A characterization of binary invertible algebras linear over a group

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Abstract. In this paper we define linear over a group and an abelian group binary invertible algebras and characterize the class of such algebras by second-order formulae, namely the $\forall\exists(\forall)$-identities.

1. Introduction

A quasigroup, $(Q;\cdot)$, of the form,

$$xy = \varphi x + a + \psi y,$$

where $(Q;+)$ is a group, φ, ψ are automorphisms (antiautomorphisms) of $(Q;+)$, and a is a fixed element of Q, is called *linear* (*alinear*) *quasigroup* over the group, $(Q;+)$, [2, 6].

All primitive linear (alinear) quasigroups form a variety [6].

A linear quasigroup over an abelian group is called a T*-quasigroup* [10]. An important subclass of the T-quasigroups is the class of medial quasigroups. A quasigroup $(Q;\cdot)$ is called *medial*, if the following identity holds: $xy \cdot uv = xu \cdot yv$. Any medial quasigroup is a T-quasigroup by Toyoda theorem, [3] – [8], with the condition, $\varphi\psi = \psi\varphi$.

Medial quasigroups have been studied by many authors, namely R.H. Bruck [8], T. Kepka, P. Nemec and J. Ježek [9]-[11], D.S. Murdoch [16], A.B. Romanowska and J.D.H. Smith [17], K. Toyoda [21] and others and this class plays a special role in the theory of quasigroups. T-quasigroups were introduced by T. Kepka and P. Nemec [10, 11]. Later G.B. Belyavskaya characterized the class of T-quasigroups by a system of two identities [5, 7].

A binary algebra $(Q;\Sigma)$ is called *invertible*, if $(Q;A)$ is a quasigroup for any operation, $A \in \Sigma$. The invertible algebras first were considered by

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R. Schauffler in touch with coding theory [19, 20]. Later such algebras were investigated by J. Aczel [1], V.D. Belousov [2, 3], Yu.M. Movsisyan [12] – [15], A. Sade [18] and others.

By analogy with linear (alinear) quasigroups we introduce the notion of a linear (alinear) invertible algebra.

Definition 1.1. An invertible algebra $(Q;\Sigma)$ is called *linear* (*alinear*) over the group $(Q;+)$ if every operation $A\in\Sigma$ has the form:

$$A(x,y)=\varphi_A x+t_A+\psi_A y, \tag{1}$$

where φ_A, ψ_A are automorphisms (antiautomorphisms) of $(Q;+)$ for all $A\in\Sigma$, and t_A are fixed elements of Q.

A linear invertible algebra over an abelian group is called an *invertible T-algebra.*

Let us recall, that the following absolutely closed second-order formulae:

$$\forall X_1,\dots,X_m\forall x_1,\dots,x_n\quad(\omega_1=\omega_2),$$
$$\forall X_1,\dots,X_k\exists X_{k+1}\dots,X_m\forall x_1,\dots,x_n\quad(\omega_1=\omega_2),$$

where ω_1,ω_2 are words (terms) written in the functional variables $X_1,...,X_m$, and in the objective variables, $x_1,\dots,x_n$, are called $\forall(\forall)$-*identity* or *hyperidentity* and $\forall\exists(\forall)$-*identity.* The satisfiability (truth) of these second order formulae in the algebra $(Q;\Sigma)$ is understood in the sense of functional quantifiers, $(\forall X_i)$ and $(\exists X_j)$, meaning: "for every value $X_i=A\in\Sigma$ of the corresponding arity" and "there exists a value $X_j=A\in\Sigma$ of the corresponding arity". It is assumed that such a replacement is possible, that is:

$$\{|X_1|,\dots,|X_m|\}\subseteq\{|A|\mid A\in\Sigma\},$$

where $|S|$ is the arity of S. Generally, hyperidentities are written without a quantifier prefix: $\omega_1=\omega_2$. For details about such formulae see [12] – [15].

The binary algebra, $(Q;\Sigma)$, is called *medial* (*abelian*) if the following hyperidentity holds:

$$X(Y(x,y),Y(u,v))=Y(X(x,u),X(y,v)).$$

Yu.M. Movsisyan proved that medial invertible algebras are a special class of invertible T-algebras, namely all automorphisms of the group $(Q;+)$,

which correspond the operations from Σ are permutable:
$\varphi_A \cdot \varphi_B = \varphi_B \cdot \varphi_A$, $\psi_A \cdot \psi_B = \psi_B \cdot \psi_A$, $\varphi_A \cdot \psi_B = \psi_B \varphi_A$ for all $A, B \in \Sigma$.

In the present paper we characterize the class of invertible linear (alinear) algebras and the class of invertible T-algebras by second-order formulae, namely, $\forall\exists(\forall)$-identities. For proofs of these results we use the methods of the papers, [6, 5].

2. Linear and alinear invertible algebras

We denote by $L_{A,a}$ and $R_{A,a}$ the left and right translations of the binary algebra $(Q;\Sigma)$: $L_{A,a} : x \mapsto A(a,x)$, $R_{A,a} : x \mapsto A(x,a)$. If the algebra $(Q;\Sigma)$ is an invertible algebra, then the translations, $L_{A,a}$ and $R_{A,a}$ are bijections for all $a \in Q$ and all $A \in \Sigma$.

The unique solution of the equality $B(a,x) = a$ ($B(x,a) = a$) is denoted by e_a^B (f_a^B), i.e., e_a^B (f_a^B) is the right (left) local identity of the element a with respect to the operation B.

