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One relator quotients of the Hecke group $\mathbf{H}(\sqrt{3})$

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Abstract. One relator quotients of the modular group Γ and of the groups $H(\sqrt{2})$ and $H(\frac{1+\sqrt{5}}{2})$, have been discussed in [3], [5], [9], [10] and [11]. In this paper we obtain one relator quotients of $H(\sqrt{3})$, by adding an extra relation to the existing ones.

1. Introduction

E. Hecke introduced Hecke groups denoted by $H(\lambda_q)$. These are finitely generated discrete subgroups of $PSL(2,\mathbb{R})$, generated by transformations $R(z) = -1/z$ and $T(z) = -1/(z+\lambda_q)$, of order 2 and q, respectively, where $\lambda_q = 2cos(\pi/q)$, $q \in \mathbb{N}$, $q \geqslant 3$. The modular group $H(\lambda_3) = H(1) = PSL(2,\mathbb{Z})$ is the most interesting, important and a well discussed Hecke group from many aspects as in [3], [5], [6] and [10]. The group for $q = 5$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$ has been discussed in [4] and [9]. And many similarities to the modular group have been observed. Other two interesting groups of this class are obtained for $q = 4$ and $q = 6$. These are denoted by $H(\sqrt{2})$ and $H(\sqrt{3})$ corresponding to $q = 4$ and $q = 6$, respectively. The group $H(\sqrt{3})$ has been discussed from some aspects in [1] and [11]. One reason for $H(\sqrt{2})$ and $H(\sqrt{3})$ to be the next most important Hecke groups is that these are the only, whose elements can be described completely [11]. One relator quotients of the Hecke groups have been an important aspect of study of Hecke groups for many mathematicians. For example one can refer to [3], [5], [9] and [10]. In [11] one relator quotients of $H(\sqrt{2})$ have been a part of discussion.

In this paper we obtain one relator quotients of the Hecke group $H(\sqrt{3})$. We have mostly used the same notations as were used in [3], [9] and [10].

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2. One relator quotients of $\mathbf{H}(\sqrt{3})$

$H(\sqrt{3})$ has a presentation $\langle a, b : a^2 = b^6 = 1\rangle$. The effect of adding a new relation to this, is the formation of a new group which is quotient group of $H(\sqrt{3})$.

By adding another relation $w = R(a, b) = 1$ in terms of a and b for a cyclically reduced word $w = ab^{\varepsilon_1}ab^{\varepsilon_2}ab^{\varepsilon_3}...ab^{\varepsilon_n}$, where $1 \leqslant \varepsilon_i \leqslant 5$, we obtain one relator quotient of $H(\sqrt{3})$.

Throughout this paper we denote by k the sum of exponents of a in w and by l the sum of exponents of b in w.

Theorem 2.1. *If $k = 0$ then $1 \leqslant l \leqslant 5$ and if $k = n$ then $n \leqslant l \leqslant 5n$.*

Proof. Immediately follows from the table given at the end. □

As in [9], a word w' is equivalent to w if it can be obtained by cutting some part of w from the beginning and pasting it to the end in the same order and vice versa. Let $N_{k,l}$ be total number of non equivalent cyclically reduced words w with k and l as defined above. Then we have the following theorem.

Theorem 2.2. $N_{n,n} = N_{n,n+1} = N_{n,5n} = N_{n,5n-1} = 1$.

Proof. Immediately follows from the table given at the end. □

To obtain cyclically reduced words for a given pair of integers k and l, we have followed the procedure as followed in [9] and [10]. We illustrate it with an example.

Example 2.3. For $k = 4$ and $l = 11$, we obtain the following non equivalent cyclically reduced words.
$abababab^4ab^5$, $abababab^5ab^4$, $abab^4abab^5$, $abab^2ab^4ab^4$, $abab^4ab^4ab^4$, $abab^4ab^2ab^4$, $abab^3ab^3ab^4$, $abab^3ab^4ab^3$, $abab^4ab^3ab^3$, $ab^2ab^2ab^3ab^4$, $ab^2ab^2ab^4ab^3$, $ab^2ab^3ab^2ab^4$, $ab^2ab^3ab^3ab^3$. □

Let us consider the first word $ababab^4ab^5$, other words $ab^5ababab^4$, ab^4ab^5abab and $abab^4ab^5ab$ are omitted since these are equivalent to it. We add a relation $ababab^4ab^5 = 1$ to the group $\langle a, b : a^2 = b^6 = 1\rangle$. Using all these relations we simplify as

$$a = babab^4ab^5, \quad a = babab^4ab^5, \quad b = ab^5a$$

and equivalently we have $abab = 1$. Thus we get $\langle a, b : a^2 = b^6 = (ab)^2 = 1\rangle$ which is finite presentation of the triangle group $\Delta(2, 6, 2)$ and is isomorphic to D_6, of order 12. The following table gives the information for different pairs of values for k and l.

