A Zariski topology for $k$-semirings

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Abstract. The prime $k$-spectrum $\text{Spec}_k(R)$ of a $k$-semiring $R$ will be introduced. It will be proven that it is a topological space, and some properties of this space will be investigated. Connections between the topological properties of $\text{Spec}_k(R)$ and possible algebraic properties of the $k$-semiring $R$ will be established.

1. Introduction

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different disciplines of applied mathematics and information sciences because semirings provides an algebraic framework for modeling. Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason; their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Let $R$ be a commutative ring with identity. The prime spectrum $\text{Spec}(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of $R$ play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $\text{Spec}(M)$, the set of all prime submodules of a module $M$ over $R$, are studied by many authors (for example see [11]). In this paper, we concentrate on Zariski topology of semirings and generalize the some well known results of Zariski topology on the sets of prime ideals of a commutative ring to prime ideals of a commutative semiring and investigate the basic properties of this topology. For example, we prove that if $R$ is a $k$-semiring, then $\text{Spec}_k(R)$ is a $T_0$-space and it is a compact space.

Throughout this paper $R$ is a commutative semiring with identity. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [1, 6, 8, 10, 11]. All semiring in this paper are commutative with non-zero identity. Allen [1] has presented the notion of $Q$-ideal $I$ in the semiring $R$ and constructed the quotient semiring $R/I$ (also see [3, 5, 7]). Let $R$ be a semiring. A subtractive ideal ($= k$-ideal) $I$ is a ideal of $R$ such that if $x, x + y \in I$, then $y \in I$ (so $\{0_R\}$ is a $k$-ideal of $R$). A prime ideal of $R$ is a proper ideal $P$ of $R$ in which $x \in P$ or $y \in P$ whenever $xy \in P$. So $P$ is prime if and only if whenever $IJ \subseteq P$ for some
ideals $I, J$ of $R$ implies that $I \subseteq P$ or $J \subseteq P$. Furthermore, the collection of all prime $k$-ideals of $R$ is called the spectrum of $R$ and denoted by $\text{Spec}_k(R)$. An ideal $I$ of $R$ is said to be semiprime if $I$ is an intersection of prime $k$-ideals of $R$. If $I$ is a proper ideal of $R$, then the radical $\text{rad}(I)$ of $I$ (in $R$) is the intersection of all prime $k$-ideals of $R$ containing $I$ (see [4]). Note that $I \subseteq \text{rad}(I)$ and that $\text{rad}(I)$ is a semiprime $k$-ideal of $R$. An ideal $I$ of $R$ is called extraordinary if whenever $A$ and $B$ are semiprime $k$-ideals of $R$ with $A \cap B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. A semiring is called a partitioning semiring, if every proper principal ideal of $R$ is a partitioning ideal (= a $Q$-ideal) (see [7]). A non-zero element $a$ of a semiring $R$ with identity is said to be a semiunit in $R$ if $1 + ra = sa$ for some $r, s \in R$.

Lemma 1.1. Let $R$ be a semiring. If $\{I_i\}_{i \in \Lambda}$ is a collection of $k$-ideals of $R$, then $\sum_{i \in \Lambda} I_i$ and $\bigcap_{i \in \Lambda} I_i$ are $k$-ideals of $R$. $\square$

2. Properties of top semirings

Let $R$ be a semiring with $1 \neq 0$. Then $R$ has at least one maximal $k$-ideal and if $I$ is a proper $Q$-ideal of $R$, then $I \subseteq P$ for some maximal $k$-ideal $P$ of $R$ (see [5]). Now by [3], $R/P$ is a semifield and hence it is a semidomain. Thus $P$ is prime and $\text{Spec}_k(R) \neq \emptyset$ (see [3]). Then we have the following

Lemma 2.1. If $P$ is a maximal $Q$-ideal of a semiring $R$, then $P$ is a prime $k$-ideal of $R$. In particular, $\text{Spec}_k(R) \neq \emptyset$. $\square$

Let $R$ be a semiring $R$ with non-zero identity. For any $k$-ideal $I$ of $R$ by $V(I)$ we mean the set of all prime $k$-ideals of $R$ containing $I$. Clearly, $V(R) = \emptyset$ and $V(\{0\}) = \text{Spec}(R)$.

Definition 2.2. A semiring is called a $k$-semiring, if every ideal of $R$ is a $k$-ideal.

