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Quotient hyper residuated lattices

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Abstract. We define the concept of regular compatible congruence on hyper residuated lattices. Then we attempt to construct quotient hyper residuated lattices. Finally, we state and prove some theorem with appropriate results such as the isomorphism theorems.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth [7], are a common structure among algebras associated with logical systems. In this definition to any bounded lattice $(\mathcal{L}, \vee, \wedge, 0, 1)$, a multiplication '$*$' and an operation '$\to$' are equipped such that $(\mathcal{L}, *, 1)$ is a commutative monoid and the pair $(*, \to)$ is an adjoint pair, i.e.,

$$x * y \leqslant z \text{ if and only if } x \leqslant y \to z, \quad \forall x, y, z \in \mathcal{L}.$$

The main examples of residuated lattices are MV-algebras introduced by Chang [2] and BL-algebras introduced by Hájek [4].

The hyperstructure theory was introduced by Marty [5], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $f : A \times A \longrightarrow P^*(A)$, of the set $A \times A$ into the set of all nonempty subsets of A, is called a binary *hyperoperation*, and the pair (A, f) is called a *hypergroupoid*. If f is associative, A is called a *semihypergroup*, and it is said to be *commutative* if f is commutative. Also, an element $1 \in A$ is called the *unit* or the *neutral element* if $a \in f(1, a)$, for all $a \in A$.

Recently, R. A. Borzooei et al. introduced and study hyper K-algebras and Sh. Ghorbani et al. applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebra, which is generalization of MV-algebra. In this paper, we want to introduced the concept of hyper residuated lattices and construct the quotient structure in hyper residuated lattices and give results as mentioned in the abstract.

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2. Preliminaries

Definition 2.1. A *residuated lattice* is a structure $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 1)$ satisfying the following axioms:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $x \odot y \leqslant z$ if and only if $x \leqslant y \rightarrow z$, for all $x, y \in L$.

Let $(L', \vee', \wedge', \odot', \rightarrow', 0', 1')$ and $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be two residuated lattices. The map $f : L \rightarrow L'$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$, for all $x, y \in L$, where $* \in \{\odot, \vee, \wedge, \rightarrow\}$

Definition 2.2. [6] A *super lattice* is a partially ordered set $(S; \leqslant)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $a, b, c \in S$,

(SL1) $a \in (a \vee a) \cap (a \wedge a)$,
(SL2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,
(SL3) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(SL4) $a \in ((a \vee b) \wedge a) \cap ((a \wedge b) \vee a)$,
(SL5) $a \leqslant b$ implies $b \in a \vee b$ and $a \in a \wedge b$,
(SL6) if $a \in a \wedge b$ or $b \in a \vee b$ then $a \leqslant b$.

Definition 2.3. Let A be a set, $\odot$ be a binary hyperoperation on A and $1 \in A$. $(A; \odot, 1)$ is called a *commutative semihypergroup* with 1 as an identity if it satisfies the following properties: for all $x, y, z \in A$,

(CSHG1) $x \odot (y \odot z) = (x \odot y) \odot z$,
(CSHG2) $x \odot y = y \odot x$,
(CSHG3) $x \in 1 \odot x$.

Proposition 2.4. *Let $(L, \leqslant)$ be a partially ordered set. Define the binary hyperoperations $\vee$ and $\wedge$ on L as follows: $a \vee b = \{c \,|\, a \leqslant c \text{ and } b \leqslant c\}$ and $a \wedge b = \{c \,|\, c \leqslant a \text{ and } c \leqslant b\}$, for all $a, b \in L$.Then $(L; \vee, \wedge)$ is a bounded super lattice.* □

Definition 2.5. Let $(P, \leqslant)$ be a partially ordered set and γ be an equivalence relation on P. Then γ is called *regular* if the set $P/\gamma = \{[x] \,|\, x \in P\}$ can be ordered in such a way that the natural map $\pi : P \rightarrow P/\gamma$ is order preserving.

Definition 2.6. Let γ be a regular equivalence relation on partially ordered set $(P, \leqslant)$.

(i) By a *γ-fence* we shall mean an ordered subset of P having the following diagram (Figure 1), where $a_i \leqslant b_{i+1}$ and three vertical lines indicate the equivalence modulo γ. We often denote this γ-fence by $\langle a_1, b_n \rangle_\gamma$ and say that a γ-fence

$\langle a_1, b_n\rangle_\gamma$ joins a_1 to b_n.

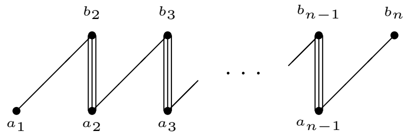


Figure 1. γ-fence

(ii) By a *γ-crown* we shall mean an ordered subset of P having the following diagram (Figure 2)

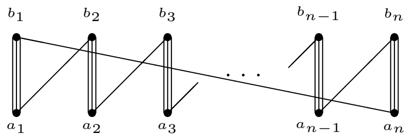


Figure 2. γ-crown

where $a_i \leqslant b_{i+1}$, $a_n \leqslant b_1$ and three vertical lines indicate the equivalence modulo γ. We often denote this γ-crown by $\langle\langle a_1, b_n\rangle\rangle_\gamma$.

(iii) A γ-crown $\langle a_1, b_n\rangle_\gamma$ is called *γ-closed*, when $a_i \gamma b_j$, for all $i, j \in \{1, 2, ..., n\}$.

Theorem 2.7. [1] *Let γ be an equivalence relation on ordered set $(P, \leqslant)$ and $\leqslant_\gamma$ be the relation on $P/\gamma = \{[x] \mid x \in P\}$ defined by $[x] \leqslant_\gamma [y]$ if and only if there is a γ-fence that joins x to y. Then the following statements are equivalent:*

(*i*) $\leqslant_\gamma$ *is an order on* P/γ,

(*ii*) γ *is regular,*

(*iii*) *every γ-crown is γ-closed.* □

3. Quotient hyper residuated lattices

Definition 3.1. By a *hyper residuated lattice* we mean a nonempty set L endowed with four binary hyperoperations $\vee$, $\wedge$, $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:

(HRL1) $(L; \vee, \wedge, 0, 1)$ is a bounded super lattice,

(HRL2) $(L; \odot, 1)$ is commutative semihypergroup with 1 as an identity,

(HRL3) $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$, for all nonempty subset A and B of L.

A hyper residuated lattice is called *nontrivial* if $0 \neq 1$. An element a of hyper residuated lattice L is called *scalar* if $|a \odot x| = 1$, for all $x \in L$.