k	l	words	quotient group	abstract structure
0	1	b	$\langle a, b : a^2 = b^6 = b = 1\rangle$	C_2
0	2	b^2	$\langle a, b : a^2 = b^6 = b^2 = 1\rangle$	*infinite group*
0	3	b^3	$\langle a, b : a^2 = b^6 = b^3 = 1\rangle$	*infinite group*
0	4	b^4	$\langle a, b : a^2 = b^6 = b^4 = 1\rangle$	*infinite group*
0	5	b^5	$\langle a, b : a^2 = b^6 = b^5 = 1\rangle$	C_2
1	0	a	$\langle a, b : a^2 = b^6 = a = 1\rangle$	C_6
1	1	ab	$\langle a, b : a^2 = b^6 = ab = 1\rangle$	C_2
1	2	ab^2	$\langle a, b : a^2 = b^6 = ab^2 = 1\rangle$	C_2
1	3	ab^3	$\langle a, b : a^2 = b^6 = ab^3 = 1\rangle$	C_6
1	4	ab^4	$\langle a, b : a^2 = b^6 = ab^4 = 1\rangle$	C_2
1	5	ab^5	$\langle a, b : a^2 = b^6 = ab^5 = 1\rangle$	C_2
2	2	$abab$	$\langle a, b : a^2 = b^6 = (ab)^2 = 1\rangle$	D_6
2	3	$abab^2$	$\langle a, b : a^2 = b^6 = abab^2 = 1\rangle$	C_6
2	4	$abab^3$	$\langle a, b : a^2 = b^6 = abab^3 = 1\rangle$	V_4
		ab^2ab^2	$\langle a, b : a^2 = b^6 = ab^2ab^2 = 1\rangle$	*infinite group*
2	5	$abab^4$	$\langle a, b : a^2 = b^6 = abab^4 = 1\rangle$	S_3
		ab^2ab^3	$\langle a, b : a^2 = b^6 = ab^2ab^3 = 1\rangle$	C_2
2	6	$abab^5$	$\langle a, b : a^2 = b^6 = abab^5 = 1\rangle$	$V_4 \times C_3$
		ab^2ab^4	$\langle a, b : a^2 = b^6 = ab^2ab^4 = 1\rangle$	*infinite group*
		ab^3ab^3	$\langle a, b : a^2 = b^6 = ab^3ab^3 = 1\rangle$	*infinite group*
2	7	ab^2ab^5	$\langle a, b : a^2 = b^6 = ab^2ab^5 = 1\rangle$	S_3
		ab^3ab^4	$\langle a, b : a^2 = b^6 = ab^3ab^4 = 1\rangle$	C_2
2	8	ab^3ab^5	$\langle a, b : a^2 = b^6 = ab^3ab^5 = 1\rangle$	V_4
		ab^4ab^4	$\langle a, b : a^2 = b^6 = ab^4ab^4 = 1\rangle$	*infinite group*
2	9	ab^4ab^5	$\langle a, b : a^2 = b^6 = ab^4ab^5 = 1\rangle$	C_6
2	10	ab^5ab^5	$\langle a, b : a^2 = b^6 = ab^5ab^5 = 1\rangle$	D_6
3	3	$ababab$	$\langle a, b : a^2 = b^6 = ababab = 1\rangle$	*infinite group*

