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On superassociative group isotopes

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Abstract

It is proved that every Menger quasigroup (grouplike Menger algebra) is an Ω-algebra. Relations between such algebraic notions as: homomorphism, subquasigroup, congruence relations and so on of a Menger quasigroup and the corresponding notions of its decomposition algebra are found. A criterion for the group isotope to be a superassociative is established. Some main algebraic notions in superassociative group isotopes are considered.

One of the well known generalization of the binary associativity is the superassociativity. It is an abstract characteristic of the class of all $(n+1)$-ary Menger algebras of n-ary transformations of a set. When $n = 1$, a Menger algebra is a semigroup of transformations and the superassociativity is the binary associativity (see [1]). In this connection works appear, where algebraic structure of Menger algebras and grouplike Menger algebras (i.e. superassociative quasigroups) are present. For example, the works of V.S. Trokhimenko [2], W.A.Dudek [3], Ya.N. Yaroker [5], H.L. Skala [6] and so on.

The main purpose of the article is to begin the study of superassotiative group isotopes, but it is necessary to consider some modification of the fundamental results of Menger algebras. We do it in sections **1** and **2**. The principal results of section 3 are: a canonical decomposition of an arbitrary group isotope and its uniqueness (**Lemma 3.1**); conditions for a group isotope to be superassociative and linear (**Theorem 3.2** and **Corollary 3.3**); conditions for a transformation of the basic set to be a homomorphism or an isomorphism of superassociative group isotopes of the same group (**Theorem 3.6**) as well as an endomorphism or an automorphism of a superassociative group isotope of a group (**Corollary 3.7**); criteria for a subset to be a subquasigroup and a normal subquasigroup of a superassociative group isotope (**Theorem 3.8**).

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1. General notes

A groupoid $(Q;f)$ of an arity $n+1$ is called *superassociative* or *Menger algebra of rang n*, if the *superassociative law* holds:

$$f(f(x,y_1,\dots,y_n),z_1,\dots,z_n)=f(x,f(y_1,z_1,\dots,z_n),\dots,f(y_n,z_1,\dots,z_n)) \quad (1)$$

Let $(Q;f)$ be a Menger algebra. A binary groupoid $(Q;\cdot)$, defined by

$$x\cdot y=f(x,y,\dots,y) \quad (2)$$

is associative. It follows from the equality (1), when $y_1=y_2=\dots=y_n=y$ and $z_1=z_2=\dots=z_n=z$. $(Q;\cdot)$ is called a *diagonal semigroup* of $(Q;f)$.

To expound the text we firstly recall (see [8]), that i-th shift defined by a of the *(n+1)*-ary groupoid $(Q;f)$ is a transformation $\lambda_{i,a}$ defined by

$$\lambda_{i,a}(x)=f(\underbrace{a,\ldots,a}_{i\ times},x,a,\ldots,a).$$

"Shift" is understood as a translation where all elements defining it are equel.

If the *i*-th shift defined by an element a is a substitution of the set Q, then the element a is called *i-invertible*, and if $\lambda_{i,a}(x)$ is an identity transformation, then it is called *i-unit* element of the groupoid $(Q;f)$; 0-invertible (0-unit) and n-invertible (n-unit) elements also will be called *right invertible (right unit) and left invertible (left unit)* respectively.

Lemma 1.1. *If a is a left (right) invertible element in a binary semigroup* $(Q;\cdot)$, *then the element* $e_l=\lambda_{0,a}^{-1}(a)$ $(e_r=\lambda_{1,a}^{-1}(a))$ *is its left (right) unit and the element* $a_l^{-1}=\lambda_{0,a}^{-2}(a)$ $(a_r^{-1}=\lambda_{1,a}^{-2}(a))$ *is a left (right) inverse of a.*

Proof. We shall prove the lemma for the "right" case only, since the proof of the "left" case is dual. The assotiative law implies the following equalities

$$\lambda_{0,a}(x\cdot y)=(x\cdot y)\cdot a=x\cdot(y\cdot a)=x\cdot\lambda_{0,a}(y).$$

