumption, J is a Q-ideal of R and $X \in (q_1 + J) \cap (q_2 + J)$; hence $q_1 = q_2$. Thus I is a Q-ideal of R. $\square$

Remark 2.10. Let $X = \mathrm{Spec}_k(R)$. For each subset S of R, by X_S we mean $X - V(S) = \{P \in X : S \nsubseteq P\}$. If $S = \{f\}$, then by X_f we denote the set $\{P \in X : f \notin P\}$. Clearly, the sets X_f are open, and they are called *basic open sets*. $\square$

Theorem 2.11. *Let R be a strong partitioning semiring and $X = \bigcup_{i\in\Lambda} X_{a_i}$. If I is the ideal of R generated by $\{a_i\}_{i\in\Lambda}$, then $I = R$.*

Proof. Suppose not. Since I is a proper Q-ideal of R by Proposition 2.9, we have $I \subseteq P$ for some maximal k-ideal P of R. By assumption, $P \notin X_{a_i}$ for every $i \in \Lambda$, which is a contradiction. $\square$

Theorem 2.12. *Let R be a strong partitioning semiring. Then the following statements hold:*

(*i*) *$X_f \cap X_e = X_{fe}$ for all $f, e \in R$.*

(*ii*) *$X_f = \emptyset$ if and only if f is nilpotent.*

(*iii*) *$X_f = X$ if and only if f is a semiunit in R.*

Proof. (*i*) If $P \in X_f \cap X_e$, then $e, f \notin P$, so $ef \notin P$, which implies that $P \in X_{fe}$. Thus $X_f \cap X_e \subseteq X_{ef}$. The other inclusion is similar.

(*ii*) Assume that an element f is nilpotent and let P be any element of X. Then $f^s = 0 \in P$ for some positive integer s. Thus P prime k-ideal gives $f \in P$; hence $P \notin X_f$ for every $P \in X$. Thus $X_f = \emptyset$. Conversely, assume that $X_f = \emptyset$. Then for each $P \in X$, we have $f \in P$; whence $f \in \bigcap_{P\in X} P = \mathrm{rad}(0)$ (see [4]). Thus f is nilpotent.

(*iii*) Let f be a semiunit. Since the inclusion $X_f \subseteq X$ is trivial, we will prove the reverse inclusion. Let P be any element of X. If $Rf \subseteq P$, then $R = P$ by [5], which is a contradiction. Thus $f \notin P$; hence $P \in X_f$, and so we have equality. Conversely, assume that $X = X_f$. Then for any $P \in X$, we must have $f \notin P$. If f is not a semiunit in R, then Rf is a Q-ideal of R and hence it is contained in a maximal k-ideal of R which is a prime k-ideal by Lemma 2.1, a contradiction. Thus f is semiunit. $\square$

Theorem 2.13. *Let R be a k-semiring. Then the set $\mathcal{A} = \{X_f : f \in R\}$ forms a base for the Zariski topology on X.*

Proof. Suppose that U is an open set in X. Then $U = X - V(I)$ for some k-ideal I of R. Let $I = \langle \{f_i : i \in \Lambda\} \rangle$, where $\{f_i : i \in \Lambda\}$ is a generator set of I. Then $V(I) = V(\sum_{i\in\Lambda} Rf_i) = \bigcap_{i\in\lambda} V(Rf_i)$ by Lemma 2.4(vi). It follows that $U = X - V(I) = X - \bigcap_{i\in\Lambda} V(Rf_i) = \bigcup_{i\in\Lambda} X_{f_i}$. Thus $\mathcal{A}$ is a base for the Zariski topology on X. □

Proposition 2.14. *Let I be an ideal of a k-semiring R. Then*

(i) *$X_I = \bigcup_{a\in I} X_a$. Moreover, if $I = \langle a_1, a_2, \ldots, a_n \rangle$, then $X_I = \bigcup_{i=1}^{n} X_{a_i}$.*

(ii) *Let $\{a_i\}_{i\in\Lambda}$ be the collection of elements of R and $a \in R$. Then $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$ if and only if there are elements $a_{i_1}, \ldots, a_{i_n} \in \{a_i\}_{i\in\Lambda}$ such that $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$.*

Proof. (i) Assume that $a \in I$ and let $P \in X_a$. Then $a \notin P$ which implies $P \in X_I$. Thus $\bigcup_{a\in I} X_a \subseteq X_I$. For the reverse inclusion, assume that $P \in X_I$. Then $P \in X_b$ for some $b \in I - P$, and so we have the equality. Finally, since the inclusion $\bigcup_{i=1}^{n} X_{a_i} \subseteq X_I$ is clear, we will prove the reverse inclusion. Let $P \in X_I$. Then there exist $a \in I - P$ and $r_i \in R$ ($1 \leqslant i \leqslant n$) such that $P \in X_a$ and $a = \sum_{i=1}^{n} r_i a_i$. It follows that there exists a positive integer j ($1 \leqslant j \leqslant n$) such that $a_j \notin P$; hence $P \in X_{a_j}$, as needed.