k	l	words	quotient group	abstract structure
3	4	$ababab^2$	$\langle a,b : a^2 = b^6 = ababab^2 = 1\rangle$	C_2
3	5	$ababab^3$	$\langle a,b : a^2 = b^6 = ababab^3 = 1\rangle$	S_3
		$abab^2ab^2$	$\langle a,b : a^2 = b^6 = abab^2ab^2 = 1\rangle$	C_2
3	6	$ababab^4$	$\langle a,b : a^2 = b^6 = ababab^4 = 1\rangle$	$A_4 \times C_2$
		$abab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^3 = 1\rangle$	C_6
		$abab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^2 = 1\rangle$	C_6
		$ab^2ab^2ab^2$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2 = 1\rangle$	*infinite group*
3	7	$ababab^5$	$\langle a,b : a^2 = b^6 = ababab^5 = 1\rangle$	S_3
		$abab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^4 = 1\rangle$	C_2
		$abab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^2 = 1\rangle$	C_2
		$abab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^3 = 1\rangle$	S_3
		$ab^2ab^2ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3 = 1\rangle$	C_2
3	8	$abab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^5 = 1\rangle$	C_2
		$abab^5ab^2$	$\langle a,b : a^2 = b^6 = abab^5ab^2 = 1\rangle$	C_2
		$abab^3ab^4$	$\langle a,b : a^2 = b^6 = abab^3ab^4 = 1\rangle$	C_2
		$abab^4ab^3$	$\langle a,b : a^2 = b^6 = abab^4ab^3 = 1\rangle$	C_2
		$ab^2ab^2ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^4 = 1\rangle$	C_2
		$ab^2ab^3ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^3 = 1\rangle$	C_2
3	9	$abab^3ab^5$	$\langle a,b : a^2 = b^6 = abab^3ab^5 = 1\rangle$	$C_9 \sim C_6$
		$abab^5ab^3$	$\langle a,b : a^2 = b^6 = abab^5ab^3 = 1\rangle$	$C_9 \sim C_6$
		$ab^2ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5 = 1\rangle$	$A_4 \times C_2$
		$abab^4ab^4$	$\langle a,b : a^2 = b^6 = abab^4ab^4 = 1\rangle$	$A_4 \times C_2$
		$ab^2ab^3ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^4 = 1\rangle$	$C_7 \sim C_6$
		$ab^2ab^4ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^3 = 1\rangle$	$C_7 \sim C_6$
		$ab^3ab^3ab^3$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^3 = 1\rangle$	*infinite group*
3	10	$abab^4ab^5$	$\langle a,b : a^2 = b^6 = abab^4ab^5 = 1\rangle$	C_2
		$abab^5ab^4$	$\langle a,b : a^2 = b^6 = abab^5ab^4 = 1\rangle$	C_2
		$ab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^5 = 1\rangle$	C_2
		$ab^2ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^3 = 1\rangle$	C_2
		$ab^2ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^4 = 1\rangle$	C_2
		$ab^3ab^3ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^4 = 1\rangle$	C_2

k	l	words	quotient group	structure
3	11	$abab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^5 = 1\rangle$	S_3
		$ab^2ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^5 = 1\rangle$	C_2
		$ab^3ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5 = 1\rangle$	S_3
		$ab^2ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^4 = 1\rangle$	C_2
		$ab^3ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^4 = 1\rangle$	C_2
3	12	$ab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^5 = 1\rangle$	$A_4 \times C_2$
		$ab^3ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^5 = 1\rangle$	C_6
		$ab^3ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^4 = 1\rangle$	C_6
2		$ab^4ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^4 = 1\rangle$	$infinite\ group$
3	13	$ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^5 = 1\rangle$	S_3
		$ab^4ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5 = 1\rangle$	C_2
3	14	$ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^5 = 1\rangle$	C_2
3	15	$ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^5ab^5ab^5 = 1\rangle$	$\Delta(2,6,3)$
4	4	$abababab$	$\langle a,b : a^2 = b^6 = abababab = 1\rangle$	$\Delta(2,6,4)$
4	5	$abababab^2$	$\langle a,b : a^2 = b^6 = abababab^2 = 1\rangle$	C_2
4	6	$abababab^3$	$\langle a,b : a^2 = b^6 = abababab^3 = 1\rangle$	$D_4 \times C_3$
		$ababab^2ab^2$	$\langle a,b : a^2 = b^6 = ababab^2ab^2 = 1\rangle$	$C_6 \times S_3$
		$abab^2abab^2$	$\langle a,b : a^2 = b^6 = abab^2abab^2 = 1\rangle$	$infinite\ group$
4	7	$abababab^4$	$\langle a,b : a^2 = b^6 = abababab^4 = 1\rangle$	$GL(2,3)$
		$ababab^2ab^3$	$\langle a,b : a^2 = b^6 = ababab^2ab^3 = 1\rangle$	C_2
		$ababab^3ab^2$	$\langle a,b : a^2 = b^6 = ababab^3ab^2 = 1\rangle$	C_2
		$abab^2abab^3$	$\langle a,b : a^2 = b^6 = abab^2abab^3 = 1\rangle$	S_3
		$abab^2ab^2ab^2$	$\langle a,b : a^2 = b^6 = abab^2ab^2ab^2 = 1\rangle$	C_2
4	8	$abababab^5$	$\langle a,b : a^2 = b^6 = abababab^5 = 1\rangle$	D_4
		$ababab^2ab^4$	$\langle a,b : a^2 = b^6 = ababab^2ab^4 = 1\rangle$	D_2
		$ababab^4ab^2$	$\langle a,b : a^2 = b^6 = ababab^4ab^2 = 1\rangle$	$infinite\ group$
		$abab^2abab^4$	$\langle a,b : a^2 = b^6 = abab^2abab^4 = 1\rangle$	$infinite\ group$
		$ababab^3ab^3$	$\langle a,b : a^2 = b^6 = ababab^3ab^3 = 1\rangle$	$GAP4(24,8)$
		$abab^3abab^3$	$\langle a,b : a^2 = b^6 = abab^3abab^3 = 1\rangle$	$infinite\ group$
		$abab^2ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^2ab^3 = 1\rangle$	D_2
		$abab^2ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^2 = 1\rangle$	$infinite\ group$
		$abab^3ab^2ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^2 = 1\rangle$	D_2
		$ab^2ab^2ab^2ab^2$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^2 = 1\rangle$	$infinite\ group$