Replacing x with $\lambda_{0,a}^{-1}(x)$ and applying $\lambda_{0,a}^{-1}(x)$ to the both side of this equality we have

$$\lambda_{0,a}^{-1}(x\cdot y)=x\cdot\lambda_{0,a}^{-1}(y).$$

Using this equality we can infer the following equalities

$$x \cdot e = x \cdot \lambda_{0,a}^{-1}(a) = \lambda_{0,a}^{-1}(x \cdot a) = \lambda_{0,a}^{-1}\lambda_{0,a}(x) = x$$

$$a \cdot a^{-1} = a \cdot \lambda_{0,a}^{-2}(a) = \lambda_{0,a}^{-1}(a \cdot \lambda_{0,a}^{-1}(a)) = \lambda_{0,a}^{-1}(a \cdot e) = \lambda_{0,a}^{-1}(a) = e$$

The lemma is proved.

Thus, the following assertion is evident.

Corollary 1.2. *A binary semigroup has a left (right) unit iff it has a left (right) invertible element. A binary semigroup has a unit iff it has an invertible element.*

It is easy to see that the right shift of a superassociative groupoid will be a right shift of its diagonal semigroup. The same is true for the right invertible and right unit elements. This permits to establish the truth of the following statement.

Corollary 1.3. *A superassociative groupoid has a right unit if and only if it has a right invertible element.*

We define n-ary operation $[...]$ on the set Q by

$$[x_1,\dots,x_n] \stackrel{def}{=} f(e,x_1,\dots,x_n), \qquad (3)$$

where e is a right unit of the superassociative groupoid $(Q;f)$. Using this relation it is easy to prove the following statement, which is a generalization of **Theorem 3.8** from [4].

Theorem 1.4. *If an $(n+1)$-ary superassociative groupoid $(Q;f)$ has at least one right invertible element a, then its diagonal semigroup operation $(\cdot)$ defined by (2), the operation $[...]$ defined by (3) with $e=\lambda_{0,a}^{-1}(a)$, and the operation f are connected by the following relations:*

$$f(x, z_1, \ldots, z_n) = x \cdot [z_1, \ldots, z_n], \tag{4}$$

$$[y_1, \ldots, y_n] \cdot z = [y_1 \cdot z, \ldots, y_n \cdot z]. \tag{5}$$

And conversely, if an associative operation $(\cdot)$ *is right distributive under some* *n-ary operation* $[\ldots]$, *then the operation* f *defined by (4) will be superassociative.*

In this case $(Q; \cdot, [\ldots])$ will be called a *decomposition algebra* of the superassociative groupoid $(Q; f)$.

Proof. By **Lemma 1.3**, the element e is a right unit of the diagonal semigroup, so the equalities (4) and (5) follow from the equalities (1) with $y_1 = y_2 = \ldots = y_n = e$ and $x = e$, $z_1 = z_2 = \ldots = z_n = z$ respectively. The converse statement is a partial case of **Lemma 3.7** from [4]. □

For example, if $(Q; +, \cdot)$ is a ring, then the ternary groupoid $(Q; f)$ defined by

$$f(x, y, z) = x \cdot (y + z)$$

is a Menger algebra. Some other examples can be found in [4].

This theorem implies immediately the following result.

Corollary 1.5. *Let a superassociative groupoid has a right invertible element* a, *then its diagonal semigroup has a left unit if and only if the operation* $[\ldots]$ *defined by the equality (3) with* $e = \lambda_{0,a}^{-1}(a)$ *is idempotent.*

Proof. Let the diagonal semigroup of the groupoid $(Q; f)$ has left unit, then it coincides with a right unit e of the semigroup. According to (2), the equality $e \cdot x = x$ is equivalent to the equality $f(e,x,\dots x)=x$, which, in turn, is equivalent to the equality $[x,\dots,x]=x$. The corollary is proved. □

Corollary 1.6. [7] *A superassociative groupoid* $(Q;f)$ *of the arity* $n+1$ *is a quasigroup if and only if there exist a group* $(Q;\cdot)$ *and an idempotent quasigroup* $(Q;[\dots])$ *such that the relations (4), (5) hold.*

Proof. In the quasigroup $(Q;f)$ every element is right invertible and, hence, according to **Theorem 1.4** the relations (4), (5) hold. Since the operation $[\dots]$ is defined by the equality (3), then it is a quasigroup as well. It remains to use the following statement, which is a corollary of **Theorem 1** from [11]. □

Proposition 1.7. *If one of functions* f, g, h *is a repetition-free superposition of two others and two of them are quasigroups, then the third one will be a quasigroup as well.*

A superassociative quasigroup is called *grouplike Menger algebra*. The following assertion is a corollary of **Lemma 1.1** and **Theorem 1.4**.