Example 2.3. Assume that $E_+$ be the set of all non-negative integers and let $R = E_+ \cup \{\infty\}$. Define $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for all $a, b \in R$. Then $R$ is a commutative semiring with $1_R = \infty$ and $0_R = 0$. An inspection will show that the list of ideals of $R$ are: $R$, $E_+$ and for every non-negative integer $n$

$E_n = \{0, 1, \ldots, n\}$.

It is clear that every ideal of $R$ is a $k$-ideal; so $R$ is a $k$-semiring. Moreover, every proper ideal of $R$ is a prime $k$-ideal; so $\text{Spec}(R) = \{E_+, I_0, \ldots\}$. $\square$

Lemma 2.4. Let $R$ be a $k$-semiring. Then the following statements hold:

(i) If $S$ is a subset of $R$, then $V(S) = V(\langle S \rangle)$.

(ii) $V(I) \cup V(J) = V(IJ) = V(I \cap J)$ for every $k$-ideals $I$ and $J$ of $R$.

(iii) If $I$ is a $k$-ideal of $R$, then $V(I) = V(\text{rad}(I))$. 

(iv) If \( V(I) \subseteq V(J) \), then \( J \subseteq \text{rad}(I) \) for every ideals \( I, J \) of \( R \).

(v) \( V(I)=V(J) \) if and only if \( \text{rad}(I)=\text{rad}(J) \) for every ideals \( I, J \) of \( R \).

(vi) If \( \{I_i\}_{i \in \Lambda} \) is a family of ideals of \( R \), then \( V(\sum_{i \in \Lambda} I_i) = \bigcap_{i \in \Lambda} V(I_i) \).

**Proof.** (i) and (iv) are obvious.

(ii) It is clear that \( V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ) \). Let \( P \in V(IJ) \). Then \( IJ \subseteq P \), and hence \( I \subseteq P \) or \( J \subseteq P \). Thus \( P \in V(I) \) or \( P \in V(J) \), i.e., \( P \in V(I) \cup V(J) \). Hence \( V(IJ) \subseteq V(I) \cup V(J) \).

(iii) Since \( I \subseteq \text{rad}(I) \), we have \( V(\text{rad}(I)) \subseteq V(I) \). For the reverse inclusion, assume that \( P \in V(I) \). Then \( I \subseteq P \). Hence \( \text{rad}(I) \subseteq P \), and so we have the equality.

(v) Let \( V(I) = V(J) \). By (iii), we have \( V(I) \subseteq V(\text{rad}(J)) \); hence \( \text{rad}(J) \subseteq \text{rad}(I) \) by (iv). Similarly, \( \text{rad}(I) \subseteq \text{rad}(J) \), and so we have the equality. The other implication is similar.

(vi) Let \( P \in \bigcap_{i \in \Lambda} V(I_i) \). Then \( I_i \subseteq P \) for every \( i \in \Lambda \), so \( \sum_{i \in \Lambda} I_i \subseteq P \), which implies that \( \bigcap_{i \in \Lambda} V(I_i) \subseteq V(\sum_{i \in \Lambda} I_i) \). The reverse inclusion is similar. \( \square \)

Let \( R \) be a \( k \)-semiring. If \( \zeta(R) \) denotes the collection of all subsets \( V(I) \) of \( \text{Spec}_k(R) \), then \( \zeta(R) \) contains the empty set and \( \text{Spec}(R) = X \) and is closed under arbitrary intersection by Lemma 2.4 (vi). If also \( \zeta(R) \) is closed under finite union, that is, for every ideals \( I \) and \( J \) of \( R \) such that \( V(I) \cup V(J) = V(L) \) for some ideal \( L \) of \( R \), for this case \( \zeta(R) \) satisfies the axioms of closed subsets of a topological spaces, which is called **Zariski topology**. The following definition is the same as that introduced by MacCasland, Moore, and Smith in [11].

**Definition 2.5.** Let \( R \) be a \( k \)-semiring. An \( R \)-semimodule \( M \) equipped with Zariski topology is called **top semimodule**. A \( k \)-semiring \( R \) which is a top semimodule as an \( R \)-semimodule is called a **top semiring**.

