About some algebraic systems related with projective planes

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Abstract

The present article is a survey of author's results on the investigations of algebraic structures related with projective planes; some new theorems are proved too.

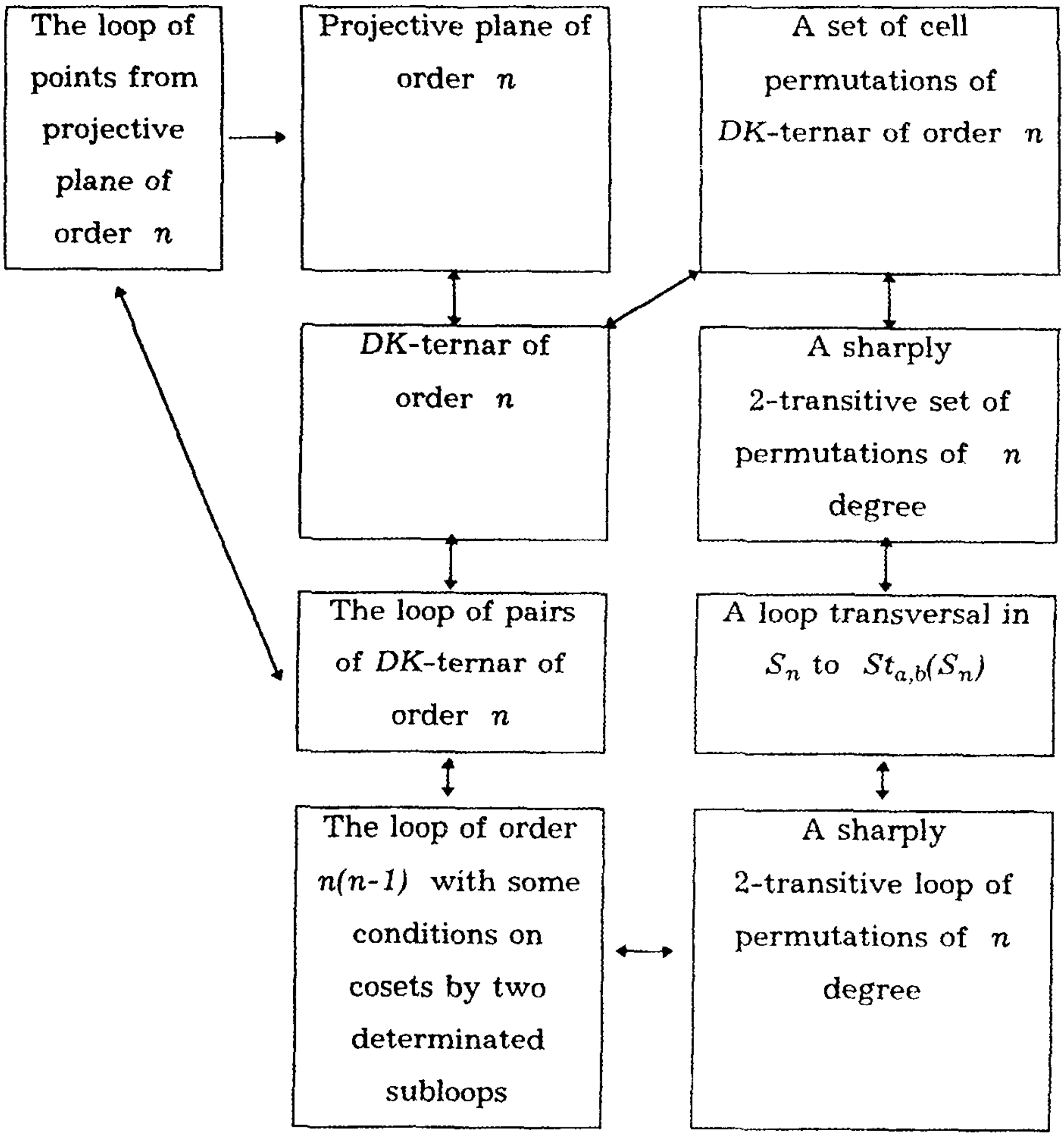
The *projective plane* is the incidence structure $< X, L, I >$ which satisfies the following axioms:

1) Given any two distinct points from X there exists just one line from L incident with both of them.

2) Given any two distinct lines from L there exists just one point from X incident with both of them.

3) There exist four points such that a line incident with any two of them is not incident with either of the remaining two.

This article is a survey of some author's results (see [2,7]) about algebraic structures related with projective planes (finite as a rule, if the contrary is not stipulated); some new theorems are proved too. The main aim of article is to demonstrate the correlations in the following scheme:



0. Necessary definitions and notations

Definition 1. [1] A system $<E,\cdot>$ is called a *quasigroup*, if for arbitrary $a,b \in E$ equations $x \cdot a = b$ and $a \cdot y = b$ have an unique solution in the set E. If in quasigroup $<E,\cdot>$ there exists element $e \in E$ such that

$$x \cdot e = e \cdot x = x$$

for any $x \in E$, then system $<E,\cdot>$ is called a *loop*.

Definition 2. [2] A system $<E,(x,t,y),0,1>$ is called a *DK-ternar* (e.g. a set E with ternary operation (x,t,y) and distinguished elements $0,1 \in E$), if the following conditions hold:

1). $(x,0,y) = x$;

2). $(x,1,y) = y$;

3). $(x,t,x) = x$;

4). $(0,t,1) = 0$;

5). If a,b,c and d are arbitrary elements from E and $a \neq b$, then the system

$$\begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases}$$

has an unique solution in $E \times E$.

6). Either set E is finite, or

a) if a,b,c are arbitrary elements from E and $c \neq 0$, $(c,a,0) \neq b$, then the system

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) \neq (c,t,0) \quad \forall t \in E; \end{cases}$$

has an unique solution in $E \times E$.

b) if a,b are arbitrary elements from E and $b \neq 0$, then inequality

$$(a,t,b) \neq (x,t,0) \quad \forall t \in E$$

has an unique solution in E.

If the set E is finite, then conditions 6a) and 6b) are corollaries of the conditions 1)-5) of **Definition 2**. Proof of this statement will be given later.

Definition 3. A set M of permutations on a set X is called *sharply (strongly) 2-transitive*, if for any two pairs (a,b) and (c,d) of different elements from X there exists an unique permutation $\alpha \in M$ satisfying the following conditions

$$\alpha(a) = c, \qquad \alpha(b) = d.$$

Definition 4. [3] Let G be a group and H be a subgroup in G. A complete system T of representatives of the left (right) cosets in G to H $(e = t_1 \in H)$ is called a *left (right) transversal in* G *to* H.

