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Isomorphisms of quasigroups isotopic to groups

Vladimir I. Izbash

Abstract

In this note quasigroups isotopic to groups are considered. The necessary and sufficient conditions for two quasigroups isotopic to the same group to be isomorphic are found. The form of isomorphism of two quasigroups isotopic to the same group and the form of automorphism of a group isotope are given. For a T-quasigroup with an idempotent the group of automorphisms is described.

1. Introduction

We consider quasigroup operations defined on the same set Q. It will be convenient to recall some of the terminology of quasigroup theory.

The *quasigroup* $(Q,\cdot)$ is a groupoid $(Q,\cdot)$ with the unique division. For each $a \in Q$ we have two transformations of the underlying set Q. They are *called the left and the right translation by* a and they are defined by

$$L_a x = a \cdot x \quad \text{and} \quad R_a x = x \cdot a$$

for every $x \in Q$.

Since $(Q,\cdot)$ is a quasigroup, both these transformations are permutations and hence they belong to the permutations group $S(Q)$ of Q.

The *left (right) loop* is a quasigroup $(Q,\cdot)$ with the left (right) unite e (f) such that $e\cdot x=x$ $(x\cdot f=x)$ for every $x\in Q$.

The element 1 of a quasigroup $(Q,\cdot)$ is said to be *a unit* of $(Q,\cdot)$ if for every x of Q

$$1\cdot x=x\cdot 1=x.$$

A *loop* is a quasigroup with the unit.

By $Aut(Q,\cdot)$ we denote the group of all automorphisms of $(Q,\cdot)$.

For each $a\in Q$ put

$$S_a(Q)=\{\alpha\in S(Q)\,|\,\alpha a=a\}.$$

Let "$\circ$" and "$*$" be two operations defined on Q. The operation "$\circ$" is said to be isotopic to "$*$", if there exist three permutations $\alpha,\beta,\gamma\in S(Q)$ such that

$$x*y=\gamma^{-1}(\alpha x\circ\beta y) \qquad (1)$$

for all $x,y\in Q$.

We also say that $(Q,*)$ and $(Q,\circ)$ are *isotopic*, or that $(Q,*)$ *is an isotope* of $(Q,\circ)$ of the form $x*y=\gamma^{-1}(\alpha x\circ\beta y)$. Shortly we write this as

$$(Q,*):\quad x*y=\gamma^{-1}(\alpha x\circ\beta y),\quad \alpha,\beta,\gamma\in S(Q),\ x,y\in Q.$$

The triple (α,β,γ) of permutations such that the relations (1) hold is called the *isotopy* of $(Q,\circ)$.

If in (1) γ is the identical permutation ε, then $(Q,*)$ is said to be the *principal isotope* of $(Q,\circ)$.

If in (1) $\alpha=\beta=\gamma$, then

$$x*y=\gamma^{-1}(\gamma x\circ\gamma y), \qquad (2)$$

which means that γ is an automorphism between $(Q,*)$ and $(Q,\circ)$.

The equality (1) is equivalent to the following equality

$$x*y=\gamma^{-1}(\alpha\gamma^{-1}\gamma x\circ\beta\gamma^{-1}\gamma y), \tag{3}$$

whence we have proved the following

Theorem 1 (**[1] Theorem 1.2**). *An isotope*

$$(Q,*):\quad x*y=\gamma^{-1}(\alpha x\circ\beta y),\quad \alpha,\beta,\gamma\in S(Q),\ x,y\in Q$$

is isomorphic to the principal isotope

$$(Q,\otimes):\quad x\otimes y=\alpha\gamma^{-1}x\circ\beta\gamma^{-1}y,\quad \alpha,\beta,\gamma\in S(Q),\ x,y\in Q,$$

and γ *is the isomorphism between them.*

Theorem 2 (**[1] Theorem 1.3**). *For arbitrary fixed elements* $a,b\in Q$ *the isotope*

$$(Q,*):\ x*y=R_a^{-1}x\cdot L_b^{-1}y,\quad x,y\in Q,$$

is a loop with unit $e=b\cdot a$, *where* R_a, L_b *are translations of a quasigroup* $(Q,\cdot)$ *by* a *and* b *respectively.*

Theorem 3 (**[1] Theorem 1.4**). *If a loop* $(Q,\circ)$ *is isotopic to a group* $(Q,\cdot)$, *then it is a group isomorphic to* $(Q,\cdot)$.