**Proposition 2.6.** Every \( k \)-semiring with a non-zero identity is a top semiring.

**Proof.** Apply Lemma 2.4. \( \square \)

**Theorem 2.7.** Every ideal of a \( k \)-semiring with a non-zero identity is extraordinary.

**Proof.** Note that \( \text{Spec}_k(R) \neq \emptyset \) by Lemma 2.1. Let \( P \) be any ideal of \( R \) and let \( I \) and \( J \) be semiprime ideals of \( R \) such that \( I \cap J \subseteq P \). By Proposition 2.6, there exists an ideal \( U \) of \( R \) such that \( V(I) \cup V(J) = V(U) \). Since \( I = \bigcap_{i \in \Lambda} P_i \), where \( P_i \) are prime \( k \)-ideals of \( R \) \( (i \in \Lambda) \), for each \( i \in \Lambda \), \( P_i \in V(I) \subseteq V(U) \), so that \( U \subseteq P_i \). Thus \( U \subseteq I \). Similarly, \( U \subseteq J \). Thus \( U \subseteq I \cap J \). Now we have \( V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(U) = V(I) \cup V(J) \), that is, \( V(I) \cup V(J) = V(I \cap J) \). Hence \( P \in V(I \cap J) \) gives \( I \subseteq P \) or \( J \subseteq P \). \( \square \)
**Definition 2.8.** A semiring is called a strong partitioning semiring, if every proper finitely generated ideal of \( R \) is a partitioning ideal (= a \( Q \)-ideal).

**Proposition 2.9.** Assume that \( R \) is a strong partitioning semiring and let \( I \) be the proper ideal of \( R \) generated by a family \( \{a_i\}_{i \in \Lambda} \) of elements \( R \). Then \( I \) is a \( Q \)-ideal of \( R \).

**Proof.** Since \( R = \bigcup \{q + Ra_t : q \in Q \} \) for some \( t \in \Lambda \), we must have \( R = \bigcup \{q + I : q \in Q \} \). Let \( X \in (q_1 + I) \cap (q_2 + I) \neq \emptyset \). Then \( X = q_1 + r_{i_1}a_{i_1} + \ldots + r_{i_n}a_{i_n} = q_2 + s_{j_1}a_{j_1} + \ldots + s_{j_m}a_{j_m} \) for some \( a_{i_k}, a_{i_k} \in I \) and \( r_{i_k}, s_{j_k} \in R \) (\( 1 \leq t \leq n, 1 \leq k \leq m \)). Let \( J \) be the ideal of \( R \) generated by \( r_{i_1}a_{i_1}, ..., r_{i_n}a_{i_n}, s_{j_1}a_{j_1}, ..., s_{j_m}a_{j_m} \).

By assumption, \( J \) is a \( Q \)-ideal of \( R \) and \( X \in (q_1 + J) \cap (q_2 + J) \); hence \( q_1 = q_2 \). Thus \( I \) is a \( Q \)-ideal of \( R \).

**Remark 2.10.** Let \( X = \text{Spec}_R(R) \). For each subset \( S \) of \( R \), by \( X_S \) we mean \( X - V(S) = \{ P \in X : S \not\subseteq P \} \). If \( S = \{f\} \), then by \( X_f \) we denote the set \( \{P \in X : f \notin P \} \). Clearly, the sets \( X_f \) are open, and they are called basic open sets.

**Theorem 2.11.** Let \( R \) be a strong partitioning semiring and \( X = \bigcup_{i \in \Lambda} X_a_i \). If \( I \) is the ideal of \( R \) generated by \( \{a_i\}_{i \in \Lambda} \), then \( I = R \).

**Proof.** Suppose not. Since \( I \) is a proper \( Q \)-ideal of \( R \) by Proposition 2.9, we have \( I \subseteq P \) for some maximal \( k \)-ideal \( P \) of \( R \). By assumption, \( P \notin X_a_i \) for every \( i \in \Lambda \), which is a contradiction.

**Theorem 2.12.** Let \( R \) be a strong partitioning semiring. Then the following statements hold:

(i) \( X_f \cap X_e = X_{fe} \) for all \( f, e \in R \).

(ii) \( X_f = \emptyset \) if and only if \( f \) is nilpotent.

(iii) \( X_f = X \) if and only if \( f \) is a semunit in \( R \).

**Proof.** (i) If \( P \in X_f \cap X_e \), then \( e, f \notin P \), so \( ef \notin P \), which implies that \( P \in X_{fe} \). Thus \( X_f \cap X_e \subseteq X_{ef} \). The other inclusion is similar.