Let T be a transversal (left or right) in G to H. We can introduce correctly the following operations on Λ (Λ is an index set; left (right) cosets in G to H are numbered by indexes from Λ):

$$i * j = v \Leftrightarrow t_i t_j = t_v h, h \in H,$$

if T is a left transversal, and

$$i * j = w \Leftrightarrow t_i t_j = h t_w, h \in H,$$

if T is a right transversal.

Definition 5. Let T be a left (right) transversal in G to H. If the system $<\Lambda, *, 1>$ $(<\Lambda, \bullet, 1>)$ is a loop, then T is called a *left (right) loop transversal in* G *to* H.

1. Projective plane and *DK*-ternar

Lemma 1. *Let π be a projective plane. It is possible to introduce coordinates $(a,b),(m),(\infty)$ for points and $[a,b],[m],[\infty]$ for lines from π (where $a,b,m \in E$, E is some set with distinguished elements 0 and 1), such that for operation (x,t,y), where*

$$(x,t,y)=z \overset{def}{\Leftrightarrow} (x,y)\in[t,z],$$

the system $<E,(x,t,y),0,1>$ is a DK-ternar.

Proof. Let π be an arbitrary projective plane. Let X,Y,O,I be arbitrary four points in the general position on π.

Suppose, by definition,

$$[XY]=[\infty]; \qquad [OI]=[0];$$
$$O=(0,0); \qquad I=(1,1).$$

Then

$$[\infty]\cap[0] \overset{def}{=} (\infty).$$

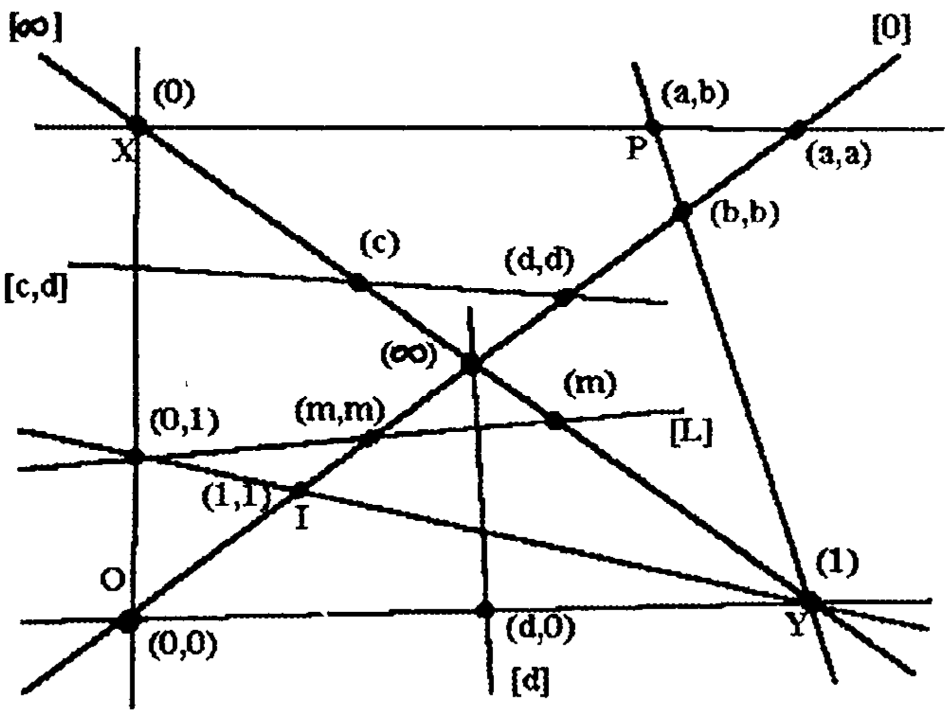
All other points of the line $[0]$ are attributed by definition by the symbols (a,a) (where $a\neq 0,1$), and different points are attributed by different symbols.

Let P be an arbitrary point from π and $P\notin[\infty]$. Let us have

$$\begin{cases} [XR]\cap[0]=(a,a); \\ [YP]\cap[0]=(b,b). \end{cases} \tag{1}$$

Then suppose, by definition,

$$P \overset{def}{=} (a,b).$$



It is evident that points of the line [0] will have their own coordinates.

Let [L] be an arbitrary line from π and $(\infty) \notin [L]$. Let us have

$$\begin{cases} [L] = (0,1) \cup (m, m); \\ [L] \cap [\infty] = Z; \end{cases} \tag{2}$$

Then suppose by definition:

$$Z \overset{def}{=} (m).$$

In particular,

$$X = (0), \qquad Y = (1).$$

Suppose by definition:

$$(\infty) \cup (d,0) \overset{def}{=} [d]. \tag{3}$$

Finally, let [S] be an arbitrary line from π and $(\infty) \notin [S]$. Let us have

$$\begin{cases} [S] \cap [\infty] = (c); \\ [S] \cap [0] = (d, d); \end{cases} \tag{4}$$

Then suppose by definition:

$$[S] \overset{def}{=} [c, d].$$

Let us define the ternary operation (x, t, y) by the condition of **Lemma**, e.g.

$$(x, t, y) = u \quad \overset{def}{\Leftrightarrow} \quad (x, y) \in [t, u],$$

and verify that conditions 1)-6) of **Definition 2** hold.

a). $(x,0,y) = x.$

$$(x,0,y) = u \quad \Leftrightarrow \quad (x,y) \in [0,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \text{(the points } (x,y), (0) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad (u,u) = [0] \cap [(x,y) \cup (0)] = (x,x) \quad \Rightarrow \quad u = x.$$

b). $(x,1,y) = y.$

The proof is analogous to that of a).

c). $(x,t,x) = x.$

$$(x,t,x) = u \quad \Leftrightarrow \quad (x,x) \in [t,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \text{(the points } (x,x), (t) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad u = x.$$

d). $(0,t,1) = t.$

$$(0,t,1) = u \quad \Leftrightarrow \quad (0,1) \in [t,u] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \text{(the points } (0,1), (t) \text{ and } (u,u) \text{ lie in a common line (see (4))} \quad \Rightarrow$$

$$\Rightarrow \quad (u = t \text{ (see(2))}.$$

e). Let a,b,c,d be arbitrary elements from E and $a \neq b$. Then we have

$$\begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases} \Leftrightarrow \begin{cases} (x,y) \in [a,c] \\ (x,y) \in [b,d] \end{cases} \Leftrightarrow \quad (x,y) = [a,c] \cap [b,d].$$

