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Congruences on ternary semigroups

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Abstract. We study congruences on ternary semigroups. We have extended Lallement's lemma for a regular ternary semigroups. We have characterized minimum group congruence and maximum idempotent pair separating congruence on a strongly regular ternary semigroups. We have also obtained a characterization for maximum idempotent pair separating congruence and smallest strongly regular congruence on an orthodox ternary semigroup.

1. Introduction

Ternary semigroups, i.e., algebras of the form $(T, [\ \ ])$, where $[\ \ ]$ is a *ternary operation* $T^3 \longrightarrow T : (x, y, z) \longrightarrow [xyz]$ satisfying the associative law

$$[xy[uvw]] = [x[yuv]w] = [[xyu]vw]$$

are studied by many authors. The study of ideals and radicals of ternary semigroups was initiated in [11]. The concept of regular ternary semigroups was introduced in [10]. In [6] regular ternary semigroups was characterized by ideals. In [8] regular ternary semigroups are characterized by idempotent pairs. Orthodox ternary semigroups are investigated in [9]. Congruences on ternary semigroups are described in [2].

In this paper we generalize to ternary semigroups some important results on congruences on binary semigroups such as the Lallement's Lemma for example. We also characterize the minimal congruence on ternary semigroup under which the quotient algebra is a ternary group and find a maximal congruence separating idempotent pairs.

2. Preliminaries

For simplicity a ternary semigroup $(T, [\ \ ])$ will be denoted by T and the symbol of an inner ternary operation $[\ \ ]$ will be deleted, i.e., instead of $[[xyz]uw]$ or $[x[yzu]w]$ or $[xy[zuw]]$ we will write $[xyzuw]$.

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Recall that an element x of a ternary semigroup T is called *regular* if there exists $y \in T$ such that $[xyx] = x$. A ternary semigroup in which each element is regular is called *regular*. An element $x \in T$ is *inverse* to $y \in T$ if $[xyx] = x$ and $[yxy] = y$. Clearly, if x is inverse to y, then y is inverse to x. Thus every regular element has an inverse. The set of all inverses of x in T is denoted by $I(x)$.

Definition 2.1. A pair (a, b) of elements of T is an *idempotent pair* if $[ab[abt]] = [abt]$ and $[[tab]ab] = [tab]$ for all $t \in T$. An idempotent pair (a, b) in which an element a is inverse to b is called a *natural idempotent pair*.

According to Post [7] two pairs (a, b) and (c, d) are *equivalent* if $[abt] = [cdt]$ and $[tab] = [tcd]$ for all $t \in T$. Equivalent pairs are denoted by $(a, b) \sim (c, d)$. If (a, b) is an idempotent pair, then $([aba], [bab])$ is a natural idempotent pair and $(a, b) \sim ([aba], [bab])$. The equivalence class containing (a, b) will be denoted by $\langle a, b\rangle$. By E_T we denote the set of all equivalence classes of idempotent pairs in T.

For $a, b \in T$ consider the maps $\mathrm{L}_{a,b} : T \longrightarrow T : x \longrightarrow [abx]$ and $\mathrm{R}_{a,b} : x \longrightarrow [xab]$. On the set

$$M = \{m(a, b) \,|\, m(a, b) = (\mathrm{L}_{a,b}, \mathrm{R}_{a,b}), a, b \in T\},$$

which can be identified with $T \times T$, we introduce a binary product by putting

$$m(a, b)m(c, d) = m([abc], d) = m(a, [bcd]).$$

Then M is a semigroup. This semigroup can be extended to the semigroup $S_T = T \cup M$ as follows. For $A, B \in S_T$ we define

$$AB = \begin{cases} m(a, b) & \text{if} \quad A = a,\ B = b \in T, \\ [abx] & \text{if} \quad A = m(a, b) \in S_T,\ B = x \in T, \\ [xab] & \text{if} \quad A = x \in T,\ B = m(a, b) \in S_T, \\ m([abc], d) & \text{if} \quad A = m(a, b),\ B = m(c, d) \in S_T. \end{cases}$$

The semigroup S_T is a covering semigroup in the sense of Post [7] (see also [1]). The product $[abc]$ in T is equal to abc in S_T. The element $m(a, b)$ in S_T is usually denoted by ab.

It is shown in [8] that T is a regular (strongly regular) ternary semigroup if and only if S_T is a regular (inverse) semigroup. There is a bijective correspondence between E_T and the set E_{S_T} of idempotents of S_T. Note that (a, b) is an idempotent pair in T if and only if $m(a, b)$ is an idempotent in S_T and $\langle a, b\rangle$ corresponds to $m(a, b)$.

Definition 2.2. A ternary semigroup T is called a *ternary group* if for $a, b, c \in T$ the equations $[abx] = c$, $[ayb] = c$ and $[zab] = c$ have (unique) solutions in T.

Definition 2.3. An element a of a ternary semigroup T is said to be *invertible* if there exists an element $b \in T$ such that $[abx] = x = [bax] = [xab] = [xba]$ for all $x \in T$.

An invertible element is regular. In ternary group each element is invertible. Moreover, directly from the definition of a ternary group it follows that in ternary groups each element is regular and invertible. An element which is inverse to x is called it skew to x and is denoted by $\overline{x}$ (see [1] or [3]). Obviously it is uniquely determined and $\overline{\overline{x}} = x$.

In this paper we will denote the unique inverse of x (also in ternary semigroups) by x^{-1}.