It is well known [3] that with each quasigroup A the next five quasigroups are connected:

$$A^{-1},\ \ {}^{-1}A,\ \ {}^{-1}(A^{-1}),\ \ ({}^{-1}A)^{-1},\ \ A^*,$$

where $A^*(x,y) = A(y,x)$. These quasigroups are called *inverse quasigroups* or *parastrophies*. Like this, with each invertible algebra $(Q;\Sigma)$ the next five invertible algebras are connected:

$$(Q;\Sigma^{-1}),\ \ (Q;{}^{-1}\Sigma),\ \ (Q;{}^{-1}(\Sigma^{-1})),\ \ (Q;({}^{-1}\Sigma)^{-1}),\ \ (Q;\Sigma^*),$$

where

$$\begin{aligned} \Sigma^{-1} &= \{A^{-1} \mid A \in \Sigma\}, \\ {}^{-1}\Sigma &= \{{}^{-1}A \mid A \in \Sigma\}, \\ {}^{-1}(\Sigma^{-1}) &= \{{}^{-1}(A^{-1}) \mid A \in \Sigma\}, \\ ({}^{-1}\Sigma)^{-1} &= \{({}^{-1}A)^{-1} \mid A \in \Sigma\}, \\ \Sigma^* &= \{A^* \mid A \in \Sigma\}. \end{aligned}$$

Each of these invertible algebras are called *parastrophies* of $(Q;\Sigma)$.

Lemma 2.1. *If an invertible algebra $(Q;\Sigma)$ satisfies the following equality:*

$$A(B(x,y),B(u,v)) = A(B(x,u),B(\alpha y,v)), \tag{2}$$

where α is a mapping from Q into Q and A, B are some operations from Σ, then α depends on u, A, B and on their inverse operations and has the form:

$$\alpha y = \alpha_u^{A,B} y =^{-1} B\big(A^{-1}(u, A(B(^{-1}B(u,u), y), u)), B^{-1}(u,u)\big)\,. \tag{3}$$

Proof. If in (2) $x = f_u^B$ and $v = e_u^B$, we obtain:

$$A\big(B\big(f_u^B, y\big), B\big(u, e_u^B\big)\big) = A\big(B\big(f_u^B, u\big), B\big(\alpha y, e_u^B\big)\big)\,,$$
$$A\big(B\big(f_u^B, y\big), u\big) = A\big(u, B\big(\alpha y, e_u^B\big)\big)\,,$$
$$A\big(L_{B,f_u^B} y, u\big) = A\big(u, R_{B,e_u^B} \alpha y\big)\,,$$
$$R_{A,u} L_{B,f_u^B} y = L_{A,u} R_{B,e_u^B} \alpha y\,,$$
$$\alpha y = R^{-1}_{B,e_u^B} L^{-1}_{A,u} R_{A,u} L_{B,f_u^B} y\,.$$

We have

$$\alpha y = R^{-1}_{B,e_u^B} L^{-1}_{A,u} R_{A,u} B\big(f_u^B, y\big) = R^{-1}_{B,e_u^B} L^{-1}_{A,u} A\big(B\big(f_u^B, y\big), u\big) =$$
$$R^{-1}_{B,e_u^B} A^{-1}\big(u, A\big(B\big(f_u^B, y\big), u\big)\big) =$$
$$^{-1}B\big(A^{-1}\big(u, A\big(B\big(^{-1}B(u,u), y\big), u\big)\big), ^{-1}B(u,u)\big),$$

since $e_u^B = B^{-1}(u,u)$, $f_u^B =^{-1} B(u,u)$, $R^{-1}_{B,y} x =^{-1} B(x,y)$, $L^{-1}_{B,y} x = B^{-1}(y,x)$. □

Lemma 2.2. *If an invertible algebra $(Q;\Sigma)$ satisfies the following equality:*

$$A(B(x,y), B(u,v)) = A(B(\beta v, y), B(u,x)), \tag{4}$$

where β is a mapping from Q into Q and A, B are some operations from Σ, then β depends on x, A, B and on their inverse operations and has the form:

$$\beta v = \beta_x^{A,B} v =^{-1} B\big(^{-1}A\big(A\big(x, B\big(^{-1}B(x,x), v\big)\big), x\big), B^{-1}(x,x)\big)\,. \tag{5}$$

Proof. If in (4) $y = e_x^B$ and $u = f_x^B$, then we obtain as in Lemma 2.1. □

Theorem 2.1. *The binary algebra $(Q;\Sigma)$ is an invertible linear algebra iff the following second order formula:*

$$X(Y(x,y), Y(u,v)) = X\big(Y(x,u), Y\big(\alpha_u^{X,Y} y, v\big)\big), \tag{6}$$

where

$$\alpha_u^{X,Y} y =^{-1} Y\big(X^{-1}\big(u, X\big(Y\big(^{-1}Y(u,u), y\big), u\big)\big), Y^{-1}(u,u)\big) \tag{7}$$

is valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)$ for all $X, Y \in \Sigma$.

Proof. Let $(Q;\Sigma)$ be an invertible linear algebra, then for every $X \in \Sigma$ we have:

$$X(x,y) = \varphi_X x + c_X + \psi_X y\,,$$

where φ_X, ψ_X are automorphisms of the group $(Q;+)$ and $c_X \in Q$. We prove that equality (6) is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup^{-1}\Sigma)$ for all $X,Y \in \Sigma$, when

$$\alpha_u^{X,Y} y = -\alpha_0^{X,Y} u + \alpha_0^{X,Y} y + u\,,$$

where $\alpha_0^{X,Y} y = \varphi_Y^{-1}\psi_X^{-1}\tilde{L}_{c_Y}^{-1}\tilde{R}_{c_X}\varphi_X\psi_Y y$, $\tilde{L}_{c_Y}x = c_Y + x$, $\tilde{R}_{c_X}x = x + c_X$. Indeed,

$$\begin{aligned}X(Y(x,y),Y(u,v)) &= \varphi_X(\varphi_Y x + c_Y + \psi_Y y) + c_X + \psi_X(\varphi_Y u + c_Y + \psi_Y v) = \\ &= \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y y + c_X + \psi_X\varphi_Y u + \psi_X c_Y + \psi_X\psi_Y v\,,\end{aligned}$$