Definition 3.2. Let $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $(L'; \vee', \wedge', \odot', \rightarrow', 0', 1')$ be two hyper residuated lattices and $f : L \rightarrow L'$ be a function. f is called a *homomorphism* if it satisfies the following conditions: for all $x, y \in L$,

(i) $f(x \vee y) \subseteq f(x) \vee' f(y)$,

(ii) $f(x \wedge y) \subseteq f(x) \wedge' f(y)$,

(iii) $f(x \odot y) \subseteq f(x) \odot' f(y)$,

(iv) $f(x \to y) \subseteq f(x) \to' f(y)$,

(v) $f(1) = 1'$ and $f(0) = 0'$.

If f satisfies (v) and the conditions (i)–(iv) holds for the equality instead of the inclusion, f is said to be a *strong homomorphism*, briefly an *S-homomorphism*.

A homomorphism which is one to one, onto or both is called a *monomorphism*, *epimorphism* or an *isomorphism*, respectively. Similarly, an S-homomorphism which is one-to-one, onto or both is called an *S-monomorphism*, *S-epimorphism* or *S-isomorphism*, respectively.

Definition 3.3. A nonempty subset F of L satisfying

(F) $x \leqslant y$ and $x \in F$ imply $y \in F$

is called a

- *hyper filter* if $x \odot y \subseteq F$, for all $x, y \in F$,
- *weak hyper filter* if $F \ll x \odot y$, for all $x, y \in F$.

A filter F of L is called *proper* if $F \neq L$ and this is equivalent to that $0 \notin F$. Let F be a proper (weak) hyper filter of L. Then F is called a *maximal* if $F \subseteq J \subseteq L$ implies $F = J$ or $J = L$, for all (weak) hyper filters J of L. Moreover, hyper residuated lattice L is called *simple* if $\{\{1\}, L\}$ is the set of all weak hyper filters of L. Obviously, in any hyper residuated lattice L, $\{1\}$ is a weak hyper filter and L is a hyper filter of L.

Remark 3.4. Clearly, any hyper filter of L is a weak hyper filter of L. Moreover, $1 \in F$, for any (weak) hyper filter F of L.

From now on, in this section, L and L' will denote two hyper residuated lattices and for convenience, we use the same notations for the hyper operations of L and L', unless otherwise stated.

In the following, we introduced the concept of regular compatible congruence relations on a hyper residuated lattices and verify some useful properties of these relations. Then we attempted to fine the S-homomorphisms, whose *ker* are regular compatible congruence relations. Then we stated and proved isomorphism theorems on hyper residuated lattices.

Definition 3.5. Let θ be an equivalence relation on L and $A, B \subseteq L$. Then

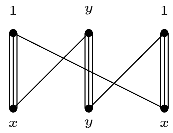
(i) $A\theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\theta b$,

(ii) $A\overline{\theta}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\theta b$ and for all $b \in B$, there exists $a \in A$ such that $a\theta b$,

Definition 3.6. An equivalence relation θ on L is called a *congruence relation* if for all $x, y, z, w \in L$, $x\theta y$ and $z\theta w$ imply $(x * z)\overline{\theta}(y * w)$, where $* \in \{\wedge, \vee, \odot, \to\}$.

Proposition 3.7. *Let θ be a regular equivalence on L. Then $[1] = \{x \in L \mid x\theta 1\}$ is a weak hyper filter of L.*

Proof. Clearly, $[1] \neq \emptyset$. Let $x, y \in [1]$. Since $(x \odot y)\overline{\theta}(1 \odot 1)$ and $1 \in 1 \odot 1$, then $(x \odot y)\theta 1$. Hence $(x \odot y) \cap [1] \neq \emptyset$ and so $[1] \ll x \odot y$. Now, let $x, y \in L$ be such that $x \in [1]$ and $x \leqslant y$. Then we have



and so $\{x, 1, y, y, x, 1\}$, forms a θ-crown on L. Since θ is regular, by Theorem 2.7, $x\theta y$ and so $y \in [1]$. Therefore, $[1]$ is a weak hyper filter of L. □

Lemma 3.8. *Let θ be a regular congruence relation on L, $L/\theta = \{[x] \mid x \in L\}$ and $\leqslant_\theta$ be the relation on L/θ defined as in Theorem 2.7. For all $x, y \in L$, define $[x]\overline{\odot}[y] = [x \odot y]$, $[x]\overline{\vee}[y] = [x \vee y]$, $[x]\overline{\wedge}[y] = [x \wedge y]$ and $[x] \rightsquigarrow [y] = [x \to y]$, where $[A] = \{[a] \mid a \in A\}$, for all $A \subseteq L$. Then*

(*i*) $\overline{\odot}, \overline{\vee}, \overline{\wedge}$ *and* $\rightsquigarrow$ *are well defined,*

(*ii*) $[x] \ll_\theta [y] \rightsquigarrow [z]$ *if and only if* $[x]\overline{\odot}[y] \ll_\theta [z]$, *where* $[A] \ll_\theta [B]$ *if and only if* $[a] \leqslant_\theta [b]$, *for some* $a \in A$ *and* $b \in B$.

Proof. (*i*) Let $[x_1] = [x_2]$ and $[y_1] = [y_2]$, for some $x_1, x_2, y_1, y_2 \in L$. Since θ is a congruence relation on L, we have $(x_1 \odot y_1)\overline{\theta}(x_2 \odot y_2)$. Let $u \in [x_1]\overline{\odot}[y_1]$. Then $[u] = [a]$, for some $a \in x_1 \odot y_1$. By $(x_1 \odot y_1)\overline{\theta}(x_2 \odot y_2)$, we conclude that $a\theta b$, for some $b \in x_2 \odot y_2$ and so $[u] = [a] = [b] \in [x_2]\overline{\odot}[y_2]$. Hence $[x_1]\overline{\odot}[y_1] \subseteq [x_2]\overline{\odot}[y_2]$. By the similar way, we can prove that $[x_2]\overline{\odot}[y_2] \subseteq [x_1]\overline{\odot}[y_1]$. Therefore, $\overline{\odot}$ is well defined. Similarly, it is proved that $\overline{\vee}, \overline{\wedge}$ and $\rightsquigarrow$ are well defined.