A permutation $\gamma\in S(Q)$ is called *a quasiautomorphism* of a group $(Q,\cdot)$ if there exist two permutations $\alpha,\beta\in S(Q)$ such that

$$x\cdot y=\gamma^{-1}(\alpha x\cdot\beta y)$$

holds for all $x,y\in Q$.

Quasiautomorphisms of a group $(Q,\cdot)$ are described by the next

Lemma 1 (**[1] Lemma 2.5**). *A permutation* $\gamma\in S(Q)$ *is a quasiautomorphism of a group* $(Q,\cdot)$ *if and only if there exist* *element* $s \in Q$, *automorphisms* $\varphi_0, \varphi_0' \in Aut(Q,\cdot)$ *such that* $\gamma = R_s\varphi_0$ *or* $\gamma = L_s\varphi_0'$.

The following equalities hold in a group $(Q,\cdot)$:

$$R_a^{-1} = R_{a^{-1}};\quad L_a^{-1} = L_{a^{-1}};\quad R_aR_b = R_{ba};\quad L_aL_b = L_{ab};$$

$$\varphi R_a = R_{\varphi a}\varphi;\quad \varphi L_a = L_{\varphi a}\varphi;\quad \varphi(a^{-1}) = (\varphi a)^{-1},$$

where $\varphi \in Aut(Q,\cdot)$, a^{-1} is the inverse of a in $(Q,\cdot)$.

In abelian group $(Q,\cdot)$ we have $R_a = L_a$ for every $a \in Q$.

2. Isomorphism of group isotopes

An isotope of a group is called *a group isotope*.

Let

$$\begin{aligned} &(Q,*):\ x*y = \alpha x\cdot\beta y,\ \ \alpha,\beta \in S(Q),\ x,y \in Q,\\ &(Q,\circ):\ x\circ y = \alpha_1 x\cdot\beta_1 y,\ \ \alpha_1,\beta_1 \in S(Q),\ x,y \in Q, \end{aligned} \tag{4}$$

be two isotopes of a group $(Q,\cdot)$.

Suppose that $(Q,*)$ and $(Q,\circ)$ are isomorphic. Then there exists a permutation $\theta \in S(Q)$ such that

$$\theta(x*y) = \theta x \circ \theta y$$

or

$$\theta(\alpha x\cdot\beta y) = \alpha_1\theta x\cdot\beta_1\theta y \tag{5}$$

for all $x,y \in Q$.

Let $1 \in Q$ be the unit of the group $(Q,\cdot)$ and $e \in Q$ be such that $\beta e = 1$. Putting in (5) $y = e$ and $\theta^{-1}x$ instead of x in (5) we obtain $\theta\alpha\theta^{-1}x = \alpha_1 x\cdot\beta_1\theta e$. Therefore

$$\alpha_1 = R_a^{-1}\theta\alpha\theta^{-1}, \tag{6}$$

where $a = \beta_1\theta e$.

From (5) using (6) we find

$$\theta(\alpha x \cdot \beta y) = R_a^{-1}\theta\alpha\theta^{-1}\theta x \cdot \beta_1\theta y = R_a^{-1}\theta\alpha x \cdot \beta_1\theta y.$$

Replacing here x for $\alpha^{-1}x$ we get

$$\theta(x \cdot \beta y) = R_a^{-1}\theta x \cdot \beta_1\theta y.$$

Putting here $x = 1$ we obtain

$$\theta\beta y = R_a^{-1}\theta 1 \cdot \beta_1\theta y.$$

Hence

$$\beta = \theta^{-1}L_b\beta_1\theta, \tag{7}$$

with $b = R_a^{-1}\theta 1$.

Analogously we will obtain a relation for α, α_1 and θ.

Let $f \in Q$ be such that $\alpha f = 1$. Putting in (5) $x = f$ and $\theta^{-1}y$ instead of y we obtain

$$\theta\beta\theta^{-1}y = \alpha_1\theta f \cdot \beta_1 y.$$

Therefore

$$\beta_1 = L_c^{-1}\theta\beta\theta^{-1} \tag{8}$$

where $c = \alpha_1\theta f$.

From (5) and (8) we find

$$\theta(\alpha x \cdot y) = \alpha_1\theta x \cdot L_c^{-1}\theta y$$

and if $y = 1$ we have

$$\theta\alpha x = \alpha_1\theta x \cdot L_c^{-1}\theta 1.$$

Hence

$$\alpha = \theta^{-1}R_d\alpha_1\theta, \tag{9}$$

where $d = L_c^{-1}\theta 1$.