on the other hand, using the expressions for $\alpha_0^{X,Y}$, we obtain

$$\begin{gathered}X\big(Y(x,u),Y\big(\alpha_u^{X,Y}y,v\big)\big) = \varphi_X(\varphi_Y x + c_Y + \psi_Y u) + c_X + \\ +\psi_X\big(\varphi_Y\alpha_u^{X,Y}y + c_Y + \psi_Y v\big) = \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y u + c_X + \\ +\psi_X\varphi_Y\big(-\alpha_0^{X,Y}u + \alpha_0^{X,Y}y + u\big) + \psi_X c_Y + \psi_X\psi_Y v = \varphi_X\varphi_Y x + \varphi_X c_Y + \\ +\varphi_X\psi_Y u + c_X - \psi_X\varphi_Y\varphi_Y^{-1}\psi_X^{-1}\tilde{L}_{c_Y}^{-1}\tilde{R}_{c_X}\varphi_X\psi_Y u + \\ +\psi_X\varphi_Y\varphi_Y^{-1}\psi_X^{-1}\tilde{L}_{c_Y}^{-1}\tilde{R}_{c_X}\varphi_X\psi_Y y + \psi_X\varphi_Y u + \psi_X c_Y + \psi_X\psi_Y v = \\ = \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y u + c_X - \tilde{L}_{c_Y}^{-1}\tilde{R}_{c_X}\varphi_X\psi_Y u + \tilde{L}_{c_Y}^{-1}\tilde{R}_{c_X}\varphi_X\psi_Y y + \\ +\psi_X\varphi_Y u + \psi_X c_Y + \psi_X\psi_Y v = \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y u + c_X - \\ -(-c_Y + \varphi_X\psi_Y u + c_X) - c_Y + \varphi_X\psi_Y y + c_X + \psi_X\varphi_Y u + \psi_X c_Y + \\ +\psi_X\psi_Y v = \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y y + c_X - c_X - \varphi_X\psi_Y u + c_Y - \\ -c_Y + \varphi_X\psi_Y y + c_X + \psi_X\varphi_Y u + \psi_X c_Y + \psi_X\psi_Y v = \\ = \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y y + c_X + \psi_X\varphi_Y u + \psi_X c_Y + \psi_X\psi_Y v.\end{gathered}$$

Thus, the right and left sides of equality (6) are equal. According to Lemma 2.1 we obtain that $\alpha_u^{X,Y}$ has the form of (7).

Conversely, let formula (6) be valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup^{-1}\Sigma)$ for all $X,Y\in\Sigma$. We prove that the algebra $(Q;\Sigma)$ is an invertible linear algebra. Let us fix (in (6)) the element $u = a$ and the operations $X = A$, $Y = B$, where $A,B\in\Sigma$, then we obtain:

$$\begin{aligned}A(B(x,y),B(a,v)) &= A\big(B(x,a),B\big(\alpha_a^{A,B}y,v\big)\big),\\ A(B(x,y),L_{B,a}v) &= A\big(R_{B,a}x,B\big(\alpha_a^{A,B}y,v\big)\big),\end{aligned}$$

or

$$A_1(A_2(x,y),v) = A_3(x, A_4(y,v)),$$

where $A_1(x,y) = A(x, L_{B,a}y)$, $A_2(x,y) = B(x,y)$, $A_3(x,y) = A(R_{B,a}x, y)$, $A_4(x,y) = B\big(\alpha_a^{A,B}x, y\big)$.

From the last equality, according to Belousov's theorem about four quasigroups which are connected through the associative law [18], all the operations A_i $(i = 1,2,3,4)$ are isotopic to the same group. Hence, the operations, A and B, are isotopic to the same group, and since the operations A and B are arbitrary we obtain that all the operations from Σ are isotopic to the same group $(Q;*)$.

For every $X \in \Sigma$, let us define the operations:

$$x \underset{X}{+} y = X\big(R^{-1}_{X,a}x, L^{-1}_{X,b}y\big), \tag{8}$$

where a, b are some elements from Q. These operations are loops with the identity element $0_X = X(b,a)$ [3], and they are isotopic to the group $(Q;*)$. Hence, by Albert's theorem [3], they are groups for every $X \in \Sigma$.

Let us rewrite equality (6) (where $X = A$, $Y = B$), (in terms of the operations $\underset{A}{+}$ and $\underset{B}{+}$) in the following way:

$$R_{A,a}\big(R_{B,a}x \underset{B}{+} L_{B,b}y\big) \underset{A}{+} L_{A,b}\big(R_{B,a}u \underset{B}{+} L_{B,b}v\big) = \\ R_{A,a}\big(R_{B,a}x \underset{B}{+} L_{B,b}u\big) \underset{A}{+} L_{A,b}\big(R_{B,a}\alpha_u^{A,B}y \underset{B}{+} L_{B,b}v\big),$$

$$R_{A,a}\big(x \underset{B}{+} y\big) \underset{A}{+} L_{A,b}\big(u \underset{B}{+} v\big) = \\ R_{A,a}\big(x \underset{B}{+} L_{B,b}R^{-1}_{B,a}u\big) \underset{A}{+} L_{A,b}\big(R_{B,a}\alpha^{A,B}_{R^{-1}_{B,a}}L^{-1}_{B,b}y \underset{B}{+} v\big).$$