(*ii*) Let $[x]\overline{\odot}[y] \ll_\theta [z]$. Then there exists $u \in x \odot y$ such that $[u] \leqslant_\theta [z]$ and so there exists a θ-fence that joins u to z. Let $\langle a_1, b_n\rangle$ be a θ-fence of L that joins u to z, where $u = a_1$ and $z = b_n$. Since $u \in x \odot y$ and $u \leqslant b_2$, then $x \odot y \ll b_2$ and so $x \leqslant c_2 \in y \to b_2$. By $b_2\theta a_2$, we get $(y \to b_2)\overline{\theta}(y \to a_2)$ whence $c_2\theta d_2$, for some $d_2 \in y \to a_2$. Now, from $d_2 \in y \to a_2$ it follows that $d_2 \ll y \to a_2$, and so $d_2 \odot y \ll a_2 \leqslant b_3$. Hence $d_2 \leqslant c_3 \in y \to b_3$. Since $(y \to b_3)\overline{\theta}(y \to a_3)$, then $c_3\theta d_3$, for some $d_3 \in y \to a_3$. Hence $x \leqslant c_2\theta d_2 \leqslant c_3\theta d_3$. By the similar way, there are $c_i \in y \to b_i$, for any $i \in \{2, 3, \ldots, n\}$ and $d_j \in y \to a_j$, for any $j \in \{2, 3, \ldots, n-1\}$ such that $x \leqslant c_2\theta d_2 \leqslant c_3\theta d_3 \leqslant \ldots \leqslant c_{n-1}\theta d_{n-1} \leqslant c_n$. Hence the set $\{x, d_2, \ldots, d_{n-1}, c_2, \ldots, c_n\}$ forms a θ-fence that joins x to c_n and so $[x] \leqslant_\theta [c_n]$. Since $c_n \in y \to b_n = y \to z$, we have $[x] \ll_\theta [y \to z] = [y] \rightsquigarrow [z]$. Conversely, let $[x] \ll_\theta [y] \rightsquigarrow [z]$. Then $[x] \leqslant_\theta [u]$, for some $u \in y \to z$. Hence there is a θ-fence, $\langle a_1, b_n\rangle_\theta$, that joins x to u, where $x = a_1$ and $u = b_n$. By $a_{n-1} \leqslant u \in y \to z$, we get $a_{n-1} \odot y \ll z$, whence $e_{n-1} \leqslant z$, for some $e_{n-1} \in a_{n-1} \odot y$. Since $a_{n-1}\theta b_{n-1}$, then $(a_{n-1} \odot y)\overline{\theta}(b_{n-1} \odot y)$ and so there exists $f_{n-1} \in b_{n-1} \odot y$ such that $f_{n-1}\theta e_{n-1}$. From $f_{n-1} \in b_{n-1} \odot y$ it follows that $b_{n-1} \odot y \ll f_{n-1}$, whence $a_{n-2} \leqslant b_{n-1} \ll y \to f_{n-1}$. Hence $a_{n-2} \odot y \ll f_{n-1}$ and so there is $e_{n-2} \in a_{n-2} \odot y$ such that $e_{n-2} \leqslant f_{n-1}$. From $(a_{n-2} \odot y)\overline{\theta}(b_{n-2} \odot y)$ it follows that $e_{n-2}\theta f_{n-2}$, for some $f_{n-2} \in b_{n-2} \odot y$. By a similar way, there are $e_i \in a_i \odot y$ and $f_i \in b_i \odot y$ such that $f_i\theta e_i$ and $e_j \leqslant f_{j+1}$, for all $i \in \{2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n-2\}$. Therefore, $\{e_1, \ldots, e_{n-1}, f_2, \ldots, f_{n-1}, z\}$ forms a θ-fence that joins e_1 to z and so $[e_1] \leqslant_\theta [z]$. Since $e_1 \in a_1 \odot y = x \odot y$, then $[x]\overline{\odot}[y] = [x \odot y] \ll_\theta [z]$. □

Definition 3.9. Let θ be a regular congruence relation on L. We say that $\leqslant_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are *compatible* if they satisfy the following conditions: for all $x, y \in L$,

(i) $[x] \in [x]\overline{\vee}[y]$ if and only if $[x] \leqslant_\theta [y]$,

(ii) $[x] \in [x]\overline{\wedge}[y]$ if and only if $[x] \leqslant_\theta [y]$.

By a *regular compatible congruence relation* on L we mean a regular congruence relation on L such that $\leqslant_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are compatible.

Theorem 3.10. *Let θ be a regular compatible congruence relation on L. Then $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.*

Proof. Since θ is regular, by Theorem 2.7, $\leqslant_\theta$ is a partially order on L. Clearly, $[0]$ and $[1]$ are the minimum and the maximum elements of $(L/\theta, \leqslant_\theta)$. Moreover, $[x]\overline{\odot}[y] = [x \odot y] = [y \odot x] = [y]\overline{\odot}[x]$, for any $x, y \in L$. By the similar way, we can show that $(L/\theta, \overline{\odot}, [1])$ is a commutative semihypergroup with $[1]$ as an identity. Hence by Lemma 3.8 and Definition 3.9, $(L/\theta, \overline{\vee}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. □

Example 3.11. Let $(\{0, a, b, c, 1\}, \leqslant)$ be a partially ordered set such that $0 < a < b < c < 1$, $L = \{0, a, b, c, 1\}$. Consider the following tables:

Table 1

$\vee$	0	a	b	c	1
0	{0,a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
a	{a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
b	{b,c,1}	{b,c,1}	{b,c,1}	{c,1}	{1}
c	{c,1}	{c,1}	{c,1}	{c,1}	{1}
1	{1}	{1}	{1}	{1}	{1}

Table 2

$\wedge$	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{a,0}	{a,0}	{a,0}	{a,0}
b	{0}	{a,0}	{b,0}	{b,a,0}	{b,a,0}
c	{0}	{a,0}	{b,a,0}	{c,a,0}	{c,a,0}
1	{0}	{a,0}	{a,b,0}	{c,a,0}	{0,1,a,c}

Table 3

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{1,a,c}	{a,1}	{b,1}	{c,1}	{1}
b	{1,b,c}	{b,1}	{b,1}	{c,1}	{1}
c	{1,c}	{c}	{c}	{c,1}	{1}
1	{1,c}	{1,c}	{1,c}	{1,c}	{1}

Let $\odot = \wedge$. It is easy to verify that $(L; \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice. Let $\theta = \{(x,x) \mid x \in L\} \cup \{(a,b),(b,a)\}$. Routine calculations show that θ is a congruence relation on L, such that $\overline{\vee}$, $\overline{\wedge}$ and $\leqslant_\theta$ are compatible. Consider the partially order relation $[0] \prec [a] \prec [c] \prec [1]$ on L/θ. Since the mapping $\pi : L \to L/\theta$ defined by $\pi(x) = [x]$, for all $x \in L$ is an ordered preserving map, then θ is regular. Therefore, by Theorem 3.10, $(L; \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. □