(ii) Assume that an element \( f \) is nilpotent and let \( P \) be any element of \( X \). Then \( f^s = 0 \in P \) for some positive integer \( s \). Thus \( P \) prime \( k \)-ideal gives \( f \in P \); hence \( P \notin X_f \) for every \( P \in X \). Thus \( X_f = \emptyset \). Conversely, assume that \( X_f = \emptyset \). Then for each \( P \in X \), we have \( f \in P \); whence \( f \in \bigcap_{P \in X} P = \text{rad}(0) \) (see [4]). Thus \( f \) is nilpotent.

(iii) Let \( f \) be a semunit. Since the inclusion \( X_f \subseteq X \) is trivial, we will prove the reverse inclusion. Let \( P \) be any element of \( X \). If \( Rf \subseteq P \), then \( R = P \) by [5], which is a contradiction. Thus \( f \notin P \); hence \( P \in X_f \), and so we have equality.

Conversely, assume that \( X = X_f \). Then for any \( P \in X \), we must have \( f \notin P \). If \( f \) is not a semunit in \( R \), then \( Rf \) is a \( Q \)-ideal of \( R \) and hence it is contained in a maximal \( k \)-ideal of \( R \) which is a prime \( k \)-ideal by Lemma 2.1, a contradiction. Thus \( f \) is semunit.
Theorem 2.13. Let $R$ be a $k$-semiring. Then the set $A = \{X_f : f \in R\}$ forms a base for the Zariski topology on $X$.

Proof. Suppose that $U$ is an open set in $X$. Then $U = X - V(I)$ for some $k$-ideal $I$ of $R$. Let $I = \langle\{a_i : i \in A\}\rangle$, where $\{f_i : i \in A\}$ is a generator set of $I$. Then $V(I) = V(\sum_{i \in A} f_i) = \cap_{i \in A} V(R f_i)$ by Lemma 2.4(vi). It follows that $U = X - V(I) = X - \cap_{i \in A} V(R f_i) = \cup_{i \in A} X_f$. Thus $A$ is a base for the Zariski topology on $X$.

Proposition 2.14. Let $I$ be an ideal of a $k$-semiring $R$. Then

(i) $X_I = \bigcup_{a \in I} X_a$. Moreover, if $I = \langle a_1, a_2, \ldots, a_n \rangle$, then $X_I = \bigcup_{i=1}^n X_{a_i}$.

(ii) Let $\{a_i\}_{i \in A}$ be the collection of elements of $R$ and $a \in R$. Then $X_a \subseteq \bigcup_{i \in A} X_{a_i}$ if and only if there are elements $a_{i_1}, \ldots, a_{i_n} \in \{a_i\}_{i \in A}$ such that $a \in \text{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$.

Proof. (i) Assume that $a \in I$ and let $P \subseteq X_a$. Then $a \notin P$ which implies $P \subseteq X_I$. Thus $\bigcup_{a \in I} X_a \subseteq X_I$. For the reverse inclusion, assume that $P \subseteq X_I$. Then $P \subseteq X_b$ for some $b \in I - P$, and so we have the equality. Finally, since the inclusion $\bigcup_{i=1}^n X_{a_i} \subseteq X_I$ is clear, we will prove the reverse inclusion. Let $P \subseteq X_I$. Then there exist $a \in I - P$ and $r_i \in R$ ($1 \leq i \leq n$) such that $P \subseteq X_a$ and $a = \sum_{i=1}^n r_i a_i$. It follows that there exists a positive integer $j$ ($1 \leq j \leq n$) such that $a_j \notin P$; hence $P \subseteq X_{a_j}$ as needed.

(ii) Let $a \in \text{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$. Then there exists a positive integer $m$ and $r_i \in R$ ($1 \leq i \leq n$) such that $a^m = \sum_{i=1}^n r_i a_i$. Now, let $P \subseteq X_a$. So $a \notin P$ gives $a^m \notin P$; hence $P \subseteq X_{a^m}$ for some $m$. Thus $X_a \subseteq \bigcup_{i \in A} X_{a_i}$.

Conversely, assume that $X_a \subseteq \bigcup_{i \in A} X_{a_i}$ and let $I$ be the ideal of $R$ generated by $\{a_i : i \in A\}$. It is clear that if $P \subseteq X_a$ and $a_i \in P$ implies that $a \in P$. Therefore we have $V(I) \subseteq V(\langle a \rangle)$. It follows that $a \in \bigcap_{P \in V(\langle a \rangle)} P \subseteq \bigcap_{P \in V(I)} P = \text{rad}(I)$. So, there exist $i_1, i_2, \ldots, i_k \in A$ and $t_1, t_2, \ldots, t_k \in R$ such that $a^m = t_1 a_{i_1} + \ldots + t_k a_{i_k}$ for some positive integer $m$; thus $a \in \text{rad}(\langle a_{i_1}, \ldots, a_{i_n} \rangle)$.