As a simple consequence of results proved in [3] and [7] we can deduce

Theorem 2.4. *A ternary semigroup T is a ternary group if and only if one of the following equivalent conditions is satisfied.*

(i) *T is regular and cancellative.*

(ii) *T is regular and all idempotent pairs are equivalent.*

(iii) *All elements of T are invertible.*

(iv) *T contains no proper one sided ideals.* □

More information on ternary groups one can find in [4] and [5].

Definition 2.5. A regular ternary semigroup T is called *orthodox* if for any two idempotents pairs (a, b) and (c, d) the pair $([abc], d)$ is also an idempotent pair.

If T is an orthodox ternary semigroup, then E_T is a band. Hence E_T is a semilattice of rectangular bands. Clearly $E_T \simeq E_{S_T}$ as bands.

For $a, b \in T$ denote by $W(a, b)$ the set of all equivalence classes $\langle u, v\rangle$ such that $(u, v) \in T \times T$ and $[abuvabt] = [abt]$, $[tabuvab] = [tab]$, $[uvabuvt] = [uvt]$, $[tuvabuv] = [tuv]$.

Clearly, $\langle x, y\rangle \in W(a, b)$ if and only if $xy \in I(ab)$ in S_T. Since E_T is a semilattice of rectangular bands, from the fact that $\langle a, b\rangle$ and $\langle c, d\rangle$ are elements of E_T it follows that $\langle [abc], d\rangle$ and $\langle [cda], b\rangle$ are in the same component of E_T and consequently $W([abc], d) = W([cda], b)$.

Proposition 2.6. $[I(c)I(b)I(a)] \subset I([abc])$ *for all elements a, b, c of each orthodox ternary semigroup.* □

Proposition 2.7. *A regular ternary semigroup is orthodox if and only if for all its elements a, b from $I(a) \cap I(b) \neq \emptyset$ it follows $I(a) = I(b)$.* □

The proofs of the above two facts are found in [9].

3. Congruences on ternary semigroups

Lemma 3.1. *If (a, b) is an idempotent pair in an orthodox ternary semigroup T, then $([uab], u')$, $([abu], u')$, $([uu'a], b)$ and $([buu'], a)$ are idempotent pairs for any $u \in T$ and $u' \in I(u)$.*

Proof. Indeed, we have $[uabu'uabu't] = [uabu'uab[u'uu']t] = [u[abu'uabu'uu']t] = [u[abu'uu']t] = [uabu't]$ for all $t \in T$. Similarly, $[tuabu'uabu'] = [tuabu'uabu'uu'] = [tu[abu'uabu'uu']] = [tuabu']$. Therefore $([uab], u')$ is an idempotent pair. For $([abu], u')$, $([uu'a], b)$ and $([buu'], a)$ the proof is analogous. □

Corollary 3.2. *If (a, b) is an idempotent pair in a strongly regular ternary semigroup T, then $([uab], u^{-1})$,$([abu], u^{-1})$ $([uu^{-1}a], b)$ and $([buu^{-1}], a)$ are idempotent pairs for any $u \in T$.* □

Lemma 3.3. *If (a, b) is an idempotent pair in an orthodox ternary semigroup T, then $([uva], [bv'u'])$ is an idempotent pair for all $u' \in I(u)$, $v' \in I(v)$ and $u, v \in T$.*

Proof. By Lemma 3.1 $([vab], v')$ is an idempotent pair and for all $u' \in I(u)$ and $v' \in I(v)$ we obtain $[uvabv'u'uvabv'u't] = [u[vabv'u'uvabv'u'uu't]] = [uvabv'u'uu't] = [uvabv'u't]$ for $t \in T$. Similarly $[tuvabv'u'uvabv'u'] = [tuvabv'u'uvabv'u'uu'] = [tuvabv'u'uu'] = [tuvabv'u']$. □

Corollary 3.4. *If (a, b) is an idempotent pair in a strongly regular ternary semigroup T, then $([uva], [bv^{-1}u^{-1}])$ is an idempotent pair for all $u, v \in T$.*

Lemma 3.5. (Generalised Lallement's Lemma)
Let ρ be a congruence on a regular ternary semigroup T. If $(a\rho, b\rho)$ is an idempotent pair in T/ρ then there exists an idempotent pair (p, q) in T such that $(a\rho, b\rho) \sim (p\rho, q\rho)$. Moreover, (p, q) satisfies the property that $[Tpq] \subseteq [Tab]$ and $[pqT] \subseteq [abT]$

Proof. It is clear that T/ρ is a ternary semigroup. Let $(a\rho, b\rho)$ be an idempotent pair in T/ρ. If b' is an inverse of b and u be an inverse of $[[aba]bb']$, then for $p = [abb']$, $q = [uab]$ and $t \in T$ we have $[pq[pqt]] = [[abb'][uab][abb'][uab]t] = [abb'[uababb'u]abt] = [[abb'][uab]t] = [pqt]$. Similarly $[[tpq]pq] = [t[abb'][uababb'u]ab] = [tpq]$. Hence (p, q) is an idempotent pair. Moreover $[p\rho q\rho x\rho] = [[abb']\rho[uab]\rho x\rho] = [a\rho b\rho b'\rho u\rho a\rho b\rho x\rho] = [a\rho b\rho a\rho b\rho b'\rho u\rho a\rho b\rho a\rho b\rho b'\rho b\rho x\rho] = [[[[aba]bb']u[[aba]bb']]bx]\rho = [[aba]bx]\rho = [a\rho b\rho a\rho b\rho x\rho] = [a\rho b\rho x\rho]$ for $x \in T$. Analogously $[x\rho p\rho q\rho] = [x\rho a\rho b\rho]$ for all $x \in T$. Thus $(a\rho, b\rho) \sim (p\rho, q\rho)$ in T/ρ. From the choice of p and q it is clear that $[Tpq] \subseteq [Tab]$ and $[pqT] \subseteq [abT]$. □

Corollary 3.6. *If T is a regular ternary semigroup and ρ is a congruence on T, then T/ρ is a regular ternary semigroup.* □

Definition 3.7. A congruence ρ on a ternary semigroup T is said to be a *ternary group congruence* if T/ρ is a ternary group.