(ii) Let $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$. Then there exists a positive integer m and $r_i \in R$ ($1 \leqslant i \leqslant n$) such that $a^m = \sum_{j=1}^{n} r_j a_{i_j}$. Now, let $P \in X_a$. So $a \notin P$ gives $a^m \notin P$; hence $P \in X_{a_{i_k}}$ for some k. Thus $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$.

Conversely, assume that $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$ and let I be the ideal of R generated by $\{a_i : i \in \Lambda\}$. It is clear that if $P \in X$ and $P \notin \bigcup_{i\in\Lambda} X_{a_i}$, then $a_i \in P$ implies that $a \in P$. Therefore we have $V(I) \subseteq V(\langle a \rangle)$. It follows that $a \in \bigcap_{P\in V(<a>)} P \subseteq \bigcap_{P\in V(I)} P = \mathrm{rad}(I)$. So, there exist $i_1, i_2, \ldots, i_s \in \Lambda$ and $t_1, t_2, \ldots, t_s \in R$ such that $a^m = t_1 a_{i_1} + \ldots + t_s a_{i_s}$ for some positive integer m; thus $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$. □

Theorem 2.15. *Let R be a k-semiring. For every $a \in R$, the set X_a is compact. Specifically the whole space $X_1 = X$ is compact.*

Proof. By Theorem 2.13, it suffices to show that every cover of basic open sets has a finite subcover. Suppose that $X_a \subseteq \bigcup_{i\in\Lambda} X_{a_i}$. By Proposition 2.14 ($ii$), there are $a_{i_1}, \ldots, a_{i_n} \in R$ such that $a \in \mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$. Since $V(\mathrm{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)) = V(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$ by Lemma 2.4 (iii), we must have $X_a \subseteq \bigcup_{i=1}^{n} X_{a_i}$ by Proposition 2.14 (i). This completes the proof. □

From Theorem 2.13 and Theorem 2.15 the next result is immediate.

Corollary 2.16. *Let R be a k-semiring. Then an open set of X is compact if and only if it is a finite union of basic open sets.* □

Let R be a k-semiring. The topological space $X = \mathrm{Spec}_k(R)$ is said to be a T_0*-space* if for every $P, P' \in X$, $P \neq P'$ there is either a neighborhood X_a of P such that $X_a \cap P' = \emptyset$ or a neighborhood X_b of P' such that $X_b \cap P = \emptyset$.

Theorem 2.17. *Let R be a k-semiring. Then the topological space $X = \mathrm{Spec}_k(R)$ is a T_0-space.*

Proof. Let $P, P' \in X$ with $P \neq P'$. We note that the set X_a is a neighborhood of P if and only if $a \notin P$. Assume that $P' \in X_a$ for all $a \notin P$. Then we conclude that $a \in P'$ implies that $a \in P$; hence $P' \subset P$. Now let $b \in P - P'$. Then $b \notin P'$ gives X_b is a neighborhood of P', but $b \in P$, so $P \notin X_b$. This completes the proof. $\square$

Quotient semimodules over a semiring R have already been introduced and studied by present authors in [6]. Chaudhari and Bonde extended the definition of Q_M-subsemimodule of a semimodule and some results given in the Section 2 in [6] to a more general quotient semimodules case in [8] (for the structure of quotient semimodules we refer [8]).

Convention. For each Q_R-subsemimodule I of the R-semimodule R, we mean I is a Q_R-ideal of R. Now If I is a Q_R-ideal of a semiring R, then R/I is a quotient semimodule of R by I. Now we give an example of semimodules over a semiring that are top semimodules.

Lemma 2.18. *Let I be a Q_R-ideal (or a Q_R-subsemimodule) of a semiring R. If J is a k-ideal of R containing I, then $(J :_R R) = (J/I :_R R/I)$.*

Proof. Let $r \in (J : R)$. If $q + I \in R/I$, then there exists a unique element q' of Q_R such that $r(q+I) = q' + I$, where $rq + I \subseteq q' + I$; so $q' \in J \cap Q_R$ since $rq \in J$ and J is a k-ideal. Thus $(J : R) \subseteq (J/I : R/I)$.