Proposition 3.12. *Let θ be a regular compatible congruence relation on L. Then*

(i) $[1]$ *is a hyper filter of L if and only if $\{[1]\}$ is a hyper filter of L/θ.*

(ii) *if $[1]$ is a maximal weak hyper filter of L, then L/θ is simple.*

Proof. (i) Let $[1]$ be a hyper filter of L. Then $\{[1]\}$ is a weak hyper filter of L/θ. It suffices to show that $[1]\overline{\odot}[1] = [1]$. Since $1 \in [1]$ and $[1]$ is a hyper filter of L, then $1 \odot 1 \subseteq [1]$ and so $[1]\overline{\odot}[1] = [1 \odot 1] = [1]$. Hence $\{[1]\}$ is a hyper filter of L. Conversely, assume that $\{[1]\}$ is a hyper filter of L/θ. By Proposition 3.7, $[1]$ is a weak hyper filter of L. Let $a, b \in [1]$. Since $[1]\overline{\odot}[1] = [1]$ and $[a] = [b] = [1]$, then $[a \odot b] = [a]\overline{\odot}[b] = [1]\overline{\odot}[1] = [1]$. Hence $a \odot b \subseteq [1]$ and so $[1]$ is a hyper filter of L/θ.

(ii) By Proposition 3.7, $[1]$ is a weak hyper filter of L. Assume $[1]$ is a maximal weak hyper filter of L and F is a weak hyper filter of L. Let $M = \cup\{[x] \mid [x] \in F\}$. Then clearly, $M \neq \emptyset$. If $u, v \in M$, then $[u] \in F$ and $[v] \in F$ and so $[u \odot v] = [u]\overline{\odot}[v] \cap F \neq \emptyset$. Hence there exists $a \in u \odot v$ such that $[a] \in F$ and so $a \in M$. Hence $(u \odot v) \cap M \neq \emptyset$. Now, let $x \in M$ and $x \leqslant y$, for some $y \in L$. Then clearly, $\{x, y\}$ formes a θ-fence that joins x to y and so $[x] \leqslant_\theta [y]$. Since $[x] \in F$ and F is a weak hyper filter of L/θ, then $[y] \in F$ and so $y \in M$. Therefore, M is a weak hyper filter of L. Clearly, $[1] \subseteq M$. Since $[1]$ is a maximal weak hyper filter of L, then $[1] = M$ or $M = L$. If $M = L$, then $F = L/\theta$. Moreover, if $[1] = M$, then $F = \{[1]\}$. Therefore, $\{\{[1]\}, L/\theta\}$ is the set of all weak hyper filters of L/θ and so L/θ is simple. □

The converse of Proposition 3.12(ii) may not be true.

Example 3.13. Let $L = \{0, a, b, c, 1\}$ and $(L, \leqslant)$ be a partially ordered set such that $0 < c < a < b < 1$. Define the binary hyperoperations $\vee$, $\odot$ and $\wedge$ on L as follows: $a \vee b = \{c \mid a \leqslant c \text{ and } b \leqslant c\}$ and $a \odot b = a \wedge b = \{c \mid c \leqslant a \text{ and } c \leqslant b\}$, for all $a, b \in L$. Now, let $\rightarrow$ be a hyperoperation on L defined by the following table.

Table 4

$\rightarrow$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0,1}	{1}	{1}	{c,1}	{1}
b	{0,1}	{b,a,1}	{1}	{c,1}	{1}
c	{1}	{1}	{1}	{1}	{1}
1	{0,1}	{a,b,1}	{b,1}	{c,1}	{1}

It is not difficult to check that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice.

Let $\theta = \{(x, x) | x \in L\} \cup \{(1, a), (a, 1), (1, b), (b, 1), (a, b), (b, a), (c, 0), (0, c)\}$. Clearly, θ is an equivalence relation on L and $L/\theta = \{[1], [0]\}$. Define a relation $\prec$ on L/θ by $[0] \prec [1]$ and $[x] \prec [x]$, for all $x \in L/\theta$. Then $\prec$ is a partially order on L/θ. Moreover, the map $f : L \rightarrow L/\theta$ defined by $f(x) = [x]$, for all $x \in L$ is an ordered preserving map and so θ is regular. Hence By Theorem 2.7, $\leqslant_\theta$ is a partially order on L/θ. It is easy to check that $\leqslant_\theta = \prec$. Clearly, $[y] \in [x]\overline{\vee}[y]$ ($[x] \in [x]\overline{\wedge}[y]$) if and only if $[x] \leqslant_\theta [y]$, for all $[x], [y] \in L/\theta$. Hence θ is a regular compatible congruence relation of L and so by Theorem 3.10, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice. Since $L/\theta = \{[0], [1]\}$, then L/θ is simple. Moreover, $F = \{1, a, b, c\}$ is a weak hyper filter of L and $[1] \subset F \subset L$ and so $[1] = \{1, a, b\}$ is not a maximal weak hyper filter of L. Therefore, the converse of Proposition 3.12 (ii) may not be true. □

Let L and L' be two hyper residuated lattices and $f : L \rightarrow L'$ be a homomorphism. It is straightforward to check that $ker(f) = \{(x, y) \in L \times L \,|\, f(x) = f(y)\}$ is an equivalence relation on L. In Theorem 3.14, we want to verify this relation.

Theorem 3.14. *Let $f : L \rightarrow L'$ be an S-homomorphism and $\theta = ker(f)$. If $f(x) \leqslant f(y)$ implies there is a θ-fence that joins x to y, for all $x, y \in L$, then*

- (i) *θ is a regular compatible congruence relation on L and $L/ker(f)$ is a hyper residuated lattice,*
- (ii) *f induces a unique S-homomorphism $\overline{f} : L/ker(f) \rightarrow L'$ by $\overline{f}([x]) = f(x)$, for all $x \in L$ such that $Im(\overline{f}) = Im(f)$ and $\overline{f}$ is an S-monomorphism.*