Definition 3.8. A congruence ρ on a regular ternary semigroup T is called *strongly regular* if T/ρ is a strongly regular ternary semigroup, and *idempotent pair separating* if (a, b) and (c, d) are equivalent in T for each idempotent pairs (a, b), (c, d) such that $(a\rho, b\rho)$ and $(c\rho, d\rho)$ are equivalent in T/ρ.

Lemma 3.9. *Let $\rho : T \longrightarrow T\rho$ be a ternary homomorphism of an orthodox ternary semigroup T. Then $T\rho$ is an orthodox ternary semigroup.* □

Lemma 3.10. *Let ρ be a ternary homomorphism of a strongly regular ternary semigroup T. Then $T\rho$ is a strongly regular ternary semigroup such that $(a\rho)^{-1} = a^{-1}\rho$ for all $t \in T$.*

Proof. For idempotent pairs $(a\rho, b\rho)$ and $(x\rho, y\rho)$ in $T\rho$, by Lemma 3.5, there exists idempotent pairs (p, q) and (u, v) such that $(p\rho, q\rho) \sim (a\rho, b\rho)$ and $(u\rho, v\rho) \sim (x\rho, y\rho)$. Thus $[a\rho b\rho x\rho y\rho t\rho] = [p\rho q\rho u\rho v\rho t\rho] = [pquvt]\rho = [uvpqt]\rho = [u\rho v\rho p\rho q\rho t\rho]$ $= [x\rho y\rho a\rho b\rho t\rho]$ and $[t\rho a\rho b\rho x\rho y\rho] = [t\rho x\rho y\rho a\rho b\rho]$. Hence the idempotent pairs $(a\rho, b\rho)$ and $(x\rho, y\rho)$ commute in T/ρ. Thus $T\rho$ is strongly regular. Moreover, for any $a \in T$ we have $[a\rho a^{-1}\rho a\rho] = a\rho$ and $[a^{-1}\rho a\rho a^{-1}\rho] = a^{-1}\rho$. Thus $a^{-1}\rho = (a\rho)^{-1}$, by [9]. □

Any congruence ρ on a ternary semigroup T can be extended to the relation ρ^e defined on $S_T = T \cup M$ in the following way:

$$(x, y) \in \rho^e \Leftrightarrow \begin{cases} (x, y) \in \rho \text{ and } x, y \in T, \text{ or} \\ x = ab,\ y = cd \in M \text{ and } ([abt], [cdt]), ([tab], [tcd]) \in \rho\ \ \forall t \in T. \end{cases}$$

Lemma 3.11. *ρ^e is a congruence on S_T.*

Proof. It is clear that ρ^e is an equivalence relation on S_T. To prove that it is a congruence suppose $x\rho^e y$ and $x, y \in S_T$.

(i) If $x, y \in T$ and $z \in T$, then $[zxt]\rho[zyt]$ and $[tzx]\rho[tzy]$ for any $t \in T$, so $zx\rho^e zy$. Similarly $[xzt]\rho[yzt]$ and $[txz]\rho[tyz]$. Hence $xz\rho^e yz$. If $z = uv$, then $zx = [uvx]$, $zy = [uvy]$ and $[uvx]\rho[uvy]$. Also $[xuv]\rho[yuv]$. Thus $zx\rho^e zy$ and $xz\rho^e yz$.

(ii) Suppose $x = ab$, $y = cd$ and $z = pq$. Then $xz = ([abp], q)$ and $yz = ([cdp], q)$. Since $x\rho^e y$, we have $[abt] = [cdt]$ and $[tab] = [tcd]$ for all $t \in T$. Therefore $[abpqt] = [cdpqt]$ and $[tabpq] = [tcdpq]$. Hence $xz\rho^e yz$. Similarly, $[pqabt] = [pqcdt]$ and $[tpqab] = [tpqcd]$. So, $zx\rho^e zy$.