If we take $u = 0_B$ and $v = L^{-1}_{A,b}0_A$ in the last equality, then we have:

$$R_{A,a}\big(x \underset{B}{+} y\big) \underset{A}{+} L_{A,b}\big(0_B \underset{B}{+} L^{-1}_{A,b}0_A\big) = \\ R_{A,a}\big(x \underset{B}{+} L_{B,b}R^{-1}_{B,a}0_B\big) \underset{A}{+} L_{A,b}\big(R_{B,a}\alpha^{A,B}_{R^{-1}_{B,a}0_B}L^{-1}_{B,b}y \underset{B}{+} L^{-1}_{A,b}0_A\big),$$

$$R_{A,a}\big(x \underset{B}{+} y\big) = \alpha_{A,B}x \underset{A}{+} \beta_{A,B}y, \tag{9}$$

where

$$\alpha_{A,B}x = R_{A,a}\big(x \underset{B}{+} L_{B,b}R^{-1}_{B,a}0_B\big),$$

$$\beta_{A,B}y = L_{A,b}\big(R_{B,a}\alpha^{A,B}_{R^{-1}_{B,a}0_B}L^{-1}_{B,b}y \underset{B}{+} L^{-1}_{A,b}0_A\big).$$

Since the operations A and B are arbitrary, we can take $A = B$ in (9), then we obtain:

$$R_{A,a}\big(x \underset{A}{+} y\big) = \alpha_{A,A}x \underset{A}{+} \beta_{A,A}y. \tag{10}$$

From (9) and (10), we have:

$$x \underset{A}{+} y = R_{A,a}\big(\alpha^{-1}_{A,A}x \underset{A}{+} \beta^{-1}_{A,A}y\big),$$
$$x \underset{A}{+} y = R_{A,a}\big(\alpha^{-1}_{A,B}x \underset{B}{+} \beta^{-1}_{A,B}y\big),$$
$$\alpha^{-1}_{A,A}x \underset{A}{+} \beta^{-1}_{A,A}y = \alpha^{-1}_{A,B}x \underset{B}{+} \beta^{-1}_{A,B}y,$$

thus, we obtain:

$$x \underset{A}{+} y = \gamma_{A,B}x \underset{B}{+} \delta_{A,B}y, \tag{11}$$

where $\gamma_{A,B} = \alpha^{-1}_{A,B}\alpha_{A,A}$ and $\delta_{A,B} = \beta^{-1}_{A,B}\beta_{A,A}$ are the permutations of the set Q. Hence, from (9), according to (11), we get:

$$R_{A,a}\big(x \underset{B}{+} y\big) = \gamma_{A,B}\alpha_{A,B}x \underset{B}{+} \delta_{A,B}\beta_{A,B}y,$$

i.e., $R_{A,a}$ is a quasiautomorphism of the group $\big(Q; \underset{B}{+}\big)$ and since the operation A is arbitrary, we have that $R_{A,a}$ is the quasiautomorphism of the group $\big(Q; \underset{B}{+}\big)$ for all operations A from Σ. We fix the operation $\underset{B}{+}$ and further will be denote it by $+$.

According to (8), for the operations $A \in \Sigma$ we have:

$$A(x, y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

According to (11), from the last equality, we get:

$$A(x, y) = \theta^{A,B}_1 x + \theta^{A,B}_2 y, \tag{12}$$

where $\theta^{A,B}_1 = \gamma_{A,B}R_{A,a}$ and $\theta^{A,B}_2 = \delta_{A,B}L_{A,b}$ are the permutations of Q.

We prove that $\theta^{A,B}_1$ and $\theta^{A,B}_2$ are quasiautomorphisms of the group $(Q; +)$. To do it we take $v = a$, $u = f^B_a$, $X = A$, $Y = B$ in equality (6) and rewrite this equality in terms of the operation $+$:

$$A(B(x, y), a) = A\big(B\big(x, f^B_a\big), B\big(\alpha^{A,B}_{f^B_a} y, a\big)\big),$$
$$\theta^{A,B}_1(R_{B,a}x + L_{B,b}y) + \theta^{A,B}_2 a = \theta^{A,B}_1 R_{B,f^B_a}x + \theta^{A,B}_2\big(R_{B,a}\alpha^{A,B}_{f^B_a} y + L_{B,b}a\big),$$
$$\theta^{A,B}_1(R_{B,a}x + L_{B,b}y) = \theta^{A,B}_1 R_{B,f^B_a}x + \theta^{A,B}_2\big(R_{B,a}\alpha^{A,B}_{f^B_a} y + L_{B,b}a\big) - \theta^{A,B}_2 a,$$

$$\theta_1^{A,B}(x+y) = \theta_1^{A,B} R_{B,f_a^B} R_{B,a}^{-1} x + \theta_2^{A,B}\big(R_{B,a}\alpha_{f_a^B}^{A,B} L_{B,b}^{-1} y + L_{B,b} a\big) - \theta_2^{A,B} a,$$
$$\theta_1^{A,B}(x+y) = \sigma_{A,B} x + \mu_{A,B} y,$$

where
$\sigma_{A,B}x = \theta_1^{A,B} R_{B,f_a^B} R_{B,a}^{-1} x$ and $\mu_{A,B}y = \theta_2^{A,B}\big(R_{B,a}\alpha_{f_a^B}^{A,B} L_{B,b}^{-1} y + L_{B,b} a\big) - \theta_2^{A,B} a$ are the permutations of Q and therefore $\theta_1^{A,B}$ is a quasiautomorphism of the group $(Q;+)$.

Now, we take $x = f_b^B$, $u = b$, $X = A$, $Y = B$ in (6) and rewrite this equality in terms of the operation $+$:

$$A\big(B\big(f_b^B, y\big), B(b,v)\big) = A\big(b, B\big(\alpha_b^{A,B} y, v\big)\big),$$
$$\theta_1^{A,B} L_{B,f_b^B} y + \theta_2^{A,B} L_{B,b} v = \theta_1^{A,B} b + \theta_2^{A,B}\big(R_{B,a}\alpha_b^{A,B} y + L_{B,b} v\big),$$
$$\theta_2^{A,B}\big(R_{B,a}\alpha_b^{A,B} y + L_{B,b} v\big) = -\theta_1^{A,B} b + \theta_1^{A,B} L_{B,f_b^B} y + \theta_2^{A,B} L_{B,b} v,$$
$$\theta_2^{A,B}(y+v) = \sigma'_{A,B} y + \mu'_{A,B} v,$$

where $\sigma'_{A,B} y = -\theta_1^{A,B} b + \theta_1^{A,B} L_{B,f_b^B} \big(\alpha_b^{A,B}\big)^{-1} R_{B,a}^{-1} y$ and $\mu'_{A,B} v = \theta_2^{A,B} v$ are the permutations of the set Q and therefore $\theta_2^{A,B}$ is a quasiautomorphism of the group $(Q;+)$.

