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On n-groupoids in which all transformations are endomorphisms

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Abstract. For an n-ary groupoid we find the neccesity and sufficient conditions under which all its transformations are endomorphisms.

It is known [1] that a semigroup in which each transformation is an endomorphism, is a left or right zero semigroup. Below we generalize this result to the case of n-ary groupoids.

Let (G, o) be an *n-ary groupoid*, i.e., a nonempty set G with an n-ary operation o. Such groupoid is also called an *n-groupoid* (cf. [2]). An element $0 \in G$ is called a *k-zero*, where $k \in \{1, 2, \ldots, n\}$, of an n-groupoid (G, o), if

$$o(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) = 0$$

holds for all $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in G$. An n-groupoid in which each element is a k-zero is called an *n-groupoid of k-zeros* or a *k-zero n-groupoid.* Following [3], an n-groupoid (G, o) in which $o(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for any $x_1, \ldots, x_n \in G$ is called *quasitrivial.*

Lemma 1. *An n-groupoid in which each transformation is an endomorphism is quasitrivial.*

Proof. Let φ_x be a transformation of an n-groupoid (G, o) such that $\varphi_x(z) = x$ for every $z \in G$. Since, by the assumption, φ_x is an endomorphism for each $x \in G$, we have

$$x = \varphi_x(o(x, \ldots, x)) = o(\varphi_x(x), \ldots, \varphi_x(x)) = o(x, \ldots, x).$$

So, every element of (G, o) is an idempotent.

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Suppose that $o(x_1,\ldots,x_n) \notin \{x_1,\ldots,x_n\}$ for some $x_1,\ldots,x_n \in G$. Consider a transformation φ of (G,o) defined by

$$\varphi(z) = \begin{cases} x_1, & \text{if } z \in \{x_1,\ldots,x_n\}, \\ z & \text{if } z \notin \{x_1,\ldots,x_n\}. \end{cases}$$

Since φ is an endomorphism we have

$$\varphi(o(x_1,\ldots,x_n)) = o(\varphi(x_1),\ldots,\varphi(x_n)) = o(x_1,\ldots,x_1) = x_1.$$

But $o(x_1,\ldots,x_n) \notin \{x_1,\ldots,x_n\}$, hence

$$\varphi(o(x_1,\ldots,x_n)) = o(x_1,\ldots,x_n).$$

Thus, $o(x_1,\ldots,x_n) = x_1$, which is a contradiction. So, an n-groupoid (G,o) is quasitrivial. $\square$

By E_k we denote the set of all equivalence relations defined on the set $\{1,2,\ldots,n\}$ having exactly k equivalence classes, where $1 \leqslant k \leqslant \min(|G|,n)$. Let $E = \bigcup\limits_{k=1}^{m} E_k$, where $m = \min(|G|,n)$. For every $\varepsilon \in E$ by H_ε we denote the set of all such n-tuples $(x_1,\ldots,x_n) \in G^n$ for which the equality $x_i = x_j$ holds if and only if $i \equiv j(\varepsilon)$, $i,j \in \{1,2,\ldots,n\}$.

Theorem 1. *Each transformation of an n-groupoid (G,o) is its endomorphism if and only if (G,o) is quasitrivial and for any $\varepsilon_1,\varepsilon_2 \in E$, where $\varepsilon_1 \subset \varepsilon_2$, there exist $i \in \{1,2,\ldots,n\}$ such that the implication*

$$o(x_1,\ldots,x_n) = x_i \longrightarrow o(y_1,\ldots,y_n) = y_i \tag{1}$$

is valid for all $(x_1,\ldots,x_n) \in H_{\varepsilon_1}$ and $(y_1,\ldots,y_n) \in H_{\varepsilon_2}$.

Proof. Let any transformation of an n-groupoid (G,o) be its endomorphism. Then, according to Lemma 1, an n-groupoid (G,o) is quasitrivial. Hence $o(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$. Consider two equivalence relations $\varepsilon_1,\varepsilon_2 \in E$ such that $\varepsilon_1 \subset \varepsilon_2$ and $(x_1,\ldots,x_n) \in H_{\varepsilon_1}$. Suppose that $(y_1,\ldots,y_n) \in H_{\varepsilon_2}$ for some $y_1,\ldots,y_n \in G$. Since each transformation of (G,o) is an endomorphism, an endomorphism is also the transformation ψ defined by $\psi(x_k) = y_k$ for $x_k \in \{x_1,\ldots,x_n\}$ and $\psi(z) = z$ for $z \notin \{x_1,\ldots,x_n\}$. Thus,

$$y_i = \psi(x_i) = \psi(o(x_1,\ldots,x_n)) = o(\psi(x_1),\ldots,\psi(x_n)) = o(y_1,\ldots,y_n).$$

So, the condition (1) is satisfied.