Proof. (i) Let $x\theta y$ and $u\theta v$, for some $x, y, u, v \in L$. Then $f(x) = f(y)$ and $f(u) = f(v)$. Since f is an S-homomorphism, then $f(x \wedge u) = f(x) \wedge f(u) = f(y) \wedge f(v) = f(y \wedge v)$ and so $(x \wedge u)\overline{\theta}(y \wedge v)$. By the similar way we can prove the other cases. Now, we show that θ is regular. Let $\langle\langle a_1, b_n\rangle\rangle_\theta$ be a θ-crown of L. Then $f(a_i) = f(b_i)$, for all $i \in \{1, 2, ..., n\}$. Since $a_i \leqslant b_{i+1}$, then $a_i \in a_i \wedge b_{i+1}$ and so $f(a_i) \in f(a_i \wedge b_{i+1}) = f(a_i) \wedge f(b_{i+1}) = f(a_i) \wedge f(a_{i+1})$. Similarly, $a_n \leqslant b_1$ implies that $f(a_n) \leqslant f(b)_1$. Hence $f(a_i) \leqslant f(a_{i+1})$, for all $i \in \{1, 2, \ldots, n-1\}$ and so $f(x) = f(a_1) \leqslant f(a_2) \leqslant f(a_3) \leqslant \cdots \leqslant f(a_{n-1}) \leqslant f(a_n) \leqslant f(b_1) = f(a_1)$. Therefore, $f(a_i) = f(b_j)$, for all $i, j \in \{1, 2, \ldots, n\}$ and so $[a_i] = [a_j] = [b_k]$, for all $i, j, k \in \{1, 2, \ldots, n\}$. By Proposition 2.7, θ is regular. In the follow, we show that $[x] \in [x]\overline{\wedge}[y] \Leftrightarrow [x] \leqslant_\theta [y] \Leftrightarrow [y] \in [x]\overline{\vee}[y]$. Let $[x] \leqslant_\theta [y]$, for some $x, y \in L$. Then there exists a θ-fence, $\langle a_1, b_n\rangle$ that joins x to y, where $x = a_1$ and $y = b_n$. By $a_1 \leqslant b_2$, it follows that $f(a_1) \in f(a_1) \wedge f(b_2) = f(a_1) \wedge f(a_2)$ and so $f(a_1) \leqslant f(a_2)$. By a similar way, we can show that $f(a_i) \leqslant f(a_{i+1})$, for all $i \in \{1, 2, \dots, n-1\}$. Since $f(a_{n-1}) \leqslant f(b_n) = f(y)$, then we conclude that $f(x) \leqslant f(y)$ and so $f(x) \in f(x) \wedge f(y) = f(x \wedge y)$ $(f(y) \in f(x) \vee f(y) = f(x \vee y))$. Hence $f(x) = f(a)$, for some $a \in x \wedge y$ $(a \in x \vee y)$, whence $[x] \in [x \wedge y] = [x]\overline{\wedge}[y]$ $([y] \in [x \vee y] = [x]\overline{\vee}[y])$. Conversely, let $[x] \in [x \wedge y] = [x]\overline{\wedge}[y]$ $([y] \in [x \vee y] = [x]\overline{\vee}[y])$, for some $x, y \in L$. Then there is $a \in x \wedge y$ $(a \in x \vee y)$ such that $[x] = [a]$ and so $f(x) = f(a) \in f(x \wedge y) = f(x) \wedge f(y)$ $(f(x) = f(a) \in f(x \vee y) = f(x) \vee f(y))$. Hence $f(x) \leqslant f(y)$, whence by hypothesis, there is a θ-fence that joins x to y. That is $[x] \leqslant_\theta [y]$. Therefore, θ is a regular congruence relation on L and θ, $\overline{\vee}$, $\overline{\wedge}$ are compatible and so by Theorem 3.10, $(L/ker(f), \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.

(*ii*) Clearly, $\overline{f} : L/ker(f) \to L'$ is an S-homomorphism and $Im(\overline{f}) = Im(f)$. Let $\overline{f}([x]) = \overline{f}([y])$, for some $x, y \in L$. Then $f(x) = f(y)$ and so $[x] = [y]$. Therefore, $\overline{f}$ is a one to one S-homomorphism. □

Example 3.15. If L and L' are two residuated lattices and $f : L \to L'$ is a homomorphism, then $f(x) \leqslant f(y)$ implies $f(x) = f(x) \wedge f(y) = f(x \wedge y)$ and so the set $\{x, x, x \wedge y, y\}$, forms a $ker(f)$-fence that joins x to y. Therefore, f satisfies the conditions (*i*) and (*ii*) in Theorem 3.14. □

Example 3.16. Let $(L = \{0, a, b, c, 1\}, \leqslant)$ and $(L' = \{0, e, 1\}, \leqslant')$ be two partially ordered sets such that $0 < a < b < c < 1$ and $0 < e < 1$. Define the binary hyperoperations $\vee, \wedge, \vee'$ and $\wedge'$ by $x \vee y = \{u \in L \,|\, x \leqslant u, y \leqslant u\}$, $a \vee' b = \{u \in L' \,|\, a \leqslant' u, b \leqslant' u\}$, $x \wedge y = \{u \in L \,|\, u \leqslant x, u \leqslant y\}$ and $a \wedge' b = \{u \in L' \,|\, u \leqslant' a, u \leqslant' b\}$, for all $x, y \in L$ and $a, b \in L'$. Then by Proposition 2.4, $(L, \vee, \wedge, 0, 1)$ and $(L', \vee', \wedge', 0, 1)$ are two bounded super lattices. Let $\odot$ and $\odot'$ are defined by

$$a \odot b = \begin{cases} \{0\} & \text{if } a = 0 \text{ or } b = 0, \\ (a \wedge b) - \{0\} & \text{if } a, b \in L - \{0\}. \end{cases}$$

$$a \odot' b = \begin{cases} \{0\} & \text{if } a = 0 \text{ or } b = 0, \\ (a \wedge' b) - \{0\} & \text{if } a, b \in L' - \{0\}. \end{cases}$$

Now, consider the following tables:

Table 5

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0}	{b,1}	{b,1}	{c,1}	{1}
b	{0}	{b,c}	{b,1}	{1}	{1}
c	{0}	{a,c}	{b,c}	{c,1}	{1}
1	{0}	{b,1}	{b,1}	{1}	{1}

Table 6

$\to'$	0	e	1
0	{1}	{1}	{1}
e	{0}	{e,1}	{1}
1	{0}	{1,e}	{1}

It is easy to verify that $(L,\vee,\wedge,\odot,\rightarrow,0,1)$ and $(L',\vee',\wedge',\odot',\rightarrow',0,1)$ are hyper residuated lattices. Define $f:L\rightarrow L'$ by $f(0)=0$, $f(a)=f(b)=e$ and $f(c)=f(1)=1$. Then f is an S-homomorphism,

$ker(f)=\{(x,x)\mid x\in L\}\cup\{(a,b),(b,a),(1,c),(c,1)\}$ and $L/ker(f)=\{[0],[a],[1]\}$.