(iii) If $x = ab$,$y = cd$, then for any $z \in T$ we have $[zab]\rho[zcd]$ and $[abz]\rho[cdz]$. Therefore $zx\rho^e zy$ and $xz\rho^e yz$. Hence ρ^e is a congruence. □

Lemma 3.12. *If T is a regular ternary semigroup, then ρ^e is an idempotent separating congruence in S_T if and only if ρ is an idempotent pair separating congruence in T.*

Proof. Let ρ^e be an idempotent separating congruence in S_T. If (a, b) and (c, d) are idempotent pairs in T such that $(a\rho, b\rho)$ and $(c\rho, d\rho)$ are equivalent in T/ρ, then $[abt]\rho[cdt]$ and $[tab]\rho[tcd]$ for all $t \in T$. Hence $ab\rho^e cd$ in S_T. Since ab and cd are idempotents in S_T and ρ^e is idempotent separating we have $ab = cd$. This means that $[abt] = [cdt]$ and $[tab] = [tcd]$ and so $(a, b) \sim (c, d)$. Conversely suppose ρ is an idempotent pair separating congruence in T. Let e, f be idempotents in S_T such that $e\rho^e f$. Let $e = ab$ and $f = cd$ for some idempotent pairs (a, b) and (c, d) in T. Then $e\rho^e f$ implies $[abt]\rho[cdt]$ and $[tab]\rho[tcd]$. Hence $(a\rho, b\rho) \sim (c\rho, d\rho)$ in T/ρ, which gives $(a, b) \sim (c, d)$ in T. So, $e = f$. Thus ρ^e is an idempotent separating congruence on S_T. □

4. Strongly regular ternary semigroups

In this section T denotes a strongly regular ternary semigroup. Below we will construct congruences on T which are analogous to the group congruence and maximum idempotent separating congruence on an ordinary inverse semigroup.

We start with the relation σ defined on T as follows:

$$(x,y) \in \sigma \iff [abx] = [aby] \text{ for some idempotent pair } (a,b) \in T.$$

Lemma 4.1. *σ is a congruence on T.*

Proof. Clearly σ is an equivalence relation on T. To prove that it is a congruence suppose $x\sigma y$ and $u,v \in T$. Then $[abx] = [aby]$ for some idempotent pair (a,b), and so $[abxuv] = [abyuv]$. Hence $([xuv],[yuv]) \in \sigma$. By Corollary 3.2, for any $u,v \in T$, $([v^{-1}u^{-1}u], v)$ is an idempotent pair and by Corollary 3.4, $([uva],[bv^{-1}u^{-1}])$ is also an idempotent pair. So,

$$\begin{aligned}
\big[[uva][bv^{-1}u^{-1}][uvx]\big] &= \big[uvabv^{-1}u^{-1}uvabv^{-1}u^{-1}uvx\big] \\
&= \big[uvabv^{-1}u^{-1}uvv^{-1}u^{-1}uvabx\big] \\
&= \big[uvabv^{-1}u^{-1}uvabx\big] = [uvabv^{-1}u^{-1}uvaby] \\
&= \big[uvababv^{-1}u^{-1}uvy\big] = [uvabv^{-1}u^{-1}uvy].
\end{aligned}$$

Therefore $([uvx],[uvy]) \in \sigma$. Similarly $([vab], v^{-1})$ and (u^{-1}, u) are idempotent pairs and they commute. Hence

$$\begin{aligned}
\big[[uva][bv^{-1}u^{-1}][uxv]\big] &= [uvabv^{-1}u^{-1}uvabv^{-1}u^{-1}uxv] \\
&= [u[vabv^{-1}u^{-1}uv]abv^{-1}u^{-1}uxv]] \\
&= \big[uu^{-1}u[vabv^{-1}vabv^{-1}u^{-1}]uxv\big] \\
&= \big[u[vabv^{-1}u^{-1}]uxv\big] = [uv[abv^{-1}u^{-1}u]xv] \\
&= [uvv^{-1}u^{-1}uabxv] = [uvv^{-1}u^{-1}uabyv] \\
&= [uvabv^{-1}u^{-1}uyv].
\end{aligned}$$

Therefore $([uxv],[uyv]) \in \sigma$. Hence σ is a congruence. $\square$

Proposition 4.2. *T/σ is a ternary group.*

Proof. By Theorem 2.4 and Lemma 3.9, it is enough to show that all idempotent pairs in T/σ are equivalent. If $(a\sigma, b\sigma)$, $(u\sigma, v\sigma)$ are two idempotent pairs in T/σ, then we have to prove $[abt]\sigma[uvt]$ and $[tab]\sigma[tuv]$ for all $t \in T$. By Lemma 3.5, without loss of generality we can assume that (a,b) and (u,v) are idempotent pairs of T. Then $([abu], v)$ and $([uva], b)$ are idempotent pairs. For any $t \in T$ we have $[[abu]v[abt]] = [ababuvt] = [abuvt] = [abuvuvt] = [[abu]v[uvt]]$ since idempotent pairs commute in T. Therefore $[abt]\sigma[uvt]$. Similarly $[[tab][uva]b] = [tabuvab] = [tuvab] = [[tuv][uva]b]$. Hence $[tab]\sigma[tuv]$. So, $(a\sigma, b\sigma)$ and $(u\sigma, v\sigma)$ are equivalent in T/σ. Thus in T/σ all idempotent pairs are equivalent and T/σ is a ternary group. $\square$

Theorem 4.3. *σ is the minimum ternary group congruence on a strongly regular ternary semigroup T.*

Proof. By Proposition 4.2, T/σ is a ternary group. Suppose θ is a congruence on T such that T/θ is a ternary group. We prove that $\sigma \subseteq \theta$. Suppose $(p,q) \in \sigma$, then $[abp] = [abq]$ for some idempotent pair (a,b) in T. Then $[a\theta b\theta p\theta] = [a\theta b\theta q\theta]$. Since T/θ is a ternary group cancellation law holds and so $p\theta = q\theta$. □

Now we consider the relation μ defined as follows:

$$(a,b) \in \mu \iff ([axx^{-1}], a^{-1}) \sim ([bxx^{-1}], b^{-1}) \;\; \forall (x, x^{-1}) \in T \times T.$$

In other words, $(a,b) \in \mu$ if $[axx^{-1}a^{-1}t] = [bxx^{-1}b^{-1}t]$ and $[taxx^{-1}a^{-1}] = [tbxx^{-1}b^{-1}]$ for every $t \in T$.