Conversely, assume that $a \in (J/I : R/I)$ and $s \in R$. Then $s = q_1 + t$ for some $q_1 \in Q_R$ and $t \in I$; so there is a unique element q_2 of Q_R with $a(q_1 + I) = q_2 + I \in J/I$, where $aq_1 + I \subseteq q_2 + I$. Thus J k-ideal gives $aq_1 \in J$. As $as = aq_1 + at \in J$, we have $a \in (J : R)$. $\square$

Proposition 2.19. *Let I be a Q_R-ideal of a semiring R. Then there is a one-to-one correspondence between prime k-subsemimodules of R-semimodule R/I and prime k-ideals of R containing I.*

Proof. Let J be a prime k-ideal of R containing I. Then it follows from [3] that J/I is a proper k-subsemimodule of R/I. Let $a(q_1 + I) = q_2 + I \in J/I$, where $q_2 \in Q_R \cap J$ and $aq_1 + I \subseteq q_2 + I$, so $aq_1 \in J$ since J is a k-ideal of R. But J is prime, hence either $q_1 \in J$ (so $q_1 + I \in J/I$) or $a \in (J : R) = (J/I : R/I)$ by Lemma 2.18. Thus, J/I is a prime k-subsemimodule of R/I.

Conversely, assume that J/I is a prime k-subsemimodule of R/I. To show that J is a prime k-ideal of R, suppose that $rx \in J$, where $r, x \in R$. We may assume that $r \neq 0$. There are elements $q \in Q_R$ and $n \in I$ such that $x = q + n$, so $rx = rq + rn \in J$; hence $rq \in J$ since J is a k-ideal. Therefore, there exists a unique element $q' \in Q_R$ such that $r(q+I) = q' + I$, where $rq + I \subseteq q' + I$; hence $q' \in J$. Thus $r(q+I) \in J/I$. Then J/I prime gives either $q + I \in J/I$ (so $x \in J$) or $r \in (J/I : R/I) = (J : R)$, and the proof is complete. $\square$

Corollary 2.20. *Let I be a Q_R-ideal of a semiring R. Then there is a one-to-one correspondence between semiprime k-subsemimodules of R/I and semiprime k-ideals of R containing I.*

Proof. Apply Theorem 2.19 (note that $(\bigcap_{i\in J} P_i)/I = \bigcap_{i\in J}(P_i/I)$, where P_i is a prime k-ideal for all $i \in J$). $\square$

Theorem 2.21. *Let I be an Q_R-ideal of a semiring R with a non-zero ideantity. Then the following statements hold:*

(*i*) *Every k-subsemimodule of R/I is extraordinary.*

(*ii*) *R/I is a top R-semimodule.*

Proof. (*i*) We may assume that $\mathrm{Spec}(R/I) \neq \emptyset$. Then any semiprime k-subsemimodule of R/I has the form A/I where A is a semiprime k-ideal of R containing I by Corollary 2.20. Let B/I be any k-subsemimodule of R/I and let U/I and L/I be semiprime k-subsemimodules of R/I such that $(L/I) \cap (U/I() \subseteq B/N$. Then $(L \cap U)/I \subseteq (L/I) \cap (U/I) \subseteq B/I$, so $U \cap L \subseteq B$; hence either $U \subseteq B$ or $L \subseteq B$ since T is extraordinary by Theorem 2.7. Thus either $U/I \subseteq B/I$ or $L/I \subseteq B/I$, as needed.

(*ii*) First we show that $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$ for any semiprime subsemimodules U/I and L/I of R/I.

Clearly $V(U/I) \cup V(L/I) \subseteq V(U/I \cap L/I)$. Let $P/I \in V(U/I \cap L/I)$, where P is a semiprime by Corollary 2.20. Then $U \cap L \subseteq P$ and hence $L \subseteq P$ or $U \subseteq P$ (see Theorem 2.7), i.e., $P/I \in V(U/I)$ or $P/I \in V(L/I)$. This proves that $V(U/I \cap L/I) \subseteq V(U/I) \cup V(L/I)$ ans hence $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$. Next, let A/I and B/I be any subsemimodules of R/I. If $V(A/I)$ is empty then $V(A/I) \cup V(B/I) = V(B/I)$. Suppose that $V(A/I)$ and $V(B/I)$ are both non-empty. Then $V(A/I) \cap V(B/I) = V(\mathrm{rad}(A/I)) \cap V(\mathrm{rad}(B/I)) = V(\mathrm{rad}(A/I) \cap \mathrm{rad}(B/I))$. This proves (*ii*). $\square$

Example 2.22. Let R be the k-semiring as described in Example 2.3. Then $\mathrm{Spec}(R)$ is compact and it is a T_0-space by Theorems 2.15 and 2.17. $\square$

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