There exists an unique such point (x,y) in the projective plane π.

f). If E is a finite set, then the proof is completed. Let E be an infinite set, a,b,c arbitrary elements from E and $c \neq 0, (c,a,0) \neq b$. Then we have

$$\begin{cases} (x,a,y) = b; \\ (x,t,y) \neq (c,t,0) \quad \forall t \in E; \end{cases} \Leftrightarrow \begin{cases} (x,y) \in [a,b] \\ (c,0) \notin [a,b] \\ (x,y) \cup (c,0) \neq [t,u] \quad \forall t,u \in E \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x,y) \in [a,b] \\ (x,y) \cup (c,0) = [c] \end{cases} \Leftrightarrow \quad (x,y) = [a,b] \cap [c].$$

There exists an unique such point (x,y) in the projective plane π.

The proof of the condition 6b) of **Definition 2** is analogous to that of 6a). Thus the system $< E,(x,t,y),0,1 >$ is a DK-ternar. □

Lemma 2. *Let* $< E,(x,t,y),0,1 >$ *be a DK-ternar and* a *be an arbitrary fixed element from E,* $a \neq 0,1$. *Then the system* $< E,(x,a,y) >$ *is a quasigroup.*

Proof. Let the conditions of **Lemma** hold. Then we have for arbitrary $b,c \in E$

$$(x,a,c) = c \quad \Leftrightarrow \quad \begin{cases} (x,a,y) = c; \\ y = b; \end{cases} \quad \Leftrightarrow \quad \begin{cases} (x,a,y) = c; \\ (x,1,y) = b; \end{cases}$$

There exists the unique solution (x_0,b) of the last system in $E \times E$. Then the equation $(x,a,b) = c$ has a unique solution x_0 in E. The reasoning for the equation $(b,a,y) = c$ is analogous. □

Lemma 3. *Let the conditions 1)-5) of* ***Definition 2*** *hold true and the set E be finite. Then conditions 6a) and 6b) of* ***Definition 2*** *hold.*

Proof. Let conditions of **Lemma** hold. Let a,b,c be arbitrary elements from E, $c \neq 0$, $(c,a,0) \neq b$. We will demonstrate that the system

$$\begin{cases} (x,a,y)=b; \\ (x,t,y)\neq(c,t,0) \quad \forall t\in E; \end{cases}$$

has an unique solution in $E\times E$ (e.g. the condition 6a) of **Definition 2** holds).

Let us study the system

$$\begin{cases} (x,a,y)=b; \\ (x,t,y)=(c,t,0); \end{cases}$$

for every fixed $t\in E-\{a\}$. This system has a unique solution (x_t, y_t) in $E\times E$. Let us assume that

$$\begin{cases} t_1\neq t_2; \\ (x_{t_1}, y_{t_1})\equiv(x_{t_2}, y_{t_2}); \end{cases}$$

Then the system

$$\begin{cases} (x,t_1,y)=(c,t_1,0); \\ (x,t_2,y)=(c,t_2,0); \end{cases}$$

has two distinct solutions $(c,0)$ and $(x_{t_1}, y_{t_1})\equiv(x_{t_2}, y_{t_2})$ in $E\times E$ ($(x_{t_1}, y_{t_1})\neq(c,0)$, since $(x_{t_1}, a, y_{t_1})=b\neq(c,a,0)$). It contradicts the condition 5) of **Definition 2**. So

$$(x_{t_1}, y_{t_1})\neq(x_{t_2}, y_{t_2}) \quad \Leftrightarrow \quad t_1\neq t_2.$$

Then

$$\operatorname{card}\{(x_t, y_t)\mid t\in E-\{a\}\}=\operatorname{card}\{E-\{a\}\}=n-1,$$

where $n=\operatorname{card}E$. From the other side,

$$\operatorname{card}\{(x,y)\mid(x,a,y)=b\}=n,$$

since this number is equal to the number of cells with the element b in table of the operation (x,a,y) (see **Lemma 2** too). So there exists an unique pair $(x_0,y_0)\in E\times E$ that satisfies the following system:

$$\begin{cases} (x_0,a,y_0)=b; \\ (x_0,y_0)\neq(x_t,y_t) \quad \forall t\in E-\{a\}; \end{cases}$$

This pair (x_0,y_0) is the unique solution of the initial system from the condition 6a) of **Definition 2**, e.g. this condition holds.

Proof of the condition 6b) is analogous to that of 6a). □

Let us introduce the following binary operation (x,∞,y) on E:

$$(x,\infty,0)\overset{def}{=}0,$$

$$\begin{cases} (x,\infty,y)=u; \\ (x,y)\neq(u,0); \end{cases} \overset{def}{\Leftrightarrow} \quad (x,t,y)\neq(u,t,0) \quad \forall t\in E.$$

As we can see from the condition 6b) of **Definition 2**, the operation (x,∞,y) is defined correctly.

Lemma 4. *Operation* (x,∞,y) *satisfies the following conditions:*

1) $$\begin{cases} (x,\infty,y)=(u,\infty,v); \\ (x,y)\neq(u,v); \end{cases} \Leftrightarrow \quad (x,t,y)\neq(u,t,v) \quad \forall t\in E. \qquad (5)$$

2) $(x,\infty,x)=0.$

3) *There exists an unique solution in* $E\times E$ *of the system*

$$\begin{cases} (x,a,y)=b; \\ (x,\infty,y)=c; \end{cases}$$

for an arbitrary fixed $a,b,c\in E$.