Corollary 1.8. *The diagonal semigroup of a superassociative groupoid is a group if and only if every element of the groupoid is right invertible and the diagonal semigroup has a left invertible element.*

To prove this corollary note that a semigroup is a group if it has a unit and every element has a right inverse.

Ya.N. Yaroker in [5] has found another criterion: a diagonal semigroup is a group if and only if the Menger algebra has no proper s- and v-ideal, that is iff the diagonal semigroup has no proper left and right ideals.

Corollary 1.9. *If in a superassociative groupoid the diagonal semigroup is a group, then its decomposition algebra is an Ω-group.*

The truth of the corollary follows directly from **Corollary 1.8** and the definition of an Ω-group: an algebra $(Q,+,\Omega)$ is called an *Ω-group*, if $(Q;+)$ is a group and $h(0,\dots,0)=0$ for all operations $h\in\Omega$.

A ring is a Ω-group as well, while the operation from $\Omega=\{\cdot\}$ is distributive under the group operation of the ring, but in the superassociative groupoid the situation is quite the reverse. From **Corollary 1.9** and the conclusions from [9] one can get a number of results for such Ω-groups.

2. On some algebraic notions

Let us consider a connection between the algebraic notions of a Menger quasigroup and its decomposition algebra.

Theorem 2.1. *A subset of a grouplike Menger algebra is a subquasigroup of it if and only if it is a subquasigroup of its decomposition algebra.*

Proof. Let $(Q;f)$ be a grouplike Menger algebra of an arity $n+1$ and let $(Q,\cdot,[\ldots])$ be its decomposition algebra.

If H is a subquasigroup of $(Q;f)$, then for arbitrary elements a, b from H $a\cdot b=f(a,b,\ldots b)\in H$ and $a^{-1}\in H$, since it is a solution of the equation

$$b=x\cdot a=f(x,a,\ldots,a).$$

So, the set H is a subgroup of the diagonal group $(Q;\cdot)$. In particular, it means that the unit e of the group is in the set H, and then for arbitrary elements $a_1,\ldots,a_n,b$ from the set H the results $[a_1,\ldots,a_n]=f(e,a_1,\ldots,a_n)\in H$ and a solution of the equation

$$b=[a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n]=f(e,a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$$

belongs to the set H. Thus, the set H is a subquasigroup of the algebra $(Q,\cdot,[\ldots])$.

Conversly, if a subset H of the set Q is a subquasigroup of the quasigroup algebra $(Q,\cdot,[\ldots])$, then the equality (4) implies that the subset H is closed under the operation f. The solution of the equation

$$b=f(x,a_1,\ldots,a_n)=x\cdot[a_1,\ldots,a_n]$$

belongs to the set H since H is a subgroup of the diagonal group. The equation

$$f(a_0,a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)=b$$

one can rewrite as

$$[a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n]=a_0^{-1}\cdot b,$$

then an element x exists, it is unique and belongs to H as soon as the elements $a_0,a_1,\ldots,a_n,b$ belong to the set H. The theorem is proved. □

Theorem 2.2. *A mapping φ from one grouplike algebra onto the other will be homomorphic (isomorphic) if and only if φ is a homomorphic (isomorphic) mapping between the corresponding decomposition algebras.*

Proof. Let $(Q;f)$, (G,h) be grouplike algebras and let $(Q,\cdot,[\ldots])$ and $(G,+,[\ldots])$ be their decomposition algebras, then

$$\varphi(x\cdot y)=\varphi f(x,y,\ldots y)=h(\varphi x,\varphi y,\ldots,\varphi y)=\varphi x+\varphi y;$$
$$\varphi([x_1,\ldots,x_n])=\varphi f(e,x_1,\ldots,x_n)=h(\varphi e,\varphi x_1,\ldots,\varphi x_n)=$$
$$=h(0,\varphi x_1,\ldots,\varphi x_n)=[\varphi x_1,\ldots,\varphi x_n].$$

The converse is evident.