Now the equality (5) can be rewritten in the following way

$$\theta(\theta^{-1}R_d\alpha_1\theta x\cdot\theta^{-1}L_b\beta_1\theta y)=\alpha_1\theta x\cdot\beta_1\theta y.$$

From this equality, replacing x for $\theta^{-1}\alpha_1^{-1}x$ and y for $\theta^{-1}\beta_1^{-1}y$ we get

$$x\cdot y=\theta(\theta^{-1}R_dx\cdot\theta^{-1}L_by). \qquad (10)$$

Therefore θ is a quasiautomorphism of the group $(Q,\cdot)$.

By **Lemma 1** there exist $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ such that $\theta=R_s\theta_0$. Then (9) and (7) can be rewritten in the following way respectively:

$$\alpha=\theta_0^{-1}R_s^{-1}R_d\alpha_1R_s\theta_0, \qquad (11)$$

where $d=L_c^{-1}R_s\theta_0 1=L_c^{-1}s,\quad c=\alpha_1R_s\theta_0 f=\alpha_1(\theta_0 f\cdot s)$, and

$$\beta=\theta_0^{-1}R_s^{-1}L_b\beta_1R_s\theta_0, \qquad (12)$$

where $b=R_a^{-1}R_s\theta_0 1=R_a^{-1}s,\quad a=\beta_1R_s\theta_0 e=\beta_1(\theta_0 e\cdot s)$.

Using (11),(12) and the equality $\theta=R_s\theta_0$ we find from (10):

$$x\cdot y=R_s\theta_0(\theta_0^{-1}R_s^{-1}R_dx\cdot\theta_0^{-1}R_s^{-1}L_by)=R_s(R_s^{-1}R_dx\cdot R_s^{-1}L_by)=$$
$$=(xd\cdot s^{-1})(by\cdot s^{-1})s=((x\cdot L_c^{-1}s)s^{-1})(R_a^{-1}s\cdot y)=((xc^{-1}s)s^{-1})(sa^{-1}\cdot y)=$$
$$=(xc^{-1})(sa^{-1}\cdot y)=(x(\alpha_1R_s\theta_0 f)^{-1}\cdot((s\cdot\beta_1R_s\theta_0 e)^{-1}))y=$$
$$=x(\alpha_1R_s\theta_0 f)^{-1}\cdot s\cdot(\beta_1R_s\theta_0 e)^{-1}\cdot y.$$

Therefore

$$(\alpha_1R_s\theta_0 f)^{-1}\cdot s\cdot(\beta_1R_s\theta_0 e)^{-1}=1. \qquad (13)$$

So, if θ is an isomorphism of quasigroups $(Q,*)$ and $(Q,\circ)$, then there exist $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ such that equalities (11)-(13) hold and $\theta=R_s\theta_0$.

Conversely, let $s\in Q$ and $\theta_0\in Aut(Q,\cdot)$ satisfy equalities (11)-(13) and $\theta=R_s\theta_0$. Then

$$R_s\theta_0(x * y) = R_s\theta_0(\theta_0^{-1}R_s^{-1}R_d\alpha_1 R_s\theta_0 x \cdot \theta_0^{-1}R_s^{-1}L_b\beta_1 R_s\theta_0 y) =$$
$$= R_s(R_s^{-1}R_d\alpha_1 R_s\theta_0 x \cdot R_s^{-1}L_b\beta_1 R_s\theta_0 y) = R_s^{-1}R_d\alpha_1 R_s\theta_0 x \cdot L_b\beta_1 R_s\theta_0 y =$$
$$= (\alpha_1(\theta_0 x \cdot s)ds^{-1})(b \cdot \beta_1(\theta_0 y \cdot s)) = (\alpha_1(\theta_0 x \cdot s)(L_c^{-1}s \cdot s^{-1})(R_a^{-1}s \cdot \beta_1(\theta_0 y \cdot s)) =$$
$$= (\alpha_1(\theta_0 x \cdot s) \cdot \alpha_1(\theta_0 f \cdot s)^{-1}s \cdot s^{-1})(s \cdot \beta_1(\theta_0 e \cdot s)^{-1}\beta_1(\theta_0 y \cdot s) =$$
$$= \alpha_1(\theta_0 x \cdot s) \cdot \beta_1(\theta_0 y \cdot s) = R_s\theta_0 x \circ R_s\theta_0 y,$$

i.e. $R_s\theta_0$ is an isomorphism of the quasigroups $(Q,*)$ and $(Q,\cdot)$.