4) *System* $<E,(x,\infty,y)>$ *is a quasigroup.*

Proof. 1). Let

$$\begin{cases}(x,\infty,y)=(u,\infty,v)=d;\\ (x,y)\neq(u,v);\end{cases}$$

Then we have by the definition of the operation (x,∞,y):

$$(x,t,y)\neq(d,t,0)\quad\forall t\in E,\tag{6}$$

$$(u,t,v)\neq(d,t,0)\quad\forall t\in E.\tag{7}$$

Assume that there exists $t_0\in E$ such that

$$(x,t_0,y)=(u,t_0,v)=w_0.\tag{8}$$

Then the system

$$\begin{cases}(x,t_0,y)=w_0;\\ (x,t,y)\neq(d,t,0)\quad\forall t\in E;\end{cases}$$

has two distinct solutions: (x,y) and (u,v) (see (6)-(8)). It contradicts condition 6a) of **Definition 2,** since

$$(x,t,y)\neq(u,t,v)\quad\forall t\in E.$$

Conversely, let

$$(x_0,t,y_0)\neq(u_0,t,v_0)\quad\forall t\in E.\tag{9}$$

Then we have (when $t=0,1$)

$$x_0\neq u_0,\qquad y_0\neq v_0,$$

i.e. $(x_0,y_0)\neq(u_0,v_0)$.

Let

$$(x_0,\infty,y_0)=d.$$

Then we have by the definition of the operation (x,∞,y):

$$(x_0,t,y_0)\neq(d,t,0)\quad\forall t\in E.\tag{10}$$

Let us assume that there exists $t_0\in E$ such that

$$(u_0,t_0,v_0)=(d,t_0,0)=z_0.\tag{11}$$

Then the system

$$\begin{cases}(x,t_0,y)=z_0;\\ (x,t,y)\neq(x_0,t,y_0)\quad\forall t\in E;\end{cases}$$

has two distinct solutions: (u_0, v_0) and $(d,0)$ (see (9)-(11)). It contradicts the condition 6a) of **Definition 2,** since

$$(u_0, t, v_0) \neq (d, t, 0) \quad \forall t \in E.$$

Then we have by the definition of the operation (x, ∞, y):

$$(u_0, \infty, v_0) = d = (x_0, \infty, y_0).$$

2). By the definition of the operation (x, ∞, y) we have

$$(0, \infty, 0) = 0.$$

If $x \neq 0$, then

$$(0, t, 0) = 0 \neq x = (x, t, x) \quad \forall t \in E,$$

(see condition 3) of **Definition 2**) and thus

$$(x, \infty, x) = (0, \infty, 0) = 0$$

(see p. 1) of this **Lemma**).

3). Let a,b,c be arbitrary fixed elements from E.

<u>Case A:</u> $c = 0$.

Then the system from the condition 3) of **Lemma** has the following form

$$\begin{cases} (x, a, y) = b; \\ (x, \infty, y) = 0; \end{cases} \qquad (12)$$

It is easy to see that the pair $(x, y) = (b, b)$ is a solution of system (12). Let us assume that there exists other solution $(x', y') \neq (b, b)$ of the system (12). Then we have

$$\begin{cases} (x', a, y') = b; \\ (x', \infty, y') = 0 = (b, \infty, b); \end{cases} \Leftrightarrow \begin{cases} (x', a, y') = b; \\ (x', t, y') \neq (b, t, b) = b \quad \forall t \in E; \end{cases}$$

It is impossible, since there exists an unique solution $(b\ b)$ of the system (12).

<u>Case B:</u> $(c, a, 0) = b$.

Then the system from the condition 3) of Lemma has the following form

$$\begin{cases}(x,a,y)=b=(c,a,0);\\(x,\infty,y)=c=(c,\infty,0);\end{cases} \tag{13}$$

It is easy to see that the pair $(x,y)=(c,0)$ is a solution of the system (13). Let us assume that there exists other solution $(x',y')\neq(c,0)$ of the system (13). Then we have

$$\begin{cases}(x',a,y')=(c,a,0);\\(x',\infty,y')=(c,\infty,0);\end{cases} \Leftrightarrow \begin{cases}(x',a,y')=(c,a,0);\\(x',t,y')\neq(c,t,0) \quad \forall t\in E;\end{cases}$$

It is impossible, since there exists an unique solution of the system (13).

<u>Case C:</u> $c\neq 0$ and $(c,a,0)\neq b$.

Then the system from the condition 3) of **Lemma** has the following form

$$\begin{cases}(x,a,y)=b;\\(x,t,y)\neq(c,t,0) \quad \forall t\in E;\end{cases} \tag{14}$$

System (14) has an unique solution in $E\times E$ (see the condition 6a) of **Definition 2**).

4). Proof is analogous to that of **Lemma 3**. □

Let us introduce points $(a,b),(m),(\infty)$ and lines $[a,b],[m],[\infty]$ (where $a,b,m\in E$) and define an incident relation I between points and lines by the following way (see [2]):

$$\begin{gathered}(a,b)I[c,d] \Leftrightarrow (a,c,b)=d,\\(a,b)I[d] \Leftrightarrow (a,\infty,b)=d,\\(a)I[c,d] \Leftrightarrow a=c,\\(a)I[\infty],\quad(\infty)I[d],\quad(\infty)I[\infty],\\(a,b)I[\infty]\Leftrightarrow(a)I[d]\Leftrightarrow(\infty)I[c,d]\Leftrightarrow \text{False}.\end{gathered} \tag{15}$$

Lemma 5. *The incidence system* $<P, L, I>$, *where*

$$P = \{(a,b),(m),(\infty) | a,b,m \in E\},$$

$$L = \{[a,b],[m],[\infty] | a,b,m \in E\},$$

I *is the incidence relation from (15)*

is a projective plane.

Proof. Let us verify the axioms of projective plane.

1). An arbitrary two distinct lines are intersected in unique point.

a). The lines $[a,b]$ and $[c,d]$:

If $a = c$, then we have from (15):

$$[a,b] \cap [c,d] \equiv [a,b] \cap [a,d] = (a).$$

If we assume that there exists a point (x,y) which lies both on lines $[a,b]$ and $[a,d]$, then

$$\begin{cases} (x,y) I [a,b]; \\ (x,y) I [a,d]; \end{cases} \Leftrightarrow \begin{cases} (x,a,y) = b; \\ (x,a,y) = d; \end{cases} \Rightarrow b = d,$$

i.e. $[a,b] \equiv [a,d]$. It is impossible since $[a,b]$ and $[a,d]$ are distinct lines.

If $a \neq c$, then we have

$$\begin{cases} (x,y) I [a,b]; \\ (x,y) I [c,d]; \end{cases} \Leftrightarrow \begin{cases} (x,a,y) = b; \\ (x,c,y) = d; \end{cases}$$

By the condition 5) from **Definition 2** there exists an unique such point (x,y).

b). The lines $[a,b]$ and $[d]$:

We have

$$\begin{cases} (x,y) I [a,b]; \\ (x,y) I [d]; \end{cases} \Leftrightarrow \begin{cases} (x,a,y) = b; \\ (x,\infty,y) = d; \end{cases}$$

As we can see from the statement 3) of **Lemma 4** there exists an unique such point (x,y).

c). The lines $[a,b]$ and $[\infty]$, $[m]$ and $[d]$, $[m]$ and $[\infty]$.