Corollary 2.3. *Any endomorphism (automorphism) of a grouplike algebra is an endomorphism (automorphism) of the corresponding decomposition algebra and vice versa*

Recall, a congruence of a quasigroup is called *normal*, if the corresponding quotient-groupoid is a quasigroup also; a subquasigroup is called *normal*, if it is a class by a normal congruence.

In a group every congruence is normal and exactly one of its classes is a normal subgroup, namely, the class containing the unit element of the group. In a quasigroup this is not so. There exist infinite quasigroups having non normal congruences and there exist quasigroups having congruence with no subquasigroup as its class. In an idempotent quasigroup every class by a normal congruence is a subquasigroup. There is the same situation in the theory of n-ary groups and polygroups as in the theory of quasigroups. The following resoning shows, that in a grouplike Menger algebra the uniqueness is the same as in the group theory. Immediatly from **Theorem 2.2** we have the truth of the following statements.

Corollary 2.4. *Any normal congruence of a grouplike algebra is a normal congruence of the corresponding decomposition algebra and vice versa.*

Corollary 2.5. *Exactly one of congruence classes of a grouplike algebra is a subquasigroup, namely, the class containing the unit element of the diagonal group.*

Corollary 2.6. *Every normal congruence of a Menger quasigroup is an equivalence relation corresponding to a partition by the normal subgroups of the diagonal group, which are normal subquasigroups of the decomposition algebra.*

A full description of all congruences (including one-side congruences) in grouplike Menger algebras was obtained by V.S.Trokhimenko in [2].

3. Superassociative group isotopes

In this section superassociative group isotopes are under consideration.

A group isotope or an *isotope of a group* $(G;\bullet)$ of the arity $n+1$ is a groupoid $(Q;f)$ defined by

$$f(x_0,x_1,\dots,x_n)=\gamma^{-1}(\gamma_0x_0\bullet\gamma_1x_1\bullet\dots\bullet\gamma_nx_n), \tag{6}$$

where $\gamma_0,\dots,\gamma_n,\gamma$ are bijections between the sets Q and G. If all of the bijections are linear transformations of the group $(G;\circ)$ (α is *linear*, iff $\alpha x = \theta x + a$ for some $a \in G$ and an automorphism θ of the group $(G;\circ)$), then the isotope $(G;f)$ is called *linear*. It is easy to prove that a groupoid being isomorphic to a linear group isotope is a linear group isotope as well. The following statement is true.

Lemma 3.1. *Let $(Q;f)$ be an $(n+1)$-ary isotope of a group, then for any element e of the set Q there exists exactly one sequence of operations $(\cdot,\alpha_0,\dots,\alpha_n,a)$ such that $(Q;\cdot)$ is a group with a unit element e; $\alpha_0,\dots,\alpha_n$ are unitary substitutions of the set Q, i.e. $\alpha_i(e)=e$, $i=0,1,\dots,n$; $a \in Q$ and the following equality*

$$f(x_0,x_1,\dots,x_n)=\alpha_0 x_0\cdot\alpha_1 x_1\cdot\ldots\cdot\alpha_n x_n\cdot a \tag{7}$$

holds. The isotope $(Q;f)$ is linear iff $\alpha_0,\dots,\alpha_n$ are aumorphisms of $(Q;\cdot)$.

In this case, let us use the following terminology: a decomposition (7) will be called *e-canonical*, permutations $\alpha_0,\dots,\alpha_n$ will be called *decomposition coefficients*, and the element a will be called a *free member.*

Proof. Let (6) holds. On the set Q we define a group operation $(+)$:

$$x+y=\gamma^{-1}(\gamma x\bullet\gamma y)$$

Then we rewrite the equality (6) as follows:

$$f(x_0,\dots,x_n)=\gamma^{-1}\gamma_0x_0+\gamma^{-1}\gamma_1x_1+\ldots+\gamma^{-1}\gamma_nx_n).$$

Replacing $(+)$ with $(\cdot)$, where $x\cdot y=x-e+y$, i.e. $x+y=(x+e)\cdot y$, and $\gamma^{-1}\gamma x_n$ by α_n, $\gamma^{-1}\gamma x_iR_e$ by α_i, $i=0,1,\dots,n-1$, we have

$$f(x_0,x_1,\dots,x_n)=\alpha_0 x_0\cdot\alpha_1 x_1\cdot\ldots\cdot\alpha_n x_n. \qquad (8)$$