Lemma 4.4. *μ is a congruence on T.*

Proof. Clearly μ is an equivalence relation. Suppose $(a,b) \in \mu$ and $u, v \in T$. For every idempotent pair (x, x^{-1}), by Corollary 3.2 $([uvx], [x^{-1}v^{-1}u^{-1}])$ is an idempotent pair and so we obtain $[auvxx^{-1}v^{-1}u^{-1}a^{-1}t] = [buvxx^{-1}v^{-1}u^{-1}b^{-1}t]$, $[tauvxx^{-1}v^{-1}u^{-1}a^{-1}] = [tbuvxx^{-1}v^{-1}u^{-1}b^{-1}]$. Hence $([auv], [buv]) \in \mu$. Since $[axx^{-1}a^{-1}t] = [bxx^{-1}b^{-1}t]$ for all $t \in T$, we have $[uvaxx^{-1}a^{-1}t] = [uvbxx^{-1}b^{-1}t]$. Replacing t by $[v^{-1}u^{-1}t]$ we get $[uvaxx^{-1}a^{-1}v^{-1}u^{-1}t] = [uvbxx^{-1}b^{-1}v^{-1}u^{-1}t]$. In a similar way we obtain $[tuvaxx^{-1}a^{-1}v^{-1}u^{-1}] = [tuvbxx^{-1}b^{-1}v^{-1}u^{-1}]$. Thus $([uva], [uvb]) \in \mu$. Hence for every idempotent pair (x, x^{-1}) also $([vxx^{-1}], v^{-1})$ is an idempotent pair. Therefore for all $t \in T$ we have $[avxx^{-1}v^{-1}a^{-1}t] = [bvxx^{-1}v^{-1}b^{-1}t]$. In particular for $t = u^{-1}$ we obtain $[avxx^{-1}v^{-1}a^{-1}u^{-1}] = [bvxx^{-1}v^{-1}b^{-1}u^{-1}]$. Hence $[[uav]xx^{-1}[v^{-1}a^{-1}u^{-1}]t] = [[ubv]xx^{-1}[v^{-1}b^{-1}u^{-1}]t]$ for $t \in T$. Analogously we obtain $[tuavxx^{-1}v^{-1}a^{-1}u^{-1}] = [tubvxx^{-1}v^{-1}b^{-1}u^{-1}]$. Hence $([uav], [ubv]) \in \mu$. Thus μ is a congruence. □

Theorem 4.5. *μ is the maximum idempotent pair separating congruence on T.*

Proof. Let (a, a^{-1}) and (b, b^{-1}) be such that $(a\mu, a^{-1}\mu)$ and $(b\mu, b^{-1}\mu)$ are equivalent idempotent pairs in T/μ. We claim that (a, a^{-1}) and (b, b^{-1}) are equivalent idempotent pairs in T. From the hypothesis it follows that in T we have $[aa^{-1}t]\mu[bb^{-1}t]$ and $[taa^{-1}]\mu[tbb^{-1}]$ for all $t \in T$. The first relation for $t = a$ and $t = b$ gives $a\mu[bb^{-1}a]$ and $[aa^{-1}b]\mu b$. Putting in the second relation $t = a^{-1}$ and $t = b^{-1}$ we obtain $a^{-1}\mu[a^{-1}bb^{-1}]$ and $[b^{-1}aa^{-1}]\mu b^{-1}$. Therefore for all idempotent pairs (z, z^{-1}) and for all $t \in T$ we have

$$[azz^{-1}a^{-1}t] = [bb^{-1}azz^{-1}a^{-1}bb^{-1}t], \tag{4.1}$$

$$[bzz^{-1}b^{-1}t] = [aa^{-1}bzz^{-1}b^{-1}aa^{-1}t]. \tag{4.2}$$

From (4.1) for $z = a^{-1}$ and $t = a$ we get $[aa^{-1}aa^{-1}a] = [bb^{-1}aa^{-1}bb^{-1}a] = [bb^{-1}a]$. Therefore

$$a = [bb^{-1}a]. \tag{4.3}$$

Thus $a^{-1} = [a^{-1}bb^{-1}]$. From (4.2) putting $z = b^{-1}$ and $t = b$ we obtain $[bb^{-1}bb^{-1}b]$ $= [aa^{-1}bb^{-1}aa^{-1}b] = [aa^{-1}b]$. Therefore

$$b = [aa^{-1}b]. \tag{4.4}$$

Hence $b^{-1} = [b^{-1}aa^{-1}]$. Now using (4.3) and (4.4) we see that

$$[aa^{-1}t] = [bb^{-1}a[a^{-1}bb^{-1}]t] = [b[b^{-1}aa^{-1}]bb^{-1}t] = [bb^{-1}bb^{-1}t] = [bb^{-1}t]$$

for all $t \in T$. Similarly

$$[taa^{-1}] = [t[bb^{-1}a][a^{-1}bb^{-1}]] = [tb[b^{-1}aa^{-1}]bb^{-1}] = [tbb^{-1}bb^{-1}] = [tbb^{-1}].$$

Therefore $(a, a^{-1}) \sim (b, b^{-1})$. Hence μ is an idempotent pair separating congruence in T.