We have

$$[a,b]\cap[\infty]=(a),$$
$$[m]\cap[d]=(\infty),$$
$$[m]\cap[\infty]=(\infty).$$

2). There exists an unique common line for arbitrary two distinct points.

a). The points (a,b) and (c,d):

If there exists an element $t_0 \in E$ such that

$$(a,t_0,b)=(c,t_0,d)=f, \tag{16}$$

then we have

$$(a,b)\cup(c,d)=[t_0,f].$$

As we can see from the condition 5) of **Definition 2**, only one element $t_0 \in E$ with the condition (16) may exist.

If

$$(a,t,b)\neq(c,t,d) \quad \forall t \in E,$$

then by the statement 1) of **Lemma 4** we have

$$(a,\infty,b)=(c,\infty,d)=h,$$

and

$$(a,b)\cup(c,d)=[h].$$

b). The points (a,b) and (m), (a,b) and (∞), (m) and (n), (m) and (∞).

We have

$$(a,b)\cup(m)=[m,(a,m,b)],$$
$$(a,b)\cup(\infty)=[(a,\infty,b)],$$
$$(m)\cup(n)=[\infty],$$
$$(m)\cup(\infty)=[\infty].$$

3). There exist four points in a common position.

These points are $(0,0),(1,0),(0)$ and (∞). Really, we have

$$(0,0)\cup(1,0)=[1,0], \qquad (1,0)\cup(0)=[0,1],$$
$$(0,0)\cup(0)=[0,0], \qquad (1,0)\cup(\infty)=[1],$$
$$(0,0)\cup(\infty)=[0], \qquad (0)\cup(\infty)=[\infty].$$

2. Cell permutations and pair loop of DK-ternar

Lemma 6. *Let the system* $<E,(x,t,y),0,1>$ *be a DK-ternar. Let* a,b *be arbitrary elements from E and* $a\neq b$. *Then any unary operation*

$$\alpha_{a,b}(t)=(a,t,b) \tag{17}$$

is a permutation on the set E.

Proof. Let the conditions of **Lemma** hold. We can prove the following: if $t_1\neq t_2$, then $\alpha_{a,b}(t_1)\neq\alpha_{a,b}(t_2)$. Let us assume that there exist $t_1,t_2\in E$ such that

$$\begin{cases} t_1\neq t_2; \\ (a,t_1,b)=(a,t_2,b)=k; \end{cases}$$

Then the system

$$\begin{cases} (x,t_1,y)=k; \\ (x,t_2,y)=k; \end{cases}$$

has two distinct solutions in $E\times E$: (a,b) and (k,k). It contradicts condition 5) of **Definition 2.**

Let us prove that for any $c\in E$ there exists $t_0\in E$ such that $c=\alpha_{a,b}(t_0)$. We have (see **Lemmas 4** and **5**):

$$c = \alpha_{a,b}(t_0) \Leftrightarrow$$
$$\Leftrightarrow \quad c = (a, t_0, b) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (a,b) \in [t_0, c] \quad \Leftrightarrow$$
$\Leftrightarrow$ (points $(a,b), (t_0)$ and (c,c) lie
in a common line in the projective plane π). $\Leftrightarrow$
$$\Leftrightarrow \quad (t_0) = [\infty] \cap [(a,b) \cup (c,c)].$$

There exists an unique such element $t_0 \in E$. □

The permutations from **Lemma 6** are called *cell permutations.*

Lemma 7. *Cell permutations satisfy of the following conditions:*

1). *All cell permutations are distinct;*

2). ($\alpha_{a,b}$ *is a fixed-point-free cell permutation)* $\Leftrightarrow$ $((a,\infty,b) = (0,\infty,1))$.

3). *There exists fixed-point-free permutation* ν *on E such that we can describe all fixed-point-free cell permutations (with the identity cell permutation* $\alpha_{0,1}(t)$*) by the following form:*

$$\alpha(t) = (a, t, \nu(a)), \qquad (\nu(0) = 1).$$

4). *The set M of all cell permutations of DK-ternar is sharply 2-transitive on the set E.*

Proof. 1). Let us have

$$\alpha_{a,b}(t) = \alpha_{c,d}(t) \quad \forall t \in E.$$

Then

$$a = (a,0,b) = \alpha_{a,b}(0) = \alpha_{c,d}(0) = (c,0,d) = c,$$
$$b = (a,1,b) = \alpha_{a,b}(1) = \alpha_{c,d}(1) = (c,1,d) = d,$$

i.e. $(a,b) \equiv (c,d)$. Thus if $(a,b) \neq (c,d)$, then $\alpha_{a,b} \neq \alpha_{c,d}$, e.g. all cell permutations are distinct.

2). ($\alpha_{a,b}$ is a fixed-point-free cell permutation) $\Leftrightarrow$

$$\Leftrightarrow \quad (a,t,b) \neq t = (0,t,1) \quad \forall t \in E \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (a,\infty,b) = (0,\infty,1)$$

(see 1) from **Lemma 4**).

3). It is a trivial corollary of 2) and the statement 4) of **Lemma 4.**

4). Let a,b,c,d be arbitrary elements of E and $a \neq b, c \neq d$. Then we have

$$\begin{cases} \alpha_{x,y}(a) = c; \\ \alpha_{x,y}(b) = d; \end{cases} \Leftrightarrow \begin{cases} (x,a,y) = c; \\ (x,b,y) = d; \end{cases}$$

By the condition 5) of **Definition 2** there exists an unique solution (x,y) of the last system; moreover, $x \neq y$, since $c \neq d$. So the set M of all cell permutations is sharply 2-transitive on E. □

Lemma 8. *Let* $M = \{\alpha_{a,b}\}_{a,b \in E}$ *be a set of permutations on the set* E *(*E *is a finite set with distinguished elements* 0 *and* 1*), and the following conditions hold:*

1) $\alpha_{0,1} \equiv \mathrm{id}$;

2) $\alpha_{a,b}(0) = a, \qquad \alpha_{a,b}(1) = b$;

3) *Set* M *is a sharply 2-transitive set of permutations on* E.