It is clear, the operations $(\cdot)$ and $(+)$ are isomorphic and the unit element in $(Q;\cdot)$ is e. Suppose, that $\alpha_0,\alpha_1,\dots,\alpha_{i-1}$ are unitary and $\alpha_i e\neq e$, $i=0,1,\dots,n$, then we consider the following notation

$$\alpha'_i x_i=\alpha_i x_i\cdot(\alpha_i e)^{-1}, \qquad \alpha'_{i+1} x_{i+1}=(\alpha_i e)\cdot(\alpha_{i+1}x_{i+1}).$$

As a result we have the relation

$$f(x_0,x_1,\dots,x_n)=\alpha_0 x_0\cdot\ldots\cdot\alpha_{i-1}x_{i-1}\cdot\alpha'_i x_i\cdot\alpha'_{i+1}x_{i+1}\cdot\alpha_{i+2}x_{i+2}\cdot\ldots\cdot\alpha_n x_n,$$

where the first substitutions $\alpha_0,\dots,\alpha_{i-1},\alpha'_i$ are unitary. After the finite number of the steps we obtain the equality (8), where the substitutions $\alpha_0,\dots,\alpha_{n-1}$ are unitary. Consider a new notation:

$$\alpha'_n x_n=\alpha_n x_n\cdot(\alpha_n e)^{-1}, \qquad \text{and} \quad a=\alpha_n e$$

Thus, we obtain the equality (7). To prove the uniqueness we assume that the decomposition

$$f(x_0,x_1,\dots,x_n)=\beta_0 x_0\oplus\beta_1 x_1\oplus\ldots\oplus\beta_n x_n\oplus b$$

is e-canonical as well, i.e. $e\oplus x=x\oplus e=x$, $\beta_0 e=\beta_1 e=\ldots=\beta_n e=e$. Then $a=f(e,\dots,e)=b$ and

$$\alpha_n x_n\cdot a=f(e,\dots,e,x_n)=\beta_n x_n\oplus b.$$

So, we obtain

$$\alpha_0 x_0\cdot\alpha_n x_n\cdot a=f(x_0,e,\dots,e,x_n)=\beta_0 x_0\oplus\beta_n x_n\oplus b=\beta_0 x_0\oplus(\alpha_n x_n\cdot a)$$

for all $x,y\in Q$. Replacing $\alpha_n x_n\cdot a$ with y we have

$$\alpha_0 x_0\cdot y=\beta_0 x_0\oplus y.$$

Since the element e is a common unit of the operations $(\cdot)$ and $(\oplus)$, then from the last equality and $y=e$ we have $\alpha_0=\beta_0$, and so the operations $(\cdot)$ and $(\oplus)$ coincide. Hence,

$$\alpha_i x\cdot a\cdot a^{-1}=f(e,\dots,x,e,\dots,e)\cdot a^{-1}=(\beta_i x\oplus b)\cdot a^{-1}=\beta_i x\cdot a\cdot a^{-1}=\beta_i x$$

for all $x\in Q$, then $\alpha_0=\beta_0$ for all $i=0,1,\dots,n$. **Lemma** is proved.

Using this lemma we may establish the truth of the following

Theorem 3.2. *A group isotope* (Q,f) *is superassociative if and only if*

$$f(x_0,x_1,\ldots,x_n)=x_0\cdot\alpha_1x_1\cdot\ldots\cdot\alpha_nx_n \tag{9}$$

for some unitary substitutions $\alpha_1,\ldots,\alpha_n$ *of the group* $(Q,\cdot)$, *which satisfy the following conditions:*

$$\alpha_1y\cdot\alpha_2y\cdot\ldots\cdot\alpha_ny=y, \tag{10}$$

$$\alpha_1y\cdot\ldots\cdot\alpha_{i-1}y\cdot\alpha_i(x\cdot y)=\alpha_ix\cdot\alpha_1y\cdot\ldots\cdot\alpha_iy,\quad i=1,\ldots,n. \tag{11}$$