Suppose that ρ is another idempotent pair separating congruence on T. If $a\rho = b\rho$, then $a^{-1}\rho = b^{-1}\rho$ by Lemma 3.10. For any idempotent pair $(x, x^{-1}) \in T$ we have $[axx^{-1}a^{-1}t]\rho = [bxx^{-1}b^{-1}t]\rho$ and $[taxx^{-1}a^{-1}]\rho = [tbxx^{-1}b^{-1}]\rho$. Hence $([axx^{-1}]\rho, a^{-1}\rho)$ and $([bxx^{-1}]\rho, b^{-1}\rho)$ are equivalent idempotent pairs in T/ρ. Since $([axx^{-1}], a^{-1})$ and $([bxx^{-1}], b^{-1})$ are idempotent pairs in T we see that they are equivalent in T. Hence $a\mu b$. Therefore $\rho \subseteq \mu$. □

5. Congruences on orthodox ternary semigroups

In this section by T will denote an orthodox ternary semigroup. By γ we denote the relation on T such that

$$(a, b) \in \gamma \Longleftrightarrow I(a) = I(b).$$

Theorem 5.1. *The relation γ is a congruence on T.*

Proof. Clearly γ is an equivalence relation. Suppose $(a, b) \in \gamma$ and $x, y \in T$. Then for any $u \in I(a) = I(b)$ and for any $v \in I(x), w \in I(y)$ it follows from Proposition 2.6, that $[uwv] \in I([xya]) \cap I([xyb])$. Hence by Proposition 2.7 we get $I([xya]) = I([xyb])$ and so $([xya], [xyb]) \in \gamma$. Similarly $[wvu] \in I([axy]) \cap I([bxy])$. Therefore $([axy], [bxy]) \in \gamma$. Also $([xay], [xby]) \in \gamma$. Hence γ is a congruence. □

Theorem 5.2. *The relation γ is the smallest congruence on T for which T/γ is a strongly regular ternary semigroup.*

Proof. $E_T = \cup E_\alpha$ is a semilattice of rectangular bands. For any $\langle a, b\rangle$, $\langle c, d\rangle$ and $\langle e, f\rangle$ in E_T, elements $([abcde], f)$ and $([cdabe], f)$ belong to the same class E_α and so $I(\langle [abcde], f\rangle) = I(\langle [cdabe], f\rangle)$ in E_T. This can be interpreted in T as $W([abcde], f) = W([cdabe], f) = W(a, [bcdef])$. Let $(a\gamma, b\gamma)$ and $(c\gamma, d\gamma)$ be two idempotent pairs in T/γ. Fix $t \in T$. If $u \in I([abcdt])$, then $[abcdtuabcdt] = [abcdt]$ and $[uabcdtu] = u$. We first show that $(t, u) \in W([cdabt], t')$, for some

$t' \in I(t)$. For all $z \in T$ we have $[tuz] = [t[uabcdtu]z] = [tuabcdtt'tuz]$ and $[abcdtt'z] = [[abcdtuabcdt]t'z] = [abcdtt'tuabcdtt'z]$. Therefore we see that (t, u) is in $W([abcdt], t') = W([cdabt], t')$. Thus, for all $z \in T$

$$[cdabtt'tucdabtt'z] = [cdabtt'z], \tag{5.1}$$

$$[tucdabtt'tuz] = [tuz], \tag{5.2}$$

$$[zcdabtt'tucdabtt'] = [zcdabtt'], \tag{5.3}$$

$$[ztucdabtt'tu] = [ztu]. \tag{5.4}$$

(5.1) for $z = t$ gives $[cdabtt'tucdabtt't] = [cdabtt't]$. Therefore

$$[cdabtucdabt] = [cdabt]. \tag{5.5}$$

Multiplying (5.2) on the left by $[uabcd]$ and on the right by u we obtain the equation $[uabcdtucdabtt'tuzu] = [uabcdtuzu]$. Therefore $[ucdabtuzu] = [uzu]$, which for $z = [abcdt]$ gives $[ucdabt[uabcdtu]] = [uabcdtu]$. Hence

$$[ucdabtu] = u. \tag{5.6}$$

From (5.5) and (5.6) we get $u \in I([cdabt])$. Thus $u \in I([abcdt]) \cap I([cdabt])$, which implies $I([abcdt]) = I([cdabt])$ (cf. [9]). Hence

$$[abcdt]\gamma[cdabt]. \tag{5.7}$$

Now we show that $I([tabcd]) = I([tcdab])$. Indeed, if $u \in I([tabcd])$, then $[tabcdutabcd] = [tabccd]$ and $[utabcdu] = u$. Moreover, for every z from T we have $[utz] = [[utabcdu]tz] = [utt'tabcdutz]$, $[zut] = [zutabcdut] = [zutt'tabcdut]$. Similarly, $[t'tabcdz] = [t'[tabcd]z] = [t'[tabcdutabcd]z] = [t'tabcdutt'tabcdz]$, $[zt'tabcd] = [zt'[tabcdutabcd]] = [zt'tabcdutt'tabcd] = [zt'tabcdutt'tabcd]$. Therefore (u, t) is in $W([t', [tabcd]) = W(t', [tcdab])$. Hence for all $z \in T$,