Thus, we have proved

Theorem 4. *A permutation $\theta \in S(Q)$ is an isomorphism of isotopes $(Q,*)$ and $(Q,\circ)$ defined by (4) if and only if there exist $s \in Q$ and $\theta_0 \in Aut(Q,\cdot)$ such that the relations (11)-(13) hold ,where 1 is the unit of the group $(Q,\cdot)$.*

Let

$$(Q,\underset{1}{\circ}):\quad x \underset{1}{\circ} y = \gamma^{-1}(\alpha x \cdot \beta y),\quad \alpha,\beta,\gamma \in S(Q),\ x,y \in Q,$$
$$(Q,\underset{2}{\circ}):\quad x \underset{2}{\circ} y = \gamma_1^{-1}(\alpha_1 x \cdot \beta_1 y),\quad \alpha_1,\beta_1,\gamma_1 \in S(Q),\ x,y \in Q, \qquad (14)$$

be two isotopes of a group $(Q,\cdot)$ and let

$$(Q,\underset{1}{*}):\quad x \underset{1}{*} y = \alpha\gamma^{-1}x \cdot \beta\gamma^{-1}y,\quad \alpha,\beta,\gamma \in S(Q),\ x,y \in Q,$$
$$(Q,\underset{2}{*}):\quad x \underset{2}{\circ} y = \alpha_1\gamma_1^{-1}x \cdot \beta_1\gamma_1^{-1}y,\quad \alpha_1,\beta_1,\gamma_1 \in S(Q),\ x,y \in Q, \qquad (15)$$

be isotopes of $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$ respectively. By **Theorem 1** it follows that $\gamma(x \underset{1}{\circ} y) = \gamma x \underset{1}{*} \gamma y$ and $\gamma_1(x \underset{2}{\circ} y) = \gamma_1 x \underset{2}{*} \gamma_1 y$ hold for all $x,y \in Q$, e.g. γ is an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{1}{*})$, and γ_1 is an isomorphism between $(Q,\underset{2}{\circ})$ and $(Q,\underset{2}{*})$. If λ is an isomorphism between $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$, i.e. $\lambda(x \underset{1}{*} y) = \lambda x \underset{2}{*} \lambda y$ holds for all $x,y \in Q$, then

$$\gamma_1^{-1}\lambda\gamma(x \underset{1}{\circ} y) = \gamma_1^{-1}\lambda(\gamma x \underset{1}{*} \gamma y) = \gamma_1^{-1}(\lambda\gamma x \underset{2}{*} \lambda\gamma y) = \gamma_1^{-1}\lambda\gamma x \underset{2}{\circ} \gamma_1^{-1}\lambda\gamma y,$$

whence $\gamma_1^{-1}\lambda\gamma$ is an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$.

Conversely, let θ be an isomorphism between $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$, e.g. $\theta(x \underset{1}{\circ} y) = \theta x \underset{2}{\circ} \theta y$ holds for all $x, y \in Q$. Then for $\mu = \gamma_1\theta\gamma^{-1}$ we obtain

$$\mu(x \underset{1}{*} y) = \gamma_1\theta(\gamma^{-1}x \underset{1}{\circ} \gamma^{-1}y) = \gamma_1(\theta\gamma^{-1}x \underset{2}{\circ} \theta\gamma^{-1}y) = \gamma_1\theta\gamma^{-1}x \underset{2}{*} \gamma_1\theta\gamma^{-1}y.$$

Since x and y are arbitrary, from the last equality we conclude that μ is an isomorphism between $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$.

So, we have proved the next

Lemma 2. *A permutation $\theta \in S(Q)$ is an isomorphism between isotopes $(Q,\underset{1}{\circ})$ and $(Q,\underset{2}{\circ})$ defined by (14), if and only if there exists an isomorphism μ between quasigroups $(Q,\underset{1}{*})$ and $(Q,\underset{2}{*})$, defined by (15), such that $\theta = \gamma_1^{-1}\mu\gamma$.*