Let us suppose by definition:

$$(x,t,x) \overset{def}{=} x,$$
$$(x,t,y) \overset{def}{=} \alpha_{x,y}(t), \quad if\ x \neq y.$$

Then system $< E,(x,t,y),0,1 >$ *is a DK-ternar.*

Proof is a trivial verification of the conditions 1)-5) of **Definition 2**.

Let the system $< E,(x,t,y),0,1 >$ be a finite DK-ternar. Let us define on set

$E \times E - \{\Delta\} = \{< a,b > | a,b \in E, a \neq b\}$ the following binary operation:

$$< x,y > \cdot < z,u > \overset{def}{=} < (x,z,y),(x,u,y) >. \tag{18}$$

Lemma 9. *The system* $< E \times E - \{\Delta\}, \cdot, < 0,1 >>$ *is a loop.*

Proof is given in [2].

This loop is called a *pair loop of the DK-ternar* $< E,(x,t,y),0,1 >$.

Lemma 10. *Let us have a finite set* E *with distinguished elements* 0 *and* 1. *Let on the set* $E \times E - \{\Delta\}$ *a binary operation* "$\cdot$" *is defined such that system* $< E \times E - \{\Delta\}, \cdot, < 0,1 >>$ *is a loop. Then the next conditions are equivalent:*

1) *The system* $< E \times E - \{\Delta\}, \cdot, < 0,1 >>$ *is a pair loop of some DK-ternar;*

2) *The following quasiidentities hold on* $< E \times E - \{\Delta\}, \cdot, < 0,1 >>$:

a) $(< x,y > \cdot < z,u > = < v,w >) \Rightarrow (< x,y > \cdot < u,z > = < w,v >)$;

b) $(< x,y > \cdot < z,u > = < v,w >, u \neq 0) \Rightarrow (< x,y > \cdot < 0,u > = < x,w >)$;

c) $(< x,y > \cdot < z,u > = < v,w >, u \neq 1) \Rightarrow (< x,y > \cdot < 1,u > = < y,w >)$;

Proof is given in [2].

3. Pair loop of DK-ternar as a loop with conditions on cosets by two subloops

Lemma 11. *Let the system* $<A,\cdot,e>$ *be a finite loop of order* $n(n-1)$. *Then the following conditions are equivalent:*

1). *The loop* $<A,\cdot,e>$ *is isomorphic to the pair loop of some finite DK-ternar.*

2). *Loop* $<A,\cdot,e>$ *satisfies the following conditions:*

a). *There exist two subloops* A_0 *and* B_1 *in the loop* A, *such that*

$$\mathrm{card}A_0 = \mathrm{card}B_1 = n-1, \qquad A_0 \cap B_1 = \{e\}.$$

b). *The loop* A *may be represented in a form of disjunctive unifications of left cosets* A_i *and* B_j *by the subloops* A_0 *and* B_1 *respectively:*

$$A = \bigcup_{i\in E} A_i = \bigcup_{j\in E} B_j,$$

where E *is an index set,* $\mathrm{card}E = n$.

c). *It is true for any* $i,j \in E$:

1) *if* $i \neq j$, *then*

$$A_i \cap B_j = \{x_{ij}\},$$

and $x_{ij} \neq x_{km}$, *when* $(i,j) \neq (k,m)$; *moreover, any* $x_0 \in A$ *may be represented in that form;*

2) *if* $i = j$, *then* $A_i \cap B_j = \varnothing$.

d). *The element* $a_0 = A_1 \cap B_0$ *satisfies the following conditions:*

1) $a_0 \in N_r(A)$, *where* $N_r(A)$ *is a right kernel of the loop* A;

2) $A_i \cdot a_0 = B_i, \qquad B_j \cdot a_0 = A_j,$

e). *It is true for any* $c_0 \in A$:

$$c_0 \cdot A_i = A_j, \qquad c_0 \cdot B_i = B_j, \qquad \forall i \in E.$$

Proof.

1) ⇒ 2). Let loop $<A, \cdot, e>$ is isomorphic to the pair loop $<E \times E - \{\Delta\}, \cdot, <0,1>>$ of some finite DK-ternar. Let us verify that the conditions 1)-5) of **Lemma** hold.

1). Let us study the following subsets of the pair loop:

$$A_0 = \{<0, x> | x \in E - \{0\}\},$$
$$B_1 = \{<x,1> | x \in E - \{1\}\}.$$

If $\mathrm{card}E = n$, then $\mathrm{card}A_0 = \mathrm{card}B_1 = n - 1$. Since

$$<0, x> \cdot <0, y> = <0, (0, y, x)>;$$
$$<x,1> \cdot <y,1> = <(x, y, 1), 1>;$$
$$<0,1> \in A_0 \cap B_1;$$

then A_0 and B_1 are subloops of the pair loop. Finally, it is evident that

$$A_0 \cap B_1 = \{<0,1>\}.$$

2). Consider the following subsets of the pair loop:

$$A_i = \{<i, y> | y \in E - \{i\}, i \text{ is a fixed element from } E\};$$
$$B_j = \{<x, j> | x \in E - \{j\}, j \text{ is a fixed element from } E\}; \tag{19}$$

It is evident that

$$\bigcup_{i \in E} A_i = \bigcup_{\substack{i, y \in E \\ i \neq y}} <i, y> = E \times E - \{\Delta\} \equiv A;$$
$$\bigcup_{j \in E} B_j = \bigcup_{\substack{j, x \in E \\ j \neq x}} <x, j> = E \times E - \{\Delta\} \equiv A;$$

By the help of **Lemma 10** we obtain

$$<i, y_0> \cdot <0, u> = <i, w> \;\Rightarrow\; <i, y_0> \cdot A_0 = A_i,$$
$$<x_0, j> \cdot <u,1> = <w, j> \;\Rightarrow\; <x_0, j> \cdot B_1 = B_j,$$

i.e. the sets A_i and B_j are left cosets by the subloops A_0 and B_1 respectively.