$$[utt'tcdabutz] = [utz], \tag{5.8}$$

$$[t'tcdabutt'tcdabz] = [t'tcdabz], \tag{5.9}$$

$$[zutt'tcdabut] = [zut], \tag{5.10}$$

$$[zt'tcdabutt'tcdab] = [zt'tcdab]. \tag{5.11}$$

Multiplying (5.10) on the left by u and on the right by $[abcdu]$ we obtain the equation $[uzutcdab[utabcdu]] = [uz[utabcdu] = [uzu]$. This for $z = [tabcd]$ gives $[[utabcdu]tcdabu] = [utabcdu] = [utabcdu]$. Therefore

$$[utcdabu] = u. \tag{5.12}$$

(5.11) for $z = t$ gives $[tt'tcdabutt'tcdab] = [tt'tcdab]$. Therefore

$$[tcdabutcdab] = [tcdab]. \tag{5.13}$$

From (5.12) and (5.13) we get $u \in I([tcdab])$. Thus $I([tabcd]) = I([tcdab])$. Hence

$$[tabcd]\gamma[tcdab]. \tag{5.14}$$

Now, from (5.7) and (5.14) it follows that $(a\gamma, b\gamma)$ and $(c\gamma, d\gamma)$ commute in T/γ and so T/γ is strongly regular.

Suppose that ρ is a congruence on T such that T/ρ is a strongly regular ternary semigroup. If $(a, b) \in \gamma$, then for any $x \in I(a) = I(b)$, $a\rho$ and $b\rho$ are both inverses of $x\rho$ in T/ρ. Since T/ρ is strongly regular, the element $x\rho$ has a unique inverse and so $a\rho = b\rho$. Hence $\gamma \subseteq \rho$. Thus γ is the smallest strongly regular ternary semigroup congruence. □

Theorem 5.3. *The relation μ defined by*

$$(a, b) \in \mu \Longleftrightarrow \begin{cases} \text{for every idempotent pair } (x, x')\ \exists a' \in I(a), \exists b' \in I(b) \\ ([axx'], a') \sim ([bxx'], b') \text{ and } ([a'xx'], a) \sim ([b'xx'], b). \end{cases}$$

is a congruence on T.

Proof. We first prove that μ is an equivalence relation. Clearly μ is reflexive and symmetric. For any $(a, b), (b, c) \in \mu$ there exists $a' \in I(a)$, b', $b'' \in I(b)$ and $c' \in I(c)$ such that for every idempotent pair (x, x') we have $[axx'a't] = [bxx'b't]$ and $[taxx'a'] = [tbxx'b']$, $[a'xx'at] = [b'xx'bt]$ and $[ta'xx'a] = [tb'xx'b]$, $[bxx'b''t] = [cxx'c't]$ and $[tbxx'b''] = [tcxx'c']$, $[b''xx'bt] = [c'xx'ct]$ and $[tb''xx'b] = [tc'xx'c]$. Put $a^* = [b''ba'bb']$. We see that $[bb'a] = [bb'aa'aa'a] = [bb'ba'ab'a] = [ba'ab'a] = [aa'aa'a] = [aa'a] = a$ and $[aa^*a] = [ab''ba'bb'a] = [bb''bb'bb'a] = [bb'bb'a] = [bb'a] = a$. Thus $[a^*aa^*] = [b''ba'bb'ab''ba'bb'] = [b''bb'bb'bb''ba'bb'] = [b''bb''ba'bb'] = [b''ba'bb'] = a^*$. Hence $a^* \in I(a)$. Similarly for $c^* = [b''bc'bb']$ we have $[cc^*c] = [cb''bc'bb'c] = [cb''bb''bb'b] = [cb''bb'b] = [cc'bb'c] = [cc'c] = c$, $[c^*cc^*] = [b''bc'bb'cb''bc'bb'] = [b''bb''bb'bb''bc'bb'] = [b''bb''bc'bb'] = [b''bc'bb'] = c^*$. Therefore $c^* \in I(c)$. Now for all idempotent pair (x, x') in T and all $t \in T$ we obtain $[a^*xx'at] = [b''ba'bb'xx'at] = [b''bb'bb'xx'bt] = [b''bb'xx'bt] = [b''bb''bb'xx'bt] = [b''bc'bb'xx'ct] = [c^*xx'ct]$ and $[ta^*xx'a] = [tc^*xx'c]$, $[axx'a^*t] = [axx'b''ba'bb't] = [bxx'b''bb'bb't] = [bxx'b''bb't] = [bxx'b''bb''bb't] = [cxx'b''bc'bb't] = [cxx'c^*t]$. Also we have $[taxx'a^*] = [tcxx'c^*]$. Hence $(a, c) \in \mu$, proving μ is a transitive relation. Thus μ is an equivalence relation.