Theorem 2.15. Let $R$ be a $k$-semiring. For every $a \in R$, the set $X_a$ is compact. Specifically the whole space $X = X$ is compact.

Proof. By Theorem 2.13, it suffices to show that every cover of basic open sets has a finite subcover. Suppose that $X_a \subseteq \bigcup_{i \in A} X_{a_i}$. By Proposition 2.14(ii), there are $a_{i_1}, \ldots, a_{i_m} \in R$ such that $a \in \text{rad}(\langle a_{i_1}, \ldots, a_{i_m} \rangle)$. Since $V(\text{rad}(\langle a_{i_1}, \ldots, a_{i_m} \rangle)) = V(\langle a_{i_1}, \ldots, a_{i_m} \rangle)$ by Lemma 2.4(iii), we must have $X_a \subseteq X_{a_{i_1}} \cup \ldots \cup X_{a_{i_m}}$ by Proposition 2.14(i). This completes the proof.

From Theorem 2.13 and Theorem 2.15 the next result is immediate.

Corollary 2.16. Let $R$ be a $k$-semiring. Then an open set of $X$ is compact if and only if it is a finite union of basic open sets.
Let $R$ be a $k$-semiring. The topological space $X = \text{Spec}_k(R)$ is said to be a $T_0$-space if for every $P, P' \in X$, $P \neq P'$ there is either a neighborhood $X_a$ of $P$ such that $X_a \cap P' = \emptyset$ or a neighborhood $X_b$ of $P'$ such that $X_b \cap P = \emptyset$.

**Theorem 2.17.** Let $R$ be a $k$-semiring. Then the topological space $X = \text{Spec}_k(R)$ is a $T_0$-space.

**Proof.** Let $P, P' \in X$ with $P \neq P'$. We note that the set $X_a$ is a neighborhood of $P$ if and only if $a \notin P$. Assume that $P' \in X_a$ for all $a \notin P$. Then we conclude that $a \in P'$ implies that $a \in P$; hence $P' \subseteq P$. Now let $b \in P - P'$. Then $b \notin P'$ gives $X_b$ is a neighborhood of $P'$, but $b \in P$, so $P \notin X_b$. This completes the proof. \(\square\)

Quotient semimodules over a semiring $R$ have already been introduced and studied by present authors in [6]. Chaudhari and Bonde extended the definition of $Q_M$-subsemimodule of a semimodule and some results given in the Section 2 in [6] to a more general quotient semimodules case in [8] (for the structure of quotient semimodules we refer [8]).

**Convention.** For each $Q_R$-subsemimodule $I$ of the $R$-semimodule $R$, we mean $I$ is a $Q_R$-ideal of $R$. Now if $I$ is a $Q_R$-ideal of a semiring $R$, then $R/I$ is a quotient semimodule of $R$ by $I$. Now we give an example of semimodules over a semiring that are top semimodules.

**Lemma 2.18.** Let $I$ be a $Q_R$-ideal (or a $Q_R$-subsemimodule) of a semiring $R$. If $J$ is a $k$-ideal of $R$ containing $I$, then $(J : R) = (J/I : R/I)$.

**Proof.** Let $r \in (J : R)$. If $q + I \in R/I$, then there exists a unique element $q'$ of $Q_R$ such that $r(q + I) = q' + I$, where $rq + I \subseteq q' + I$; so $q' \in J \cap Q_R$ since $rq \in J$ and $J$ is a $k$-ideal. Thus $(J : R) \subseteq (J/I : R/I)$.

Conversely, assume that $a \in (J/I : R/I)$ and $s \in R$. Then $s = q_1 + t$ for some $q_1 \in Q_R$ and $t \in I$; so there is a unique element $q_2$ of $Q_R$ with $a(q_1 + I) = q_2 + I \in J/I$, where $aq_1 + I \subseteq q_2 + I$. Thus $J$ $k$-ideal gives $aq_1 \in J$. As $as = a(q_1 + at) \in J$, we have $a \in (J : R)$. \(\square\)

**Proposition 2.19.** Let $I$ be a $Q_R$-ideal of a semiring $R$. Then there is a one-to-one correspondence between prime $k$-subsemimodules of $R$-semimodule $R/I$ and prime $k$-ideals of $R$ containing $I$.