According to [3, lemma 2.5] we have:

$$\theta_1^{A,B} x = \varphi_A x + s_A,$$
$$\theta_2^{A,B} x = t_A + \psi_A y,$$

where φ_A, ψ_A are automorphisms of the group $(Q;+)$ and t_A, s_A are some elements of the set Q. Hence, from (12), it follows that

$$A(x,y) = \varphi_A x + c_A + \psi_A y, \tag{13}$$

where $c_A = s_A + t_A$.

Since the operation A is arbitrary, we obtain that all the operations from Σ can be presented in the form of (13) through the operation $+$. □

Theorem 2.2. *The binary algebra $(Q;\Sigma)$ is an invertible alinear algebra iff the following second order formula:*

$$X(Y(x,y), Y(u,v)) = X\big(Y\big(\beta_x^{X,Y} v, y\big), Y(u,x)\big), \tag{14}$$

where

$$\beta_x^{X,Y} v = {}^{-1}Y\big({}^{-1}X\big(X\big(x, Y\big({}^{-1}Y(x,x), v\big)\big), x\big), Y^{-1}(x,x)\big) \tag{15}$$

is valid in the algebra $(Q; \Sigma \cup \Sigma^{-1} \cup {}^{-1}\Sigma)$ for all $X, Y \in \Sigma$.

Proof. Let $(Q;\Sigma)$ be an invertible alinear algebra, then for every $X\in\Sigma$

$$X(x,y)=\varphi_X x+c_X+\psi_X y\,,$$

where φ_X, ψ_X are antiautomorphisms of the group $(Q;+)$ and $c_X\in Q$. We prove that equality (14) is valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ for all $X,Y\in\Sigma$, if:

$$\beta_x^{X,Y}v=x+\beta_0^{X,Y}v-\beta_0^{X,Y}x\,,$$

where $\beta_0^{X,Y}v=\varphi_Y^{-1}\varphi_X^{-1}\tilde{R}_{c_Y}^{-1}\tilde{L}_{c_X}\psi_X\psi_Y v$, $\tilde{R}_{c_Y}x=x+c_Y$, $\tilde{L}_{c_X}x=c_X+x$. Indeed,

$$\begin{aligned}X(Y(x,y),Y(u,v))&=\varphi_X(\varphi_Y x+c_Y+\psi_Y y)+c_X+\psi_X(\varphi_Y u+c_Y+\psi_Y v)=\\&=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+c_X+\psi_X\psi_Y v+\psi_X c_Y+\psi_X\varphi_Y u\,,\end{aligned}$$

on the other hand, using the expressions for $\beta_0^{X,Y}$, and taking into account that $\varphi_X\varphi_Y$ is an automorphism of the group $(Q;+)$ we obtain:

$$\begin{gathered}X\big(Y\big(\beta_x^{X,Y}v,y\big),Y(u,x)\big)=\varphi_X(\varphi_Y\beta_x^{X,Y}v+c_Y+\psi_Y y)+c_X+\\+\psi_X\big(\varphi_Y u+c_Y+\psi_Y x\big)=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y\beta_x^{X,Y}v+c_X+\\+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\varphi_X\psi_Y y+\varphi_X c_Y+\\+\varphi_X\varphi_Y\big(x+\beta_0^{X,Y}v-\beta_0^{X,Y}x\big)+c_X+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\\=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+\varphi_X\varphi_Y\beta_0^{X,Y}v-\varphi_X\varphi_Y\beta_0^{X,Y}x+c_X+\\+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+\\+\varphi_X\varphi_Y\varphi_Y^{-1}\varphi_X^{-1}\tilde{R}_{c_Y}^{-1}\tilde{L}_{c_X}\psi_X\psi_Y v-\varphi_X\varphi_Y\varphi_Y^{-1}\varphi_X^{-1}\tilde{R}_{c_Y}^{-1}\tilde{L}_{c_X}\psi_X\psi_Y x+\\+c_X+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+c_X+\\+\psi_X\psi_Y v-c_Y-(c_X+\psi_X\psi_Y x-c_Y)+c_X+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\\=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+c_X+\psi_X\psi_Y v-c_Y+c_Y-\psi_X\psi_Y x-\\-c_X+c_X+\psi_X\psi_Y x+\psi_X c_Y+\psi_X\varphi_Y u=\\=\varphi_X\psi_Y y+\varphi_X c_Y+\varphi_X\varphi_Y x+c_X+\psi_X\psi_Y v+\psi_X c_Y+\psi_X\varphi_Y u.\end{gathered}$$

Thus, the right and left sides of equality (14) are equal. According to Lemma 2.2, we get that $\beta_x^{X,Y}$ has the form of (15).

Conversely, let the formula (14) be valid in the algebra $(Q;\Sigma\cup\Sigma^{-1}\cup{}^{-1}\Sigma)$ for all $X,Y\in\Sigma$. We prove that the algebra $(Q;\Sigma)$ is an invertible alinear algebra. Fixing the element $x=p$ and the operations $X=A$, $Y=B$, where $A, B \in \Sigma$ in (14), we obtain:

$$A(B(p,y), B(u,v)) = A\big(B\big(\beta_p^{A,B} v, y\big), B(u,p)\big),$$
$$A(L_{B,p} y, B(u,v)) = A\big(B\big(\beta_p^{A,B} v, y\big), R_{B,p} u\big),$$
$$A^*(B(u,v), L_{B,p} y) = A^*\big(R_{B,p} u, B\big(\beta_p^{A,B} v, y\big)\big)$$

or

$$A_1(A_2(u,v), y) = A_3(u, A_4(v,y)),$$

where $A_1(x,y) = A^*(x, L_{B,p} y)$, $A_2(x,y) = B(x,y)$, $A_3(x,y) = A^*(R_{B,p} x, y)$, $A_4(x,y) = B\big(\beta_p^{A,B} x, y\big)$.