From **Lemma 2** and **Theorem 4** we obtain

Theorem 5. *A permutation $\theta \in S(Q)$ is an isomorphism between isotopes $(Q,*)$ and $(Q,\circ)$ defined by (14), if and only if there exist $s \in Q$ and $\theta_0 \in Aut(Q,\cdot)$ such that $\theta = \gamma_1^{-1}R_s\theta_0\gamma$ and relations*

$$\begin{aligned} \alpha\gamma^{-1} &= \theta_0^{-1}R_s^{-1}R_d\alpha_1\gamma_1^{-1}R_s\theta_0, \\ \beta\gamma^{-1} &= \theta_0^{-1}R_s^{-1}L_b\beta_1\gamma_1^{-1}R_s\theta_0, \\ (\alpha_1\gamma_1^{-1}R_s\theta_0 f)^{-1} &\cdot s \cdot (\beta_1\gamma_1^{-1}R_s\theta_0 e)^{-1} = 1, \end{aligned} \tag{16}$$

hold, with $d = (\alpha_1\gamma_1^{-1}R_s\theta_0 f)^{-1}\cdot s$, $b = s\cdot(\beta_1\gamma_1^{-1}R_s\theta_0 e)^{-1}$, $f = \gamma\alpha^{-1}1$, $e = \gamma\beta^{-1}1$, where 1 is the unit of the group $(Q,\cdot)$.

If in **Theorem 5** the quasigroup $(Q,*)$ is replaced by $(Q,\circ)$, then we obtain the following

Corollary 1. *A permutation* $\theta \in S(Q)$ *is an automorphism of the isotope* $(Q,\circ): x\circ y = \gamma(\alpha x \cdot \beta y)$, $\alpha,\beta,\gamma \in S(Q)$, $x,y \in Q$ *of a group* $(Q,\cdot)$ *if and only if there exist* $s \in Q$ *and* $\theta_0 \in Aut(Q,\cdot)$ *such that* $\theta = \gamma^{-1}R_s\theta_0\gamma$ *and relations (16) hold for* $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$.

We will adapt some of the above results for right loops principal isotopic to the same group.

Let $(Q,\cdot)$ be a group with the unit 1 and a principal isotope

$$(Q,*): x*y = \alpha x \cdot \beta y, \quad \alpha,\beta, \in S(Q), \quad x,y \in Q$$

be a right loop with the unit e. Then we have

$$x = x*e = \alpha x \cdot \beta e$$

for all $x \in Q$, from which it follows that

$$\alpha = R_{(\beta e)^{-1}}.$$

Therefore

$$x*y = R_{(\beta e)^{-1}}x \cdot \beta y = x(\beta e)^{-1}\beta y = x \cdot L_{(\beta e)^{-1}}\beta y.$$

Let us consider the isotope

$$(Q,\circ): x\circ y = x \cdot L_{(\beta e)^{-1}}\beta L_e y, \quad x,y \in Q.$$

We note that

$$L_{(\beta e)^{-1}}\beta L_e 1 = 1,$$

i.e. $(Q,\circ)$ is a right loop with the unit 1. The right loop $(Q,\circ)$ is isomorphic to the right loop $(Q,*)$, since

$$L_e(x\circ y) = L_e(x \cdot L_{(\beta e)^{-1}}\beta L_e y) = L_e x \cdot L_{(\beta e)^{-1}}\beta L_e y = L_e x * L_e y.$$

From this equality and **Theorem 1** we have

Proposition 1. *Every right loop which is isotopic to a group* $(Q,\cdot)$ *with the unit 1 is isomorphic to a right loop*

$$(Q,\circ): x\circ y = x\cdot \alpha y, \quad \alpha,\in S_1(Q), \quad x,y\in Q$$

with the same unit 1.

The following statement is a direct corollary from **Theorem 5.**

Proposition 2. *A permutation* $\theta\in S(Q)$ *is an isomorphism between right loops*

$$(Q,*):\quad x*y = x\cdot\alpha y,\ \alpha\in S_1(Q),\ x,y\in Q$$

and

$$(Q,\circ):\quad x\circ y = x\cdot\beta y,\ \beta\in S_1(Q),\ x,y\in Q$$

if and only if $\theta\in Aut(Q,\cdot)$ *and* $\alpha=\theta^{-1}\beta\theta.$

Corollary 2. *For a right loop*

$$(Q,*):\quad x*y = x\cdot\alpha y,\ \alpha\in S_1(Q),\ x,y\in Q,$$

which is isotopic to a group $(Q,\cdot)$ *with the unit* 1 *the equalities*

$$Aut(Q,*) = Z_{Aut(Q,\cdot)}\{\alpha\} = \{\varphi\in Aut(Q,\cdot)\mid \varphi\alpha=\alpha\varphi\}$$

hold.