Conversely, let (G, o) be an n-groupoid satisfying all conditions of the above theorem. Then, obviously, for arbitrary $x_1, \dots, x_n \in G$ there exist $\varepsilon_1 \in E$ such that $(x_1, \dots, x_n) \in H_{\varepsilon_1}$. Since (G, o) is quasitrivial, we have $o(x_1, \dots, x_n) = x_i$ for some $i \in \{1, 2, \dots, n\}$. Therefore,

$$\varphi(o(x_1, \dots, x_n)) = \varphi(x_i) \tag{2}$$

for each transformation φ of (G, o). Let $(\varphi(x_1), \dots, \varphi(x_n)) \in H_{\varepsilon_2}$, where $\varepsilon_2 \in E$. Then $\varepsilon_1 \subset \varepsilon_2$. Thus, $o(x_1, \dots, x_n) = x_i$, by (1), implies

$$o(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(x_i). \tag{3}$$

From (2) and (3) we obtain $\varphi(o(x_1, \dots, x_n)) = o(\varphi(x_1), \dots, \varphi(x_n))$. This means that φ is an endomorphism. $\square$

Corollary 1. *If $|G| \geqslant n$, then each transformation of an n-groupoid (G, o) is its endomorphism if and only if (G, o) is a k-zero n-groupoid (for some $k \in \{1, 2, \dots, n\}$).*

Proof. Let (G, o) be a k-zero n-groupoid. Then $o(x_1, \dots, x_n) = x_k$ and $\varphi(o(x_1, \dots, x_n)) = \varphi(x_k)$ for any transformation φ of G. On the other hand $o(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(x_k)$. So, $\varphi(o(x_1, \dots, x_n)) = o(\varphi(x_1), \dots, \varphi(x_n))$, which means that φ is an endomorphism of (G, o).

Conversely, let each transformation of an n-groupoid (G, o) be its endomorphism. Then (G, o) satisfies all conditions of Theorem 1. Because $|G| \geqslant n$ there exists n-tuple $(g_1, \dots, g_n)$ of pairwise different elements from G. Moreover, in this case $m = \min(|G|, n) = n$, $E_n = \{\triangle\}$ and $(g_1, \dots, g_n) \in H_{\triangle}$, where $\triangle$ denotes the identity binary relation on the set $\{1, \dots, n\}$. By quasitriviality we have $o(g_1, \dots, g_n) = g_k$ for some $k \in \{1, \dots, n\}$. Let $(x_1, \dots, x_n)$ be an arbitrary n-tuple from G^n and let ε be an equivalence relation from E such that $(x_1, \dots, x_n) \in H_{\varepsilon}$. Since $\triangle \subset \varepsilon$, from $o(g_1, \dots, g_n) = g_k$, by (1), we obtain $o(x_1, \dots, x_n) = x_k$. So, (G, o) is a k-zero n-groupoid. $\square$

Corollary 2. *If all transformations of a binary groupoid are its endomorphisms, then this groupoid is a left or right zero groupoid.*

Proof. For $|G| = 1$ it is obvious. For $|G| \geqslant 2$, by Corollary 1, this groupoid is a k-zero groupoid for some $k \in \{1, 2\}$. So, it is either a left or right zero groupoid. $\square$

Note that each left (right) zero groupoid is a semigroup. Thus Corollary 2 is also valid for semigroups.

Theorem 2. *Each transformation of an n-groupoid (G, o) with $1 \leqslant |G| < n$ is its endomorphism if and only if for arbitrary $\varepsilon_1, \varepsilon_2 \in E$, where $\varepsilon_1 \subset \varepsilon_2$, there exist $i \in \{1, 2, \ldots, n\}$ such that* (1) *is true for all $(x_1, \ldots, x_n) \in H_{\varepsilon_1}$ and $(y_1, \ldots, y_n) \in H_{\varepsilon_2}$.*

Proof. In view of Theorem 1 it is enough to show that an n-groupoid (G, o) satisfying all conditions of Theorem 2 is quasitrivial.

Since $1 \leqslant |G| < n$, we have $m = \min(|G|, n) = |G|$. Let $x_1, \ldots, x_n \in G$ and $\varepsilon \in E$ be such that $(x_1, \ldots, x_n) \in H_\varepsilon$. Since the set E is finite, it has minimal elements. Clearly, all minimal elements of E belong to E_m. From elements of E_m we can choose ε_0 such that $\varepsilon_0 \subset \varepsilon$. Now let $(g_1, \ldots, g_n) \in H_{\varepsilon_0}$. Then $\{g_1, \ldots, g_n\} = G$ because $|G| = m < n$. Therefore $o(g_1, \ldots, g_n) \in \{g_1, \ldots, g_n\}$, which according to (1), implies $o(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. So, an n-groupoid (G, o) is quasitrivial. Hence, by Theorem 1, all transformations of this n-groupoid are endomorphisms. So, the necessity of the above conditions is proved.

The proof of the sufficiency of these conditions is based on Lemma 1 and is analogous to the corresponding part of the proof of Theorem 1. $\square$

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