Assume $\prec=\{(x,x)\mid x\in L/ker(f)\}\cup\{([0],[a]),([a],[1]),([0],[1])\}$. Then clearly, $\prec$ is a partially order on $L/ker(f)$. Since the map $\pi:L\rightarrow L/\theta$ defined by $\pi(x)=[x]$ is an order preserving map, then $ker(f)$ is regular. Easy calculations show that $f(x)\leqslant f(y)$ implies there exists a θ-fence on L that joins x to y, for any $x,y\in L$ and so by Theorem 3.14, $\overline{f}:L/\theta\rightarrow L'$ is a one to one homomorphism. □

Lemma 3.17. *Let θ and χ be two regular compatible congruence relations on L such that $\theta\subseteq\chi$. Then χ/θ is a regular compatible congruence relation on L/θ, where $\chi/\theta=\{([x]_\theta,[y]_\theta)\in L/\theta\times L/\theta\mid(x,y)\in\chi\}$.*

Proof. By Theorem 3.10, $(L/\theta,\overline{\vee},\overline{\wedge},\overline{\odot},\rightsquigarrow,[0],[1])$ is a hyper residueted lattice. Clearly, χ/θ is an equivalence relation on L/θ. Let $([x]_\theta,[y]_\theta)([a]_\theta,[b]_\theta)\in\chi/\theta$. Then $(x,y),(a,b)\in\chi$. Since χ is a congruence relation on L we have $(a\wedge x)\overline{\chi}(b\wedge y)$ and so by definition of $\overline{\wedge}$ we get $([a]_\theta\overline{\wedge}[x]_\theta)\overline{\chi/\theta}([b]_\theta\overline{\wedge}[y]_\theta)$. By the similar way, we can show that

$$([a]_\theta\overline{\vee}[x]_\theta)\overline{\chi/\theta}([b]_\theta\overline{\vee}[y]_\theta,\ ([a]_\theta\overline{\odot}[x]_\theta)\overline{\chi/\theta}([b]_\theta\overline{\odot}[y]_\theta),\ ([a]_\theta\rightsquigarrow[x]_\theta)\overline{\chi/\theta}([b]_\theta\rightsquigarrow[y]_\theta).$$

Hence χ/θ is a congruence relation on L/θ. Let $R=\chi/\theta$ and $(L/\theta)/R=\{[[x]_\theta]_R\mid[x]_\theta\in L/\theta\}$. Define the hyperoperations $\sqcup$, $\sqcap$, $\otimes$ and $\mapsto$ by

$$[[x]_\theta]_R\sqcup[[y]_\theta]_R=[[x]_\theta\overline{\vee}[y]_\theta]_R,\quad [[x]_\theta]_R\sqcap[[y]_\theta]_R=[[x]_\theta\overline{\wedge}[y]_\theta]_R,$$

$$[[x]_\theta]_R\otimes[[y]_\theta]_R=[[x]_\theta\overline{\odot}[y]_\theta]_R \text{ and } [[x]_\theta]_R\mapsto[[y]_\theta]_R=[[x]_\theta\rightsquigarrow[y]_\theta]_R$$

for all $[[x]_\theta]_R,[[y]_\theta]_R\in(L/\theta)/R$. Since R is a congruence relation on L/θ, then these hyperoperations are well defined. Now, we show that R is regular. Let $\langle\langle[a_1]_\theta,[b_n]_\theta\rangle\rangle_R$ be an R-crown in L/θ. Then $[a_n]_\theta\leqslant_\theta[b_1]_\theta$, $[a_i]_\theta\overline{R}[b_i]_\theta$ and $[a_j]_\theta\leqslant_\theta[b_{j+1}]_\theta$, for all $i\in\{1,2,\ldots,n\}$ and $j\in\{1,2,\ldots,n-1\}$. Hence there are $n_i\in\mathbb{N}$ such that $a_{2,i},a_{3,i},\ldots,a_{n_i-1,i},b_{2,i},b_{3,i},\ldots,b_{n_i-1,i}\in L/\theta$ such that

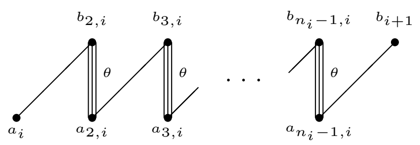


Figure 3. θ-fence joins a_i to b_{i+1}

for all $i\in\{1,2,\ldots,n-1\}$. Moreover, there exists a θ-fence $\langle x_1,y_n\rangle_\theta$, that joins a_n to b_1. Since $[a_i]_\theta\overline{R}[b_i]_\theta$, for all $i\in\{1,2,\ldots,n\}$ and $\theta\subseteq\chi$, then we can obtain the following χ-crown.

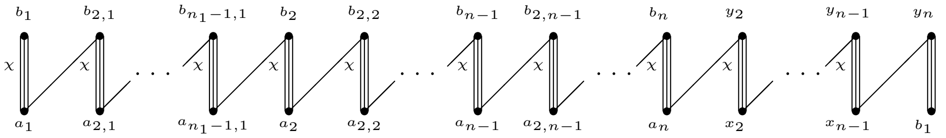


Figure 4. χ-crown

Since χ is regular, then by Theorem 2.7, $(a_i, b_j) \in \chi$ and so $[a_i]_\theta R[b_j]_\theta$, for all $i, j \in \{1, 2, \ldots, n\}$, whence $\langle [a_1]_\theta, [b_n]_\theta\rangle_R$ is χ/θ closed. Now, by Theorem 2.7, R is regular. Finally, we show that R is compatible. Let $x, y \in L$ such that $[x]_\theta \leqslant_R [y]_\theta$. Then there is an R-fence $\langle [a_1]_\theta, [b_n]_\theta\rangle_R$ that joins $[x]_\theta$ to $[y]_\theta$, where $[x]_\theta = [a_1]_\theta$ and $[y]_\theta = [b_n]_\theta$. By $[a_j]_\theta R[b_j]_\theta$, we get $(a_j, b_j) \in \chi$, for all $j \in \{2, 3, \ldots, n-1\}$. Since $[a_i]_\theta \leqslant_\theta [b_{i+1}]_\theta$, for all $i \in \{1, 2, \ldots, n-1\}$, then there exists θ-fence $\langle a_{1,i}, b_{n_i,i}\rangle_\theta$ joins a_i to b_{i+1}, where $a_i = a_{1,i}$ and $b_{i+1} = b_{n_i,i}$, for all $i \in \{1, 2, \ldots, n-1\}$. Hence by $\theta \subseteq \chi$, we can obtain the following χ-fence that joins x to y.