Proof. The statement follows from **Proposition 2** replacing $(Q,\circ)$ by $(Q,*)$. □

Corollary 3. *For a right loop*

$$(Q,*):\quad x*y = x\cdot\alpha y,\ \alpha\in S_1(Q),\ x,y\in Q$$

which is isotopic to a group $(Q,\cdot)$ *with the unit* 1 *we have*

$$Aut(Q,*) = \{\varepsilon\},$$

provided that

$$\alpha \notin Z_{S_1(Q)}\{Aut(Q,\cdot)\} = \{\psi \in S_1(Q) \mid \psi\varphi = \varphi\psi,\ \varphi \in Aut(Q,\cdot)\}.$$

Proof. It follows directly from **Corollary 2.** □

Corollary 4. *For a right loop*

$$(Q,*):\quad x*y = x\cdot\alpha y,\ \alpha \in S_1(Q),\ x,y \in Q$$

which is isotopic to a group $(Q,\cdot)$ *with the unit* 1 *we have*

$$Aut(Q,*) = Aut(Q,\cdot),$$

provided that

$$\alpha \in Z_{S_1(Q)}\{Aut(Q,\cdot)\} = \{\psi \in S_1(Q) \mid \psi\varphi = \varphi\psi,\ \varphi \in Aut(Q,\cdot)\}.$$

Proof. It follows from **Corollary 2.** □

3. Automorphisms of a T-quasigroup

We will use **Corollary 1** to describe automorphisms of a T-quasigroup.

We recall that a T-quasigroup $(Q,*)$ is an isotope

$$x*y = \varphi x + \psi y + g = \varphi x + R_g\psi y \tag{17}$$

of an abelian group $(Q,+)$, where $\varphi,\psi \in Aut(Q,+)$, $g \in Q$, $R_g x = x + g$ [2].

Let 0 be the zero in $(Q,+)$. Then, by **Corollary 1** a permutation $\theta \in S(Q)$ is an automorphism of the quasigroup (17) if and only if there exist $s \in Q$ and $\theta_0 \in Aut(Q,+)$ such that $\theta = R_s\theta_0$ and relations

$$\varphi=\theta_0^{-1}R_s^{-1}R_d\varphi R_s\theta_0,$$
$$R_g\psi=\theta_0^{-1}R_s^{-1}L_bR_g\psi R_s\theta_0,$$
$$-\varphi s+s-R_g\psi R_s\theta_0\psi^{-1}(-g)=0, \tag{18}$$
$$b=s-R_g\psi R_s\theta_0\psi^{-1}(-g),$$
$$d=-\varphi s+s.$$

hold.

Here $\psi^{-1}(-g)$ is a solution of the equation $R_g\psi x=0$. Relations (18) can be simplified.

The first equality of (18) can be written as follows:

$$\varphi=\theta^{-1}R_s^{-1}R_d\varphi R_s\theta_0=\theta^{-1}R_{-s}R_dR_{\varphi s}\varphi\theta_0=$$
$$=\theta^{-1}R_{s-\varphi s+s+\varphi s}\varphi\theta_0=\theta^{-1}\varphi\theta_0.$$

Thus $\varphi\theta_0=\theta_0\varphi$.

Further we have

$$b=s-R_g\psi R_s\theta_0\psi^{-1}(-g)=s-R_g(\psi\theta_0\psi^{-1}(-g)+\psi s)=$$
$$=s+\psi\theta_0\psi^{-1}(-g)=\psi s-g,$$

$$R_g\psi=\theta_0^{-1}R_s^{-1}L_bR_g\psi R_s\theta_0=\theta_0^{-1}R_{-s}R_bR_gR_{\psi s}\psi\theta_0=$$
$$=\theta_0^{-1}R_{-s+b+g+\psi s}\psi\theta_0=\theta_0^{-1}R_{\psi\theta_0\psi^{-1}(g)}\psi\theta_0=\theta_0^{-1}\psi\theta_0R_{\psi^{-1}(g)}.$$

Therefore

$$\theta_0^{-1}\psi\theta_0=R_g\psi R^{-1}_{\psi^{-1}(g)}=R_g\psi R_{\psi^{-1}(-g)}=R_gR_{-g}\psi=\psi,$$

i.e. $\psi\theta_0=\theta_0\psi$. Consequently θ_0 is an element of the centralizer

$$C=Z_{Aut(Q,+)}\{\varphi,\psi\}$$

of automorphisms φ and ψ in the group $Aut(Q,+)$. Granting this we obtain

$$0=-\varphi s+s-R_g\psi R_s\theta_0\psi^{-1}(-g)=-\varphi s+s-R_gR_{\psi s}\psi\theta_0\psi^{-1}(-g)=$$
$$=-\varphi s+s+\theta_0g-\psi s-g,$$

from which the equality

$$\varphi s + \psi s - s = \theta_0 g - g$$

holds.