Proof. Let a superassociative quasigroup (Q,f) of arity $n+1$ be an isotope of some group and let (7) be its e-canonical decomposition, where e is a unit of its diagonal group. From the uniqueness of the e-canonical decomposition of the group isotope $(Q;+)$, from the equalities (4), (7) and the idempotence of the operation [...] the relations $\alpha_0=e$, $(\bullet)=(\cdot)$ and

$$[y_1,\ldots,y_n]=\alpha_1y_1\cdot\alpha_2y_2\cdot\ldots\cdot\alpha_ny_n, \tag{12}$$

follow, since

$$e\overset{Corol.1.5}{=}[e,\ldots,e]\overset{(3)}{=}f(e,e,\ldots,e)\overset{Lemmma\ 3.1}{=}a.$$

The idempotence of [...] is equivalent to the identity (10). Thus, the distributivity relation (5) will be rewritten as follows:

$$\alpha_1x_1\cdot\alpha_2x_2\cdot\ldots\cdot\alpha_nx_n\cdot y=\alpha_1(x_1\cdot y)\cdot\alpha_2(x_2\cdot y)\cdot\ldots\cdot\alpha_n(x_n\cdot y).$$

Setting $x_j=e$ for all $j\neq i$ in the last equality we obtain

$$\alpha_ix\cdot y=\alpha_1(y)\cdot\ldots\cdot\alpha_{i-1}(y)\cdot\alpha_i(x\cdot y)\cdot\alpha_{i+1}(y)\cdot\ldots\cdot\alpha_n(y)$$

In the left side of it we replace y with its value from the equality (10), and then we cancel it out of $\alpha_{i+1}y\cdot\ldots\cdot\alpha_ny$. As a result we obtain the relation (11).

Converse, let the operation f on the set Q be defined by the equality (9) by such unitary substitutions $\alpha_0,\dots,\alpha_n$ of the group $(Q,\cdot)$, that the relations (10) and (11) hold. To prove the superassociativity of the operation f we defined an operation $[\dots]$ by (12). Hence, we obtain the following equalities:

$$[x_1,\dots,x_n]\cdot y =$$
$$\overset{(10)}{=} \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_n x_n\cdot y \overset{(11)}{=}$$
$$= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_n x_n\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_n y \overset{(11)}{=}$$
$$= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_{n-1}x_{n-1}\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_{n-1}y\cdot\alpha_n(x_n\cdot y) \overset{(11)}{=}$$
$$= \alpha_1 x_1\cdot\alpha_2 x_2\cdot\ldots\alpha_{n-2}x_{n-2}\cdot\alpha_1 y\cdot\alpha_2 y\cdot\ldots\alpha_{n-2}y\cdot\alpha_{n-1}(x_{n-1}\cdot y)\cdot\alpha_n(x_n\cdot y) =$$
$$\overset{(11)}{=}\ldots\overset{(11)}{=} \alpha_1 x_1\cdot\alpha_1 y\cdot\alpha_2(x_2\cdot y)\cdot\ldots\cdot\alpha_n(x_n\cdot y) =$$
$$= \alpha_1(x_1\cdot y)\cdot\alpha_2(x_2\cdot y)\cdot\ldots\cdot\alpha_n(x_n\cdot y) \overset{(12)}{=}$$
$$= [x_1\cdot y,\dots,x_n\cdot y].$$

By **Theorem 1.4** the triple $(Q;\cdot,[\dots])$ is a decomposition of the superassociative groupoid (Q,f), which is a quasigroup, since the operation f is a repetition-free superposition of the quasigroup operations $(\cdot)$ and $[\dots]$ (see **Proposition 1.7**). **Theorem** is proved. □

Corollary 3.3. *For any superassociative group isotope (Q,f) with a canonical decomposition (9) the following statements are equivalent:*

1) *the isotope (Q,f) is linear;*
2) *α_2 is an automorphism of the group $(Q,\cdot)$;*
3) *the group $(Q,\cdot)$ is commutative.*

Proof. The relations (11) with $i = 2$ give the following equality:

$$\alpha_1 y \cdot \alpha_2 (x \cdot y) = \alpha_2 x \cdot \alpha_1 y \cdot \alpha_2 y,$$

which implies that 2) and 3) are equivalent. If, in addition, the group $(Q;\cdot)$ is commutative, then the relations (11) mean that every of the substitutions α_i is an automorphism of the group.