3). It is evident since

$$\{<i,j>\} = A_i \cap B_j;$$
$$<i,i> \notin E\times E-\{\Delta\}.$$

4). We have

$$A_1 \cap B_0 = \{<1,0>\}$$

and by the help of **Lemma 10** we obtain

$$(<x,y>\cdot<u,z>)\cdot<1,0> = <v,w>\cdot<1,0> = <w,v> =$$
$$= <x,y>\cdot<z,u> = <x,y>\cdot(<u,z>\cdot<1,0>),$$

i.e. $<1,0> \in N_r(A)$. We have too

$$<i,y>\cdot<1,0> = <y,i> \quad\Rightarrow\quad A_i\cdot<1,0> = B_i;$$
$$<x,j>\cdot<1,0> = <j,x> \quad\Rightarrow\quad B_j\cdot<1,0> = A_j;$$

5). Let $<a_0,b_0>$ be an arbitrary element from $E\times E-\{\Delta\}$. Then we have for any $i_0 \in E$:

$$<a_0,b_0>\cdot<i_0,y> = <(a_0,i_0,b_0),(a_0,y,b_0)> = <j_0,w>,$$

i.e.

$$<a_0,b_0>\cdot A_{i_0} = A_{j_0} \quad \text{for some} \quad j_0\in E.$$

Analogously we obtain

$$<a_0,b_0>\cdot B_j = B_k \quad \text{for some} \quad k\in E.$$

2) $\Rightarrow$ 1).

Let the conditions 1)-5) of the present **lemma** hold for the loop $<A,\cdot,e>$. Let us define the following reflection

$$\varphi: A\to E\times E-\{\Delta\};$$
$$\varphi(A_i\cap B_j) \stackrel{def}{=} <i,j>.$$

The reflection φ is a bijection (see the condition 3) of **lemma**). Let us define the following operation "$\cdot$" on the set $E\times E-\{\Delta\}$:

$$<i,j>\cdot<k,m> \overset{def}{=} \varphi(x_{ij}\cdot x_{km}),$$

where $x_{uv}=A_u\cap B_v$. Operation "$\cdot$" is defined correctly, since φ is a bijection. Moreover, since

$$\varphi(x_{ij}\cdot x_{km})=<i,j>\cdot<k,m>=\varphi(x_{ij})\cdot\varphi(x_{km}),$$

then φ is an isomorphism of the loop $<A,\cdot,e>$ on some pair loop $<E\times E-\{\Delta\},\cdot,<0,1>>$ (and $\varphi(e)=\varphi(A_0\cap B_1)=<0,1>$).

Let us prove that this pair loop is a pair loop of some finite DK-ternar. It is necessary to verify that the conditions 1)-3) of **Lemma 10** hold.

a). Let us have

$$<x,y>\cdot<z,u>=<v,w>.$$

Then

$$\begin{aligned} x_{vw}&=\varphi^{-1}(<v,w>)=\varphi^{-1}(<x,y>\cdot<z,u>)=\\ &=\varphi^{-1}(<x,y>)\cdot\varphi^{-1}(<z,u>)=x_{xy}\cdot x_{zu}. \end{aligned} \tag{20}$$

By the help of the condition 4) we obtain

$$x_{vw}\cdot a_0=(A_v\cap B_w)\cdot a_0=(A_v\cdot a_0)\cap(B_w\cdot a_0)=B_v\cap A_w=x_{wv}, \tag{21}$$

and

$$(x_{xy}\cdot x_{zu})\cdot a_0=x_{xy}\cdot(x_{zu}\cdot a_0). \tag{22}$$

From (20)-(22) we obtain

$$x_{wv}=x_{vw}\cdot a_0=(x_{xy}\cdot x_{zu})\cdot a_0=x_{xy}\cdot(x_{zu}\cdot a_0)=x_{xy}\cdot x_{uz},$$

i.e.

$$<w,v>=\varphi(x_{wv})=\varphi(x_{xy}\cdot x_{uz})=<x,y>\cdot<u,z>.$$

The quasiidentity 1) from **Lemma 10** holds.

b). Let us have

$$<x,y>\cdot<z,u>=<v,w>,\quad u\neq 0.$$

Then

$$x_{vw}=x_{xy}\cdot x_{zu}.$$

By means of the condition 5) we obtain

$$A_v \cap B_w = x_{vw} = x_{xy} \cdot x_{zu} = x_{xy} \cdot (A_z \cap B_u) = \\ = (x_{xy} \cdot A_z) \cap (x_{xy} \cdot B_u) = A_m \cap B_t. \tag{23}$$

By virtue of the condition 3) we obtain for any $i_0 \in E$:

$$A_{i_0} = \left\{z \in A_{i_0} \cap B_j \mid j \in E - \{i_0\}\right\} \equiv \left\{x_{i_0 j} \mid j \in E - \{i_0\}\right\}.$$

But the set A_{i_0} is a left coset by the subloop A_0 and so there exists $x_{pq} \in A$ such that

$$x_{pq} \cdot A_0 = A_{i_0}. \tag{24}$$

Since $e \in A_0$ then $x_{pq} \in A_{i_0}$; e.g. $x_{pq} \equiv x_{i_0 j_0}$ for some $j_0 \in E$. Then we obtain from (24)

$$x_{i_0 j_0} \cdot A_0 = A_{i_0},$$

and since i_0 was an arbitrary element from E, then

$$x_{xy} \cdot A_0 = A_x \tag{25}$$

for any $x \in E$. From (23) and (25) it follows that

$$x_{xy} \cdot x_{0u} = x_{xy} \cdot (A_0 \cap B_u) = (x_{xy} \cdot A_0) \cap (x_{xy} \cdot B_u) = A_x \cap B_t = x_{xt}. \tag{26}$$

By the help of the conditions 2) and 3) of this **lemma** and the identities (23)-(26) we obtain

$$A_v = A_m, \qquad B_w = B_t,$$

i.e. $v = m, w = t$. In accord with (15)

$$x_{xy} \cdot x_{0u} = x_{xw},$$

i.e.

$$< x, y > \cdot < 0, u >= \varphi(x_{xy}) \cdot \varphi(x_{0u}) = \varphi(x_{xy} \cdot x_{0u}) = \varphi(x_{xw}) =< x, w >.$$

The quasiidentity 2) of **Lemma 10** holds.

c). Proof of quasiidentity 3) of **Lemma 10** is analogously to that of b). □

§4. Sharply 2-transitive sets of permutations degree n and loop transversals in S_n to $St_{a,b}(S_n)$

Let us return to the set of cell permutations of some finite *DK*-ternar. The following statement is true.

Lemma 12. *Let* E *be a finite set and* $|E|=n$. *The following conditions are equivalent:*

1). *A set* T *is a loop transversal in* S_n *to* $St_{a,b}(S_n)$, *where* $a,b \in E$ *are arbitrary fixed distinct elements;*

2). *A set* T *is a sharply 2-transitive set of permutations on* E;

3) *A set* T *is a sharply 2-transitive permutation loop on* E;

The permutation loop is defined in [6].