From the last equality, according to Belousov's theorem about four quasigroups which are connected with the associative law [18], all the operations A_i $(i = 1, 2, 3, 4)$ are isotopic to the same group. Since the operation B is arbitrary, we obtain that all the operations from Σ are isotopic to the same group $(Q; *)$.

For every $X \in \Sigma$ let us define the operations:

$$x \underset{X}{+} y = X\big(R_{X,a}^{-1} x, L_{X,b}^{-1} y\big), \tag{16}$$

where a, b are some elements from Q. These operations are loops with the identity element $0_X = X(b,a)$ [3], and they are isotopic to the group $(Q; *)$. Hence by Albert's theorem [3] they are groups for every $X \in \Sigma$.

Let us rewrite the equality (14) (where $X = A$, $Y = B$) in terms of the operations $\underset{A}{+}$ and $\underset{B}{+}$

$$R_{A,a}\big(R_{B,a} x \underset{B}{+} L_{B,b} y\big) \underset{A}{+} L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} v\big) =$$
$$R_{A,a}\big(R_{B,a} \beta_x^{A,B} v \underset{B}{+} L_{B,b} y\big) \underset{A}{+} L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} x\big).$$

If we take $y = a$ and $x = R_{B,a}^{-1} b = d$ in the last equality, we have:

$$R_{A,a}\big(R_{B,a} R_{B,a}^{-1} b \underset{B}{+} L_{B,b} a\big) \underset{A}{+} L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} v\big) =$$
$$R_{A,a}\big(R_{B,a} \beta_d^{A,B} v \underset{B}{+} L_{B,b} a\big) \underset{A}{+} L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} d\big),$$
$$R_{A,a}\big(b \underset{B}{+} 0_B\big) \underset{A}{+} L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} v\big) =$$
$$R_{A,a}\big(R_{B,a} \beta_d^{A,B} v \underset{B}{+} 0_B\big) \underset{A}{+} L_{A,b} B(u,d),$$
$$R_{A,a} b + L_{A,b}\big(R_{B,a} u \underset{B}{+} L_{B,b} v\big) = R_{A,a} R_{B,a} \beta_d^{A,B} v \underset{A}{+} L_{A,b} R_{B,d} u,$$

$$L_{A,b}\big(R_{B,a}u \underset{B}{+} L_{B,b}v\big) = R_{A,a}R_{B,a}\beta_d^{A,B}v \underset{A}{+} L_{A,b}R_{B,d}u,$$

or

$$L_{A,b}\big(u \underset{B}{+} v\big) = \alpha_{A,B}v \underset{A}{+} \beta_{A,B}u \tag{17}$$

where

$$\alpha_{A,B} = R_{A,a}R_{B,a}\beta_d^{A,B}L_{B,b}^{-1} \qquad \text{and} \qquad \beta_{A,B} = L_{A,b}R_{B,d}R_{B,a}^{-1}$$

are permutations of the set Q.

Since the operations A and B are arbitrary, we can take $A = B$ in (17), and get:

$$L_{A,b}\big(u \underset{A}{+} v\big) = \alpha_{A,A}v \underset{A}{+} \beta_{A,A}u. \tag{18}$$

From (17) and (18) we have:

$$\begin{aligned}
&v \underset{A}{+} u = L_{A,b}\big(\beta_{A,B}^{-1}u \underset{B}{+} \alpha_{A,B}^{-1}v\big),\\
&v \underset{A}{+} u = L_{A,b}\big(\beta_{A,A}^{-1}u \underset{A}{+} \alpha_{A,A}^{-1}v\big),\\
&\beta_{A,B}^{-1}u \underset{B}{+} \alpha_{A,B}^{-1}v = \beta_{A,A}^{-1}u \underset{A}{+} \alpha_{A,A}^{-1}v,
\end{aligned}$$

and thus, we obtain:

$$u \underset{A}{+} v = \gamma_{A,B}u \underset{B}{+} \delta_{A,B}v, \tag{19}$$

where $\gamma_{A,B} = \beta_{A,B}^{-1}\beta_{A,A}$ and $\delta_{A,B} = \alpha_{A,B}^{-1}\alpha_{A,A}$ are the permutations of the set Q.

According to (16), for the operations $A \in \Sigma$, we have:

$$A(x, y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

According to (19), from the last equality, we get:

$$A(x, y) = \theta_1^{A,B}x \underset{B}{+} \theta_2^{A,B}y, \tag{20}$$

where $\theta_1^{A,B} = \gamma_{A,B}R_{A,a}$ and the $\theta_2^{A,B} = \delta_{A,B}L_{A,b}$ are the permutations of the set Q. Thus, we can represent every operations from Σ by the operation $\underset{B}{+}$. We fix the operation $\underset{B}{+}$ and further denote it by $+$.