Corollary 3.4. *An $(n+1)$-ary isotope (Q, f) of an abelian group will be superassociative if and only if there exist an abelian group $(Q;+)$ and a sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of its automorphisms such that the following equalities are true*

$$f(x_0, x_1, \ldots, x_n) = x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n, \quad \alpha_1 + \ldots + \alpha_n = \varepsilon$$

By **Theorem 2.2**, an isomorphism of grouplike algebras implies an isomorphism of its diagonal groups. It is easy to prove that the converse is true as well.

Proposition 3.5. *If groups are isomorphic, then there exists a bijection between the sets of superassociative isotopes of this groups such that the corresponding isotopes are isomorphic.*

Hence, from here on it suffices to consider superassociative isotopes of the same arbitrary fixed group $(Q;\cdot)$. An isomorphism criterion for them is in the following

Theorem 3.6. *A transformation θ of the set Q is a homomorphism (isomorphism) of a superassociative isotope (Q, f) in a superassociative isotope (Q, h) of a group $(Q;\cdot)$ with* *the canonical decompositions* $(\varepsilon,\alpha_1,\dots,\alpha_n)$ *and* $(\varepsilon,\beta_1,\dots,\beta_n)$ *respectively if and only if* θ *is an endomorphism (automorphism) of the group* $(Q,\cdot)$ *and the following relations are true*

$$\theta\beta_i = \alpha_i\theta, \quad i = 1,2,\dots,n.$$

Proof. From **Theorem 2.2** and the relation (12) it follows that the group isotopes are homomorphic (isomorphic) if and only if the transformation θ is an endomorphism (automorphism) of the diagonal group $(Q,\cdot)$ and the following relation fulfils:

$$\theta(\beta_1x_1\cdot\beta_2x_2\cdot\ldots\cdot\beta_nx_n) = \alpha_1\theta x_1\cdot\alpha_2\theta x_2\cdot\ldots\cdot\alpha_n\theta x_n$$

for all $x_1,x_2,\dots,x_n \in Q$. In particular, when $x_j = e$ for all $j \neq i$, we obtain the necessary relations. The converse is evident. □

Corollary 3.7. *A transformation* θ *of a set* Q *is an endomorphism (automorphism) of a superassociative isotope* (Q,f) *with a canonical decomposition (9) if and only if* θ *is an endomorphism (automorphism) of the diagonal group* $(Q,\cdot)$ *and commutes with every coefficient of the canonical decomposition.*

In the following assertion we shall describe the subquasigroups of the superassociative group isotopes.

Theorem 3.8. *A subset of a superassociative group isotope is a subquasigroup of it if and only if it is a subgroup of the diagonal group and invariant under all components of its canonical decomposition.*

Proof. Let (Q,f) be a superassociative isotope and (9) be its canonical decomposition. After **Theorem 2.1** it remains to elucidate which conditions are necessary for a subgroup H of the diagonal group to be a subquasigroup of the quasigroup $(Q;[...])$, where (12) defines the operation $[...]$. Hence, the subgroup H is a subquasigroup of $(Q;[...])$ iff in the equality

$$\alpha_1 x_1 \cdot \alpha_2 x_2 \cdot ... \cdot \alpha_n x_n = h$$

any n elements from the set H uniquely determine the $(n+1)$-th element which is in the set H as well. The first part of this assertion is fulfilled, since $(Q;[...])$ is a quasigroup. It remains to show the belonging of this element to the set H. From the last equality with $h_j = e$ for all $j \neq i$ we have the following statement: in the equality $\alpha_j h_j = h$ the elements h_j and h belong to the set H simultaneously, that is $\alpha_i H = H$. The reverse is obvious. The **theorem** is proved. □

Corollary 3.9. *A subset of a superassociative group isotope is its normal subquasigroup if and only if it is a normal subgroup of the diagonal group and it is invariant under all components of its canonical decomposition.*

Comparing the results of the article [12] with the assertions given here we have the following properties for superassociative isotopes of the cyclic groups:

1) *superassociative cyclic group isotopes are nonisomorphic, if their canonical decomposition groups coincide and the corresponding sequences of coefficients are different;*

2) *endomorphisms, automorphisms, subquasigroups, normal subquasigroups, congruences of a superassociative isotope of a cyclic group are the same as in the group;*

3) *there exist exactly* $\left((p-1)^n+(-1)^n(p-1)\right)/p$ *of* $(n+1)$*-ary* $(n>1)$ *superassociative group isotopes of prime order* p *up to isomorphism.*

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