Suppose $(a, b) \in \mu$ and $u, v \in T$ so that for every idempotent pair (x, x') in T and for all $t \in T$,

$$[axx'a't] = [bxx'b't], \tag{5.15}$$

$$[taxx'a'] = [tbxx'b'], \tag{5.16}$$

$$[a'xx'at] = [b'xx'bt], \tag{5.17}$$

$$[ta'xx'a] = [tb'xx'b]. \tag{5.18}$$

In (5.15), replacing (x, x') by $([uvx], [x'v'u'])$ we get $[auvxx'v'u'a't] = [buvxx'v'u'b't]$. Similarly, (5.16) becomes $[tauvxx'v'u'a'] = [tbuvxx'v'u'b']$. In (5.17) replacing

t by $[uvt]$ and multiplying on the left by v' and u' we get $[v'u'a'xx'auvt] = [v'u'b'xx'buvt]$ $\forall t \in T$. In (5.18) replacing t by $[tv'u']$ and multiplying on the right by u and v, we get $[tv'u'a'xx'auv] = [tv'u'b'xx'buv]$. Since $[v'u'a'] \in I([auv])$ and $[v'u'b'] \in I([buv])$ we have $([auv],[buv]) \in \mu$. Similarly we can show that $([uva],[uvb]) \in \mu$ and $([uav],[ubv]) \in \mu$. □

Theorem 5.4. *μ is the maximum idempotent pair separating congruence on T.*

Proof. Let $(a\mu, a'\mu)$ and $(b\mu, b'\mu)$ be two equivalent idempotent pairs in T/μ so that $[aa't]\mu[bb't]$, $[taa']\mu[tbb']$, $[a'at]\mu[b'bt]$ and $[ta'a]\mu[tb'b]$ $\forall t \in T$. Putting $t = a$ and $t = b$ in the first relation we get $a\mu[bb'a]$ and $[aa'b]\mu b$. Putting $t = a'$ and $t = b'$ in the second relation we get $a'\mu[a'bb']$ and $[b'aa']\mu b'$. Hence for every idempotent pair (x, x') and for all $t \in T$ we have

$$[axx'a't] = [bb'axx'[bb'a]'t], \tag{5.19}$$

$$[bxx'b't] = [aa'bxx'[aa'b]'t], \tag{5.20}$$

$$[ta''xx'a'] = [t[a'bb']'xx'a'bb'], \tag{5.21}$$

$$[tb''xx'b'] = [t[b'aa']'xx'b'aa'] \tag{5.22}$$

for some $[bb'a]' \in I([bb'a])$. From (5.19) for $(x, x') = (a', a)$ and $t = a$ we get $a = [bb'aa'a[bb'a]'t] = [[bb'a][bb'a]'t]$. Multiplying on the left by b and b' we have $[bb'a] = [bb'[bb'a][bb'a]'t] = a$. Therefore $[bb'a] = a$. Putting $(x, x') = (b', b)$ and $t = b$ in (5.20) we obtain $b = [aa'bb'b[aa'b]'b] = [aa'b[aa'b]'b]$. Multiplying on the left by a and a' we get $[aa'b] = [aa'aa'b[aa'b]'b] = [aa'b[aa'b]'b] = a$. Therefore $[aa'b] = a$. Replacing in (5.21) x by a' and x' by a'' we obtain $[ta''a'] = [t[a'bb']'a'a''a'bb'] = [t[a'bb']'a'bb']$ for every $t \in T$, which for $t = a'$ implies $a' = [a'[a'bb']'a'bb']$. Multiplying this on the right by b and b' we get $[a'bb'] = [a'[a'bb']'a'bb'bb'] = a'$. Therefore $[a'bb'] = a'$. (5.22) for $x = b'$ and $x' = b''$ gives $[tb''b'] = [t[b'aa']'b'b''b'aa'] = [t[b'aa']'b'aa']$, $\forall t \in T$. In particular, for $t = b'$ we get $b' = [b'[b'aa']'b'aa']$. Multiplying this on the right by a and a' we obtain $[b'aa'] = [b'[b'aa']'b'aa'aa'] = [b'[b'aa']'b'aa'] = b'$. Therefore $[b'aa'] = b'$ and $[aa't] = [[bb'a][a'bb']t] = [b[b'aa']bb't] = [bb'bb't] = [bb't]$, $\forall t \in T$, Similarly $[taa'] = [t[bb'a][a'bb']] = [tb[b'aa']bb'] = [tbb'bb'] = [tbb']$. Hence $(a, a') \sim (b, b')$. Thus μ is an idempotent pair separating congruence on T.

Suppose that θ is an idempotent pair separating congruences on T and θ_e is the congruence induced on S_T by θ. If $x\theta y$, then $x\theta_e y$ in S_T. S_T is orthodox and by Lemma 3.12, θ_e is an idempotent separating congruences on S_T. Hence $\theta_e \subset \mathcal{H}$, where $\mathcal{H}$ is the Green's equivalence on S_T. Hence $x\mathcal{H}y$ in S_T we can find inverse x' of x and y' of y such that $xx' = yy'$ and $x'x = y'y$ in S_T. Therefore for all $t \in T$, $[xx't] = [yy't]$ and $[txx'] = [tyy']$. Similarly, $[x'xt] = [y'yt]$ and $[tx'x] = [ty'y]$ in T. Therefore $x = [xx'x] = [yy'x]$, $x' = [x'xx'] = [x'yy']$, $y' = [y'yy']$. Thus $x' = [x'yy']\theta[x'xy'] = y'$. Hence for every idempotent pair (u, v) in T, $[x'uvxt]\theta[y'uvyt]; [xuvx't]\theta[yuvy't]$. $([x'uv]\theta, x\theta) \sim ([y'uv]\theta, y\theta)$ and $([xuv]\theta, x'\theta) \sim ([yuv]\theta, y'\theta)$ in T/θ. Since θ is idempotent pair separating we have

$([xuv], x') \sim ([yuv], y')$. In a similar way we can show that $([x'uv], x) \sim ([y'uv], y)$. Thus $x\mu y$. Hence $\theta \subseteq \mu$ and so μ is the maximum idempotent pair separating congruences on T. □

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