We shall prove that $\theta_1^{A,B}$ and $\theta_2^{A,B}$ are antiquasiautomorphisms of the group $(Q;+)$. To do it we take $x = a$, $u = f_a^B$, $X = A$, $Y = B$, in equality (14) and rewrite this equality in terms of the operation, $+$:

$$\begin{aligned}
&A\big(B(a,y),B\big(f_a^B,v\big)\big)=A\big(B\big(\beta_a^{A,B}v,y\big),a\big),\\
&\theta_1^{A,B}(R_{B,a}a+L_{B,b}y)+\theta_2^{A,B}L_{B,f_a^B}v=\theta_1^{A,B}\big(R_{B,a}\beta_a^{A,B}v+L_{B,b}y\big)+\theta_2^{A,B}a,\\
&\theta_1^{A,B}(R_{B,a}\beta_a^{A,B}v+L_{B,b}y)=\theta_1^{A,B}\big(R_{B,a}a+L_{B,b}y\big)+\theta_2^{A,B}L_{B,f_a^B}v-\theta_2^{A,B}a,\\
&\theta_1^{A,B}(v+y)=\theta_1^{A,B}\big(R_{B,a}a+y\big)+\theta_2^{A,B}L_{B,f_a^B}\big(\beta_a^{A,B}\big)^{-1}R_{B,a}^{-1}v-\theta_2^{A,B}a,\\
&\theta_1^{A,B}(v+y)=\sigma_{A,B}y+\mu_{A,B}v,
\end{aligned}$$

where
$\sigma_{A,B}y=\theta_1^{A,B}\big(R_{B,a}a+y\big)$ and $\mu_{A,B}v=\theta_2^{A,B}L_{B,f_a^B}\big(\beta_a^{A,B}\big)^{-1}R_{B,a}^{-1}v-\theta_2^{A,B}a$ are the permutations of the set Q and therefore, $\theta_1^{A,B}$ is an antiquasiautomorphism of the group $(Q;+)$.

If we take $x=a$, $y=e_a^B$, $X=A$, $Y=B$ in the equality (14), we can similarly prove that $\theta_2^{A,B}$ is an antiquasiautomorphism of the group $(Q;+)$.

Thus, we have [2]

$$\begin{aligned}
\theta_1^{A,B}x&=\varphi_A x+s_A,\\
\theta_2^{A,B}x&=t_A+\psi_A y,
\end{aligned}$$

where φ_A, ψ_A are antiautomorphisms of the group $(Q;+)$ and t_A, s_A are some elements of the set Q. Hence, from (20) we get that:

$$A(x,y)=\varphi_A x+c_A+\psi_A y, \tag{21}$$

where $c_A=s_A+t_A$.

Since the operation A is arbitrary, we obtain that all the operations from Σ can be presented in the form of (21). □

3. Invertible T-algebras

It is known [10, 11] that T-quasigroups are invariant under parastrophies. We have the same result for parastrophies of invertible T-algebras.

Proposition 3.1. *Let $(Q;\Sigma)$ be an invertible T-algebra. Then all parastrophies of the algebra, $(Q;\Sigma)$, are invertible T-algebras.*

Also, as in the case of quasigroups [6], we have the following result:

Proposition 3.2. *If an invertible algebra is linear and alinear then it is T-algebra.*

Lemma 3.1. *If the algebra* $(Q; \Sigma \cup \Sigma^{-1} \cup^{-1} \Sigma)$, *where* $(Q;\Sigma)$ *is an invertible* T*-algebra, satisfies equality* (6) *for all* $X, Y \in \Sigma$, *then this equality is also valid in the algebra* $\big(Q; \Sigma \cup^{-1} \Sigma \cup \Sigma^{-1} \cup ({}^{-1}\Sigma)^{-1} \cup^{-1} (\Sigma^{-1}) \cup \Sigma^*\big)$ *for all* $X, Y \in \Sigma \cup^{-1} \Sigma \cup \Sigma^{-1} \cup ({}^{-1}\Sigma)^{-1} \cup^{-1} (\Sigma^{-1}) \cup \Sigma^*$.

Proof. We must check equalities for all $A, B \in \Sigma \cup^{-1} \Sigma \cup \Sigma^{-1} \cup ({}^{-1}\Sigma)^{-1} \cup^{-1} (\Sigma^{-1})$. For example, let us check the following equality:

$$A\big({}^{-1}B(x,y), {}^{-1}B(u,v)\big) = A\big({}^{-1}B(x,u), {}^{-1}B\big(\alpha_u^{A,{}^{-1}B} y, v\big)\big).$$

In this case, we have:

$$\alpha_u^{A,{}^{-1}B} y = B\big(A^{-1}\big(u, A({}^{-1}B(B(u,u),y),u)\big), ({}^{-1}B)^{-1}(u,u)\big).$$

It follows from (1):

$$A^{-1}(x,y) = \psi_A^{-1}(-c_A - \varphi_A x + y),$$
$${}^{-1}B(x,y) = \varphi_B^{-1}(x - \psi_B y - c_B),$$
$$({}^{-1}B)^{-1}(x,y) = \psi_B^{-1}(-c_B - \varphi_B y + x).$$

Let us calculate $\alpha_u^{A,{}^{-1}B} y$:

$$\begin{aligned}\alpha_u^{A,{}^{-1}B} y &= \varphi_B \psi_A^{-1}\big(\varphi_A \varphi_B^{-1} \psi_B u - \varphi_A \varphi_B^{-1} \psi_B y + \psi_A u\big) + u - \varphi_B u - c_B + c_B\\ &= \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} \psi_B u - \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} \psi_B y + \varphi_B u + u - \varphi_B u\\ &= \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1} (\psi_B u - \psi_B y) + u.\end{aligned}$$

Therefore

$$\begin{aligned}&A\big({}^{-1}B(x,u), {}^{-1}B\big(\alpha_u^{A,{}^{-1}B} y, v\big)\big)\\ &= A\big(\varphi_B^{-1}(x - \psi_B u - c_B), \varphi_B^{-1}\big(\alpha_u^{A,{}^{-1}B} y - \psi_B v - c_B\big)\big)\\ &= \varphi_A \varphi_B^{-1}(x - \psi_B u - c_B) + \psi_A \varphi_B^{-1}\big(\alpha_u^{A,{}^{-1}B} y - \psi_B v - c_B\big) + c_A\\ &= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} \psi_B u - \varphi_A \varphi_B^{-1} c_B + \psi_A \varphi_B^{-1} \varphi_B \psi_A^{-1} \varphi_A \varphi_B^{-1}\big(\psi_B u - \psi_B y\big)\\ &\qquad + \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A\\ &= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} c_B - \varphi_A \varphi_B^{-1} \psi_B y + \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A\end{aligned}$$