Conversely, for $\theta_0 \in C$ and $s \in Q$ such that

$$\varphi s + \psi s - s = \theta_0 g - g$$

we find

$$R_s\theta_0(x * y) = R_s\theta_0(\varphi x + \psi y + g) = R_s(\theta_0\varphi x + \theta_0\psi y + \theta_0 g) =$$
$$= \varphi\theta_0 x + \psi\theta_0 y + \theta_0 g + s = \varphi\theta_0 x + \psi\theta_0 y + \varphi s + \psi s + g =$$
$$= \varphi(\theta_0 x + s) + \psi(\theta_0 y + +s) + g = \varphi R_s\theta_0 x + \psi R_s\theta_0 y + g =$$
$$= R_s\theta_0 x * R_s\theta_0 y.$$

Thus we have proved the following

Proposition 3. *A permutation* $\theta \in S(Q)$ *is an automorphism of a T-quasigroup (17) if and only if there exist* $s \in Q$ *and*

$$\theta_0 \in C = Z_{Aut(Q,+)}\{\varphi, \psi\} = \{\alpha \in Aut(Q,+) \mid \alpha\varphi = \varphi\alpha;\ \alpha\psi = \psi\alpha\},$$

such that $\theta = R_s\theta_0$ *and* $\varphi s + \psi s - s = \theta_0 g - g$.

In some cases the automorphism group of the T-quasigroup (17) can be described.

Let us denote

$$N = \{s \in Q \mid \varphi s + \psi s - s = 0\};$$
$$R_N = \{R_n \mid n \in N\};$$
$$A_0 = \{\theta_0 \in C = Z_{Aut(Q,+)}\{\varphi, \psi\} \mid \theta_0 g = g\}.$$

It is easy to see that $(N,+)$ is a subgroup of $(Q,+)$. From **Proposition 3** it follows that A_0 and R_N are subgroups of $Aut(Q,*)$ and A_0 is a subgroup of $Aut(Q,+)$. Since

$$0 = \theta_0 0 = \theta_0(\varphi s + \psi s - s) = \varphi\theta_0 s + \psi\theta_0 s - \theta_0 s$$

and

$$0 = \theta_0^{-1} 0 = \theta_0^{-1}(\varphi s + \psi s - s) = \varphi\theta_0^{-1} s + \psi\theta_0^{-1} s - \theta_0^{-1} s$$

for $\theta_0 \in C$ and $s \in N$, then we get $\theta_0 N = N$, where

$$\theta_0 N = \{\theta_0 n \mid n \in N\}.$$

Let

$$R_s\theta_0 \in Aut(Q,*), \quad s \in Q, \quad \theta_0 \in Aut(Q,+), \quad n \in N.$$

Then

$$R_s\theta_0 R_n = R_s R_{\theta_0 n}\theta_0 = R_{\theta_0 n} R_s \theta_0.$$

So, $\alpha R_N \alpha^{-1} \subseteq R_N$ for every $\alpha \in Aut(Q,*)$, hence R_N is a normal subgroup of $Aut(Q,*)$.

It is evident that a permutation R_c is an automorphism of the group $(Q,+)$ if and only if $c = 0$, e.g. R_c is the identical permutation ε. Therefore $R_N \cap A_0 = \{\varepsilon\}$ and a semidirect product $R_N \times A_0$ is a subgroup of $Aut(Q,*)$. Also we have $Aut(Q,*) = R_N \times C$ if $g = 0$.

So we have proved the following

Proposition 4. *Let* $(Q,+)$ *be an abelian group with zero* 0, *and*

$$(Q,\otimes): \quad x \otimes y = \varphi x + \psi y, \quad \varphi, \psi \in Aut(Q,+), \quad x, y \in Q$$

be a T*-quasigroup. Then*

$$Aut(Q,\otimes) = R_N \times C,$$

where

$$C = Z_{Aut(Q,+)}\{\varphi, \psi\},$$
$$R_N = \{R_s \mid s \in Q, \ \varphi s + \psi s = 0\}.$$

We note that medial quasigroups that contain at least one idempotent [3] and transitive distributive quasigroups [4] satisfy **Proposition 4.**

4. Examples

In the following example the efficiency of **Theorem 5** and **Corollary 1** is visually demonstrated.