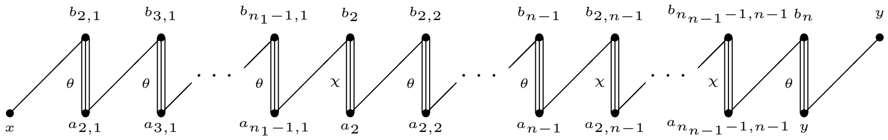


Figure 5. χ-fence joins x to y

Therefore, $[x]_\chi \leqslant_\chi [y]_\chi$. Since χ is a compatible regular congruence relation on L, then $[x]_\chi \in [x]_\chi \overline{\overline{\wedge}}[y]_\chi = [x \wedge y]_\chi$ and $[y]_\chi \in [y]_\chi \overline{\overline{\vee}}[y]_\chi = [x \vee y]_\chi$, where $\overline{\overline{\wedge}}$ and $\overline{\overline{\vee}}$ are hyper operation induced by χ in Lemma 3.8. Hence

$$[[y]_\theta]_R \in [[x \vee y]_\theta]_R = [[x]_\theta \overline{\vee} [y]_\theta]_R = [[x]_\theta]_R \sqcup [[y]_\theta]_R.$$

By the similar way, $[[x]_\theta]_R \in [[x]_\theta]_R \sqcap [[y]_\theta]_R$. Conversely, let $[[x]_\theta]_R \in [[x]_\theta]_R \sqcap [[y]_\theta]_R$. Then $[[x]_\theta]_R \in [[x \wedge y]_\theta]_R$ and so $[[x]_\theta]_R = [[u]_\theta]_\chi$, for some $u \in x \wedge y$. By definition of R, we conclude that $(x, u) \in \chi$ and so $[x]_\chi \in [x \wedge y]_\chi = [x]_\chi \overline{\overline{\wedge}}[y]_\chi$. Since χ is a compatible regular congruence relation on L, then $[x]_\chi \leqslant_\chi [y]_\chi$ and so there exists a χ-fence $\langle a_1, b_n\rangle_\chi$, that joins x to y, where $x = a_1$ and $y = b_n$. Clearly, $\langle [a_1]_\theta, [b_n]_\theta\rangle_R$ is a R-fence on L/θ and so $[[x]_\theta]_R \leqslant_R [[y]_\theta]_R$. By a similar way, $[[y]_\theta]_R \in [[x]_\theta]_R \sqcup [[y]_\theta]_R$ implies $[[x]_\theta]_R \leqslant_R [[y]_\theta]_R$. Therefore, R is a compatible regular congruence relation on L/θ. □

Theorem 3.18. *Let θ and χ be two regular compatible congruence relations on L such that $\theta \subseteq \chi$. Then $\frac{L/\theta}{\chi/\theta}$ and L/χ are S-isomorphic.*

Proof. By Theorem 3.10, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0]_\theta, [1]_\theta)$ and $(L/\chi, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0]_\chi, [1]_\chi)$ are two hyper residuated lattices. Let $\sqcup$, $\sqcap$, $\otimes$ and $\mapsto$ be the hyperoperations defined in Lemma 3.17. Then by Lemma 3.17 and Theorem 3.10, we see that $(\frac{L/\theta}{\chi/\theta}, \sqcup, \sqcap, \otimes, \mapsto, [[0]_\theta]_{\chi/\theta}, [[1]_\theta]_{\chi/\theta})$ is a hyper residuated lattice.

Define $f : \frac{L/\theta}{\chi/\theta} \to L/\chi$ by $f([[x]_\theta]_{\chi/\theta}) = [x]_\chi$. Let $[[x]_\theta]_{\chi/\theta} = [[y]_\theta]_{\chi/\theta}$, for some $x, y \in L$. Then by definition of χ/θ, we get $(x, y) \in \chi$ and so $[x]_\chi = [y]_\chi$. Hence f is well defined. Let $x, y \in L$. Then

$$\begin{aligned} f([[x]_\theta]_{\chi/\theta} \sqcap [[y]_\theta]_{\chi/\theta}) &= f([[x]_\theta \overline{\wedge} [y]_\theta]_{\chi/\theta}) \\ &= f([[x \wedge y]_\theta]_{\chi/\theta}) = \{f([[u]_\theta]_{\chi/\theta}) | u \in x \wedge y\} \\ &= \{[u]_\chi | u \in x \wedge y\} = [x \wedge y]_\chi = [x]_\chi \overline{\wedge} [y]_\chi \\ &= f([[x]_\theta]_{\chi/\theta}) \overline{\wedge} f([[y]_\theta]_{\chi/\theta}). \end{aligned}$$

By the similar way, we can show that

$$f([[x]_\theta]_{\chi/\theta} \sqcup [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \overline{\vee} f([[y]_\theta]_{\chi/\theta}),$$

$$f([[x]_\theta]_{\chi/\theta} \otimes [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \overline{\odot} f([[y]_\theta]_{\chi/\theta}),$$

$$f([[x]_\theta]_{\chi/\theta} \mapsto [[y]_\theta]_{\chi/\theta}) = f([[x]_\theta]_{\chi/\theta}) \rightsquigarrow f([[y]_\theta]_{\chi/\theta}).$$

Hence f is an S-homomorphism. Now, we show that f is one to one and onto. Clearly, f is an onto map. Let $f([[x]_\theta]_{\chi/\theta}) = f([[y]_\theta]_{\chi/\theta})$, for some $x, y \in L$. Then $[x]_\chi = [y]_\chi$ and so $(x, y) \in \chi$. Hence $[[x]_\theta]_{\chi/\theta} = [[y]_\theta]_{\chi/\theta}$ and so f is one to one. Therefore, f is an S-isomorphism. □