On the other hand

$$\begin{aligned}&A\big({}^{-1}B(x,u), {}^{-1}B(u,v)\big) = \varphi_A \varphi_B^{-1}(x - \psi_B y - c_B) + \psi_A \varphi_B^{-1}(u - \psi_B v - c_B) + c_A\\ &= \varphi_A \varphi_B^{-1} x - \varphi_A \varphi_B^{-1} \psi_B y - \varphi_A \varphi_B^{-1} c_B + \psi_A \varphi_B^{-1} u - \psi_A \varphi_B^{-1} \psi_B v - \psi_A \varphi_B^{-1} c_B + c_A.\end{aligned}$$

Thus, the right and left sides are equal. Similarly, we can check the other cases. □

Lemma 3.2. *Let* $(Q;\Sigma)$ *be an invertible* T*-algebra. If the algebra,* $(Q;\Sigma\cup\Sigma^{-1}\cup^{-1}\Sigma)$*, satisfies equality* (14) *for all* $X,Y\in\Sigma$*, then this equality is valid in the algebra* $\big(Q;\Sigma\cup^{-1}\Sigma\cup\Sigma^{-1}\cup({}^{-1}\Sigma)^{-1}\cup^{-1}(\Sigma^{-1})\cup\Sigma^{*}\big)$ *for all* $X,Y\in\Sigma\cup^{-1}\Sigma\cup\Sigma^{-1}\cup({}^{-1}\Sigma)^{-1}\cup^{-1}(\Sigma^{-1})\cup\Sigma^{*}$.

Proof. Similarly as Lemma 3.1. □

Theorem 3.1. $(Q;\Sigma)$ *is an invertible* T*-algebra iff* (6) *and* (14) *are valid in the algebra* $\big(Q;\Sigma\cup^{-1}\Sigma\cup\Sigma^{-1}\cup({}^{-1}\Sigma)^{-1}\cup^{-1}(\Sigma^{-1})\cup\Sigma^{*}\big)$ *for all* $X,Y\in\Sigma\cup^{-1}\Sigma\cup\Sigma^{-1}\cup({}^{-1}\Sigma)^{-1}\cup^{-1}(\Sigma^{-1})\cup\Sigma^{*}$.

Proof. As in the proof of Theorems 2.1 and 2.2, the invertible T-algebra satisfies formulae (6) and (14). The rest follows from Lemmas 3.1 and 3.2. The converse statement is a consequence of Proposition 3.2. □

Corollary 3.1. *Let* $(Q;\Sigma)$ *be an invertible* T*-algebra. If* $(Q;\Sigma)$ *satisfies the following second-order formula:*

$$\forall X_1,X_2\,\forall x_1,x_2,x_3\,\exists x_4$$
$$\big(X_1\big(X_2(x_1,x_2),X_2(x_4,x_3)\big)=X_1\big(X_2(x_1,x_4),X_2(x_2,x_3)\big)\big), \tag{22}$$

then in $(Q;\Sigma)$ *the following hyperidentity is valid:*

$$X_1\big(X_2(x_1,x_2),X_2(x_4,x_3)\big)=X_1\big(X_2(x_1,x_4),X_2(x_2,x_3)\big).$$

Proof. Let $(Q;\Sigma)$ be an invertible T-algebra. Then it satisfies (6). If we rewrite (6), in terms of the operation $+$, then after cancellations we obtain

$$\psi_X\varphi_Y u+\varphi_X\psi_Y y=\varphi_X\psi_Y u+\psi_X\varphi_Y\alpha_u^{X,Y}y\,, \tag{23}$$

which for $u=0$ gives $\varphi_X\psi_Y=\psi_X\varphi_Y\alpha_0^{X,Y}$. This together with (23) implies

$$u+\alpha_0^{X,Y}y=\alpha_0^{X,Y}u+\alpha_u^{X,Y}y\,, \tag{24}$$

where $\alpha_0^{X,Y}$ is the permutation which corresponds to the identity element of the group, $(Q;+)$.

If (22) is valid in $(Q;\Sigma)$, then for every $X,Y\in\Sigma$ and every $x,y,v\in Q$ there exists an element $h\in Q$ such that the following equality is valid:

$$X(Y(x,y,Y(h,v))=X(Y(x,h),Y(y,v)).$$

Therefore, $\alpha_h^{X,Y}$ is the identity permutation of the set Q.

From the proof of Theorem 2.1, it follows that the loops $x \underset{X}{+} y = X\big(R_{X,a}^{-1}x, L_{X,b}^{-1}y\big)$ are groups for all $a, b \in Q$ and all operations $X \in \Sigma$ and also, we can take any of the groups, $\underset{X}{+}$ $(X \in \Sigma)$ as a group $+$.

Let us choose the elements a, b such that $h = Y(b, a)$ is an identity element of the group $(Q; +)$, then $\alpha_h^{X,Y}$ is the identity permutation of the set Q. Therefore, from (24), we have $\alpha_u^{X,Y} y = y$ since $\alpha_0^{X,Y} = \alpha_h^{X,Y}$ is the identity permutation. Hence $\alpha_u^{X,Y}$ is the identity permutation for all $u \in Q$ and all $X, Y \in \Sigma$. $\square$

Corollary 3.2. *The quasigroup,* $(Q; \cdot)$, *is a* T*-quasigroup iff formulae* (6) *and* (14) *are valid in the quasigroup,* $(Q; \cdot, /, \backslash)$, *for all* $X, Y \in \{\cdot, \backslash, /\}$.

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