Let $(Q,\cdot)=\langle h\rangle$ be the infinite cyclic group which is generated by an element g, $\alpha=(h^{-1}h^{-2})$ be the transposition of elements h^{-1} and h^{-2} (i.e., $\alpha(h^{-1})=h^{-2}$, $\alpha(h^{-2})=h^{-1}$, $\alpha(x)=x$ forever $x\in Q$, $h^{-1}\neq x\neq h^{-2}$, $\alpha_1=(hh^2)$ be the transposition of elements h and h^2, I be the permutation of $(Q,\cdot)$ defined by $Ix=x^{-1}$ Consider isotopes

$$(Q,\circ):\quad x\circ y=\alpha_1 x\cdot Iy,\quad x,y\in Q,$$
$$(Q,*):\quad x*y=\alpha x\cdot Iy,\quad x,y\in Q.$$

We will prove that $(Q,*)$ and $(Q,\circ)$ are isomorphic.

Remark that in this case elements f and e from the **Theorem 5** are equal to the unit 1 of the group $(Q,\cdot)$, whence the third relation of (16) is equivalent to the equality

$$(\alpha_1 s)^{-1}s(Is)^{-1}=1.$$

Element 1 and h and only they satisfy the above equality. It is also known that $Aut(Q,\cdot)=\{\varepsilon,I\}$. By **Theorem 5** permutations $R_1\varepsilon=\varepsilon$, $R_1 I=I$, $R_h\varepsilon=R_h$, $R_h I$ and only they can be isomorphisms between $(Q,*)$ and $(Q,\circ)$. Let us verify the first two conditions of (16) for $s=1$ and $\theta_0=I$. We have $d=1$, $b=1$ and $I\alpha_1 I=\alpha$, $I=III$. The validity of the equality $I\alpha_1 I=\alpha$ is verified directly and the equality $I=III$ is trivial. Hence I is an isomorphism between $(Q,*)$ and $(Q,\circ)$.

Now we will find $Aut(Q,\circ)$. As above we can show that permutations $R_1\varepsilon=\varepsilon$, $R_1 I=I$, $R_h\varepsilon=R_h$, $R_h I$ and only they can be automorphisms of $(Q,\circ)$. The identical permutation is automorphism in every algebraic system. It is easy to see nevertheless that $R_1\varepsilon$ satisfies **Corollary 1.** As it was shown above $I\alpha_1 I=\alpha\neq\alpha_1$ then $I\notin Aut(Q,\circ)$.

For $R_h\varepsilon$ we obtain

$$b=h(Ih)^{-1}=hh=h^2,\quad d=(\alpha h)^{-1}h=h^{-2}h=h^{-1},\quad \alpha_1\neq R_h^{-1}R_h^{-1}\alpha_1 R_h$$

since

$$R_{h^{-1}}R_{h^{-1}}\alpha_1 R_h h=R_{h^{-1}}R_{h^{-1}}\alpha_1 h^2=R_{h^{-1}}R_{h^{-1}}h=$$
$$=R_{h^{-1}}1=h^{-1}\neq\alpha_1 h=h^2$$

So, $R_h\notin Aut(Q,\circ)$. For $R_h I$ we have

$$d=(\alpha_1 R_h I1)^{-1}\cdot h=(\alpha_1 h)^{-1}\cdot h=h^{-1}h=1$$

and

$$IR_h^{-1}R_{h^{-1}}\alpha_1 R_h I=IR_{h^{-2}}\alpha_1 R_h I\neq\alpha_1,$$

since

$$IR_{h^{-2}}\alpha_1 R_h Ih^2=IR_{h^{-1}}\alpha_1 R_h h^{-2}=IR_{h^{-2}}\alpha_1 h^{-1}=$$
$$=IR_{h^{-2}}h^{-1}=Ih^{-3}=h^3\neq h=\alpha_1 h^2,$$

i.e. $R_h I\notin Aut(Q,\circ)$. Therefore

$$Aut(Q,\circ)=\{\varepsilon\}.$$

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Izbash V.I. **Ph.D.**
department of quasigroup theory,
Institute of Mathematics,
Academy of Sciences of Moldova,
5, Academiei str.,
Kishinau, 277028,
MOLDOVA.

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