Remark 3.19. Let $(L_1; \vee_1, \wedge_1, \odot_1, \to_1, 0_1, 1_1)$ and $(L_2; \vee_2, \wedge_2, \odot_2, \to_2, 0_2, 1_2)$ be two hyper residuated lattices. We define the hyperoperations $\vee$, $\wedge$, $\to$ and $\odot$ on $L = L_1 \times L_2$ as follows:

$$\begin{aligned} (x_1, x_2) \vee (y_1, y_2) &= (x_1 \vee_1 y_1, x_2 \vee_2 y_2), \\ (x_1, x_2) \wedge (y_1, y_2) &= (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2), \\ (x_1, x_2) \odot (y_1, y_2) &= (x_1 \odot_1 y_1, x_2 \odot_2 y_2), \\ (x_1, x_2) \to (y_1, y_2) &= (x_1 \to_1 y_1, x_2 \to_2 y_2). \end{aligned}$$

where $(A, B) = \{(a, b) \mid a \in A, b \in B\}$, for all subsets $A \subseteq L_1$ and $B \subseteq L_2$. Then $(L_1 \times L_2, \leqslant)$ satisfies (HRL1)-(HRL3) in which the order $\leq$ is given by

$$(a, b) \leqslant (c, d) \Leftrightarrow a \leqslant c, bd, \quad \forall a, c \in L_1, b, d \in L_2.$$

Hence $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper residuated lattice, where $1 = (1, 1)$ and $0 = (0, 0)$.

Theorem 3.20. *If θ_1 and θ_2 are two regular compatible congruence relations on L_1 and L_2, respectively, and θ is a relation on $L_1 \times L_2$ defined by $(a, b)\theta(u, v)$ if and only if $(a, u) \in \theta_1$ and $(b, v) \in \theta_2$. Then θ is a regular compatible congruence relation on L and*

$$L/\theta \cong (L_1/\theta_1) \times (L_2/\theta_2).$$

Proof. Since θ_1 and θ_2 are regular compatible congruence relations on L_1 and L_2, respectively, then by Theorem 3.10, $(L_1/\theta_1, \leqslant_{\theta_1})$ and $(L_2/\theta_2, \leqslant_{\theta_2})$ are hyper residuated lattices. Let $\leqslant'$ be a partial order on $(L_1/\theta_1) \times (L_2/\theta_2)$, where $([x],[y]) \leqslant' ([a],[b])$ means that $[x] \leqslant_{\theta_1} [a]$ and $[y] \leqslant_{\theta_2} [b]$. Clearly, θ is a congruence relation on $L = L_1 \times L_2$. Let $\langle\langle (a_1, b_1), (c_1, d_1)\rangle\rangle_\theta$ be a θ-crown in L. Then by definition of $\leqslant$, we get $\langle\langle a_1, c_n\rangle\rangle$ is a θ_1-crown on L_1 and $\langle\langle b_1, d_n\rangle\rangle$ is a θ_2 crown on L_2. Since θ_1 is regular, then by Theorem 2.7, $a_i \cong c_j$, for all $i, j \in \{1, 2, \ldots, n\}$. By a similar way, we can show that $b_i \cong d_j$, for all $i, j \in \{1, 2, \ldots, n\}$. Hence $(a_i, b_i)\theta(c_i, d_i)$, for all $i, j \in \{1, 2, \ldots, n\}$ and so by Theorem 2.7, θ is regular. Now, we show that θ is compatible. Let $[x]_i = \{a \in L_i \mid x\theta_i a\}$, for all $i \in \{1, 2\}$. If $x, a \in L_1$, $y, b \in L_2$ and $\overline{\vee}\ \overline{\wedge}$ are the hyperoperations on L induced by $\vee$ and $\wedge$, then we have

$$\begin{aligned}
[(x,y)] \in [(x,y)]\overline{\wedge}[(a,b)] &\Leftrightarrow [(x,y)] \in [(x \wedge_1 a, y \wedge_2 b)]\\
&\Leftrightarrow [x] \in [x \wedge_1 a]_1 \text{ and } [y] \in [y \wedge_1 b]_2\\
&\Leftrightarrow x \leqslant_1 a, y \leqslant_2 b, \text{ since } \theta_1 \text{ and } \theta_2 \text{ are compatible}\\
&\Leftrightarrow (x,y) \leqslant (a,b).
\end{aligned}$$

By a similar way, we can show that $[(x,y)] \in [(x,y)]\overline{\vee}[(a,b)] \Leftrightarrow (x,y) \leqslant (a,b)$. Hence θ is compatible and so by Theorem 3.10, L/θ is a hyper residuated lattice. Define the map $f : L \to (L_1/\theta_1) \times (L_2/\theta_2)$, by $f((x,y)) = ([x]_1, [y]_2)$, for any $(x,y) \in L$. Let $* \in \{\vee, \wedge, \odot, \to\}$. Then

$$\begin{aligned}
f((x,y) * (a,b)) &= f(x * a, y * b)\\
&= ([x *_1 a]_1, [y *_2 b]_2)\\
&= ([x]_1 *_1 [a]_1, [y]_2 *_2 [b]_2)\\
&= ([x]_1, [y]_2) * ([a]_1, [b]_2)\\
&= f((x,y)) * f((a,b)).
\end{aligned}$$

Hence f is a S-homomorphism. Clearly, f is onto. Now, we show that $ker(f) = \theta$.

$$\begin{aligned}
ker(f) &= \{((x,y),(a,b)) \in L \times L \mid f((x,y)) = f((a,b))\}\\
&= \{((x,y),(a,b)) \in L \times L \mid ([x]_1,[y]_2) = ([a]_1,[b]_2)\}\\
&= \{((x,y),(a,b)) \in L \times L \mid [x]_1 = [a]_1, \quad [y]_1 = [b]_1\}\\
&= \theta.
\end{aligned}$$

Now, let $f((x,y)) \leqslant' f((a,b))$. Then $([x]_1,[y]_2) \leqslant' ([a]_1,[b]_2)$ and so $[x]_1 \leqslant_{\theta^1} [a]_1$ and $[y]_2 \leqslant_{\theta_2} [b]_2$. Hence by definition of $\leqslant_{\theta_1}$ and $\leqslant_{\theta_2}$, there are $\langle u_1, v_n\rangle_{\theta_1}$, that joins x to a and $\langle w_1, z_m\rangle_{\theta_2}$, that joins y to b. Without loss of generality, we assume that $n \leqslant m$. Then the set

$$\{(u_1,w_1),(v_2,z_2)\ldots,(v_n,z_n),(v_n,w_{n+1})(v_n,z_{n+1}),\ldots,\\
\ldots,(v_n,z_{m-1}),(v_n,w_{m-1}),(v_n,z_m)\}$$

is a θ-fence that joins (x,y) to (a,b). Hence by Theorem 3.14 we obtain $L/\theta = L/ker(f) \cong (L_1/\theta_1) \times (L_2/\theta_2)$, which completes the proof. □

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