Proof is given in [7]. □

§5. Loop of points of a projective plane

In this paragraph it will be proved the definition of such binary operation on the set of points of a projective plane, which is identical to the operation of pair loop of *DK*-ternar corresponding to that plane. This operation will be a loop (see §2) and since the loop of points mentioned above will be called a *loop of points of a projective plane.*

Let us have a projective plane π and a *DK*-ternar corresponding to it (see §1). Let us demonstrate the method of a purely geometrical construction (with the help of an incidence relation only) of the point (v,w) by the points (x,y) and (z,u) (where $x \neq y, z \neq u$), where $<v,w>=<x,y>\cdot<z,u>$ in the pair loop of the *DK*-ternar mentioned above. The sequence of the construction will be described step by step below.

1). $$X=(0), \quad O=(0,0), \quad Y=(1), \quad I=(1,1)$$

are four points in a common position on the plane π.

2). $$(1)\cup(1,1)=[1,1]; \quad (0,0)\cup(1,1)=[0]; \quad (0)\cup(0,0)=[0,0]; \quad (0)\cup(1)=[\infty].$$

3). $$[0,0]\cap[1,1]=(0,1).$$

4). $$(0)\cup(z,u)=[0,z]; \quad (1)\cup(z,u)=[1,u].$$

5). $$[0,z]\cap[0]=(z,z); \quad [1,u]\cap[0]=(u,u).$$

6). $$(0,1)\cup(u,u)=[u,u]; \quad (0,1)\cup(z,z)=[z,z].$$

7). $$[u,u]\cap[\infty]=(u); \quad [z,z]\cap[\infty]=(z).$$

8). $$(x,y)\cup(u)=[u,(x,u,y)]\equiv[u,w]; \quad (x,y)\cup(z)=[z,(x,z,y)]\equiv[z,v].$$

9). $$[u,w]\cap[0]=(w,w); \quad [z,v]\cap[0]=(v,v).$$

10). $$(0)\cup(v,v)=[0,v]; \quad (1)\cup(w,w)=[1,w].$$

11). $[0,v]\cap[1,w]=(v,w)$.

The point (v,w) is constructed.

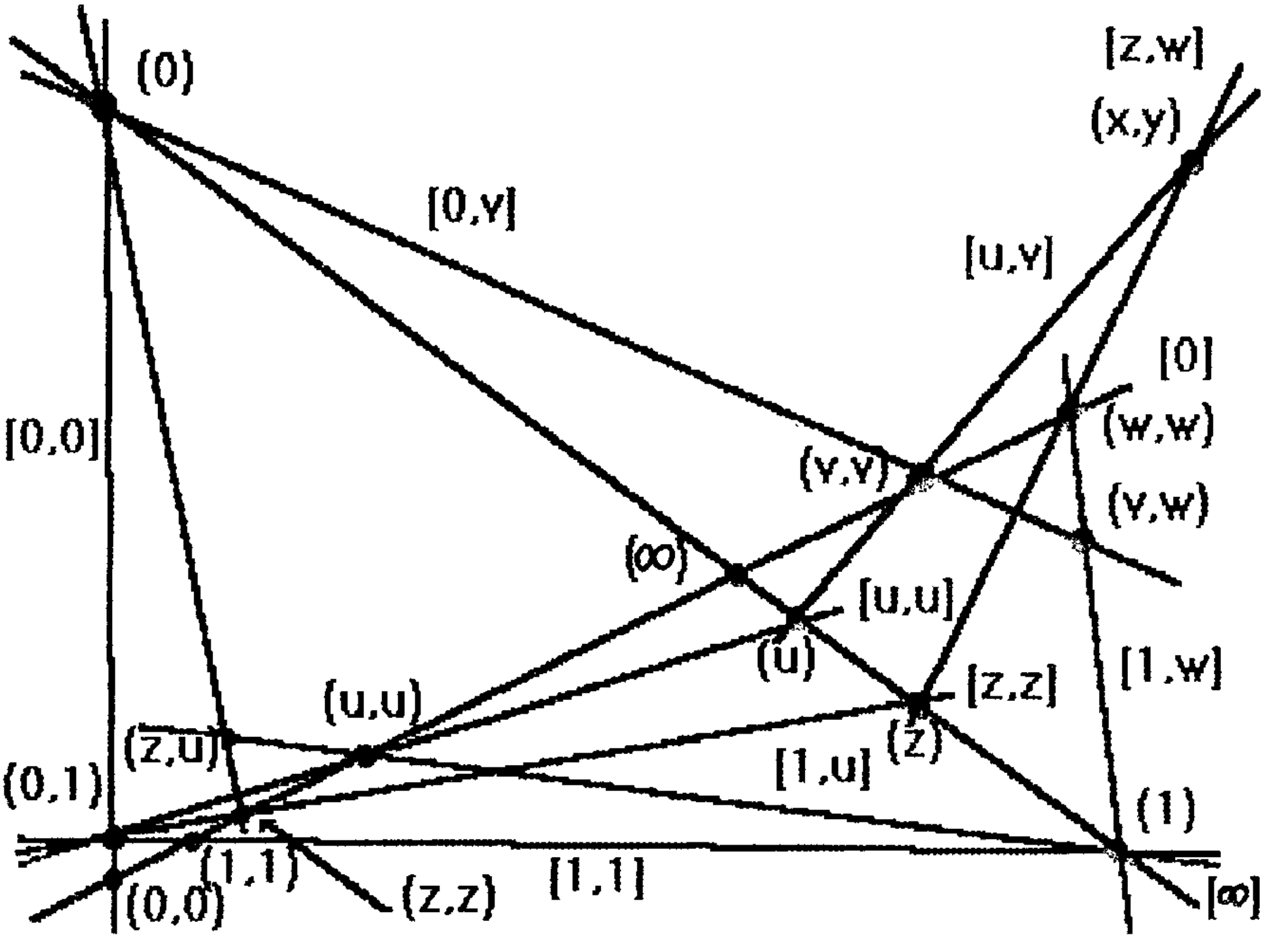


Fig. 1.

It is easy to see that we used the incidence relation only in the construction described above. Then this construction is independent from a coordinatization on the plane π and could be done without some coordinates on π.

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