**Proof.** Let $J$ be a prime $k$-ideal of $R$ containing $I$. Then it follows from [3] that $J/I$ is a proper $k$-subsemimodule of $R/I$. Let $a(q_1 + I) = q_2 + I \in J/I$, where $q_2 \in Q_R \cap J$ and $aq_1 + I \subseteq q_2 + I$, so $aq_1 \in J$ since $J$ is a $k$-ideal of $R$. But $J$ is prime, hence either $q_1 \in J$ (so $q_1 + I \in J/I$) or $a \in (J : R) = (J/I : R/I)$ by Lemma 2.18. Thus, $J/I$ is a prime $k$-subsemimodule of $R/I$.

Conversely, assume that $J/I$ is a prime $k$-subsemimodule of $R/I$. To show that $J$ is a prime $k$-ideal of $R$, suppose that $rx \in J$, where $r, x \in R$. We may assume that $r \neq 0$. There are elements $q \in Q_R$ and $n \in I$ such that $x = q + n$, so $rx = rq + rn \in J$; hence $rq \in J$ since $J$ is a $k$-ideal. Therefore, there exists a
unique element $q' \in Q_R$ such that $r(q + I) = q' + I$, where $rq + I \subseteq q' + I$; hence $q' \in J$. Thus $r(q + I) \in J/I$. Then $J/I$ prime gives either $q + I \in J/I$ (so $x \in J$) or $r \in (J/I : R/I) = (J : R)$, and the proof is complete.

**Corollary 2.20.** Let $I$ be a $Q_R$-ideal of a semiring $R$. Then there is a one-to-one correspondence between semiprime $k$-subsemimodules of $R/I$ and semiprime $k$-ideals of $R$ containing $I$.

**Proof.** Apply Theorem 2.19 (note that $(\bigcap_{i \in I} P_i)/I = \bigcap_{i \in I}(P_i/I)$, where $P_i$ is a prime $k$-ideal for all $i \in I$).

**Theorem 2.21.** Let $I$ be an $Q_R$-ideal of a semiring $R$ with a non-zero identity. Then the following statements hold:

(i) Every $k$-subsemimodule of $R/I$ is extraordinary.

(ii) $R/I$ is a top $R$-semimodule.

**Proof.** (i) We may assume that $\text{Spec}(R/I) \neq \emptyset$. Then any semiprime $k$-subsemimodule of $R/I$ has the form $A/I$ where $A$ is a semiprime $k$-ideal of $R$ containing $I$ by Corollary 2.20. Let $B/I$ be any $k$-subsemimodule of $R/I$ and let $U/I$ and $L/I$ be semiprime $k$-subsemimodules of $R/I$ such that $(L/I) \cap (U/I) \subseteq B/N$. Then $(L \cap U)/I \subseteq (L/I) \cap (U/I) \subseteq B/I$, so $U \cap L \subseteq B$; hence either $U \subseteq B$ or $L \subseteq B$ since $I$ is extraordinary by Theorem 2.7. Thus either $U/I \subseteq B/I$ or $L/I \subseteq B/I$, as needed.

(ii) First we show that $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$ for any semiprime subsemimodules $U/I$ and $L/I$ of $R/I$.

Clearly $V(U/I) \cup V(L/I) \subseteq V(U/I \cap L/I)$. Let $P/I \in V(U/I \cap L/I)$, where $P$ is a semiprime by Corollary 2.20. Then $U \cap L \subseteq P$ and hence $L \subseteq P$ or $U \subseteq P$ (see Theorem 2.7), i.e., $P/I \in V(U/I)$ or $P/I \in V(L/I)$. This proves that $V(U/I \cap L/I) \subseteq V(U/I) \cup V(L/I)$ as $V(U/I) \cup V(L/I) = V(U/I \cap L/I)$. Next, let $A/I$ and $B/I$ be any subsemimodules of $R/I$. If $V(A/I)$ is empty then $V(A/I) \cup V(B/I) = V(B/I)$. Suppose that $V(A/I)$ and $V(B/I)$ are both non-empty. Then $V(A/I) \cap V(B/I) = V(\text{rad}(A/I)) \cap V(\text{rad}(B/I)) = V(\text{rad}(A/I) \cap \text{rad}(B/I))$. This proves (ii).

**Example 2.22.** Let $R$ be the $k$-semiring as described in Example 2.3. Then $\text{Spec}(R)$ is compact and it is a $T_0$-space by Theorems 2.15 and 2.17.

**References**


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