k	l	words	quotient group	structure
4	9	$ababab^2ab^5$	$\langle a,b : a^2 = b^6 = ababab^2ab^5 = 1\rangle$	$S_3 \times C_3$
		$ababab^5ab^2$	$\langle a,b : a^2 = b^6 = ababab^5ab^2 = 1\rangle$	$S_3 \times C_3$
		$abab^2abab^5$	$\langle a,b : a^2 = b^6 = abab^2abab^5 = 1\rangle$	$GAP4(48,33)$
		$ababab^3ab^4$	$\langle a,b : a^2 = b^6 = ababab^3ab^4 = 1\rangle$	C_6
		$ababab^4ab^3$	$\langle a,b : a^2 = b^6 = ababab^4ab^3 = 1\rangle$	C_6
		$abab^3abab^4$	$\langle a,b : a^2 = b^6 = abab^3abab^4 = 1\rangle$	C_6
		$abab^2ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^2ab^4 = 1\rangle$	$S_3 \times C_3$
		$abab^2ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^2 = 1\rangle$	$GAP4(48,33)$
		$abab^4ab^2ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^2 = 1\rangle$	$S_3 \times C_3$
		$abab^2ab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^3 = 1\rangle$	C_6
		$abab^3ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^3 = 1\rangle$	$infinite\ group$
		$abab^3ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^3ab^2 = 1\rangle$	C_6
		$ab^2ab^2ab^2ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^3 = 1\rangle$	C_6
4	10	$ababab^3ab^5$	$\langle a,b : a^2 = b^6 = ababab^3ab^5 = 1\rangle$	D_4
		$ababab^5ab^3$	$\langle a,b : a^2 = b^6 = ababab^5ab^3 = 1\rangle$	D_4
		$abab^3abab^5$	$\langle a,b : a^2 = b^6 = abab^3abab^5 = 1\rangle$	$GAP4(24,8)$
		$ababab^4ab^4$	$\langle a,b : a^2 = b^6 = ababab^4ab^4 = 1\rangle$	$GAP4(96,190)$
		$abab^4abab^4$	$\langle a,b : a^2 = b^6 = abab^4abab^4 = 1\rangle$	$infinite\ group$
		$abab^2ab^3ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^4 = 1\rangle$	$infinite\ group$
		$abab^2ab^4ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^3 = 1\rangle$	D_6
		$abab^3ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^4 = 1\rangle$	D_2
		$abab^3ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^4ab^2 = 1\rangle$	D_6
		$abab^4ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^3 = 1\rangle$	D_2
		$abab^4ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^3ab^2 = 1\rangle$	$infinite\ group$
		$ab^2ab^2ab^2ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^4 = 1\rangle$	$infinite\ group$
		$abab^3ab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^3ab^3 = 1\rangle$	D_4
		$ab^2ab^2ab^3ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3ab^3 = 1\rangle$	D_6
		$ab^2ab^3ab^2ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^2ab^3 = 1\rangle$	$infinite\ group$
4	11	$ababab^4ab^5$	$\langle a,b : a^2 = b^6 = ababab^4ab^5 = 1\rangle$	C_2
		$ababab^5ab^4$	$\langle a,b : a^2 = b^6 = ababab^5ab^4 = 1\rangle$	C_2
		$abab^4abab^5$	$\langle a,b : a^2 = b^6 = abab^4abab^5 = 1\rangle$	C_2
		$abab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^3ab^5 = 1\rangle$	S_3
		$abab^2ab^5ab^3$	$\langle a,b : a^2 = b^6 = abab^2ab^5ab^3 = 1\rangle$	C_2
		$abab^3ab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^3ab^2ab^5 = 1\rangle$	C_2
		$abab^3ab^5ab^2$	$\langle a,b : a^2 = b^6 = abab^3ab^5ab^2 = 1\rangle$	C_2

k	l	words	quotient group	structure
		$abab^5ab^2ab^3$	$\langle a,b : a^2 = b^6 = abab^5ab^2ab^3 = 1\rangle$	C_2
		$abab^5ab^3ab^2$	$\langle a,b : a^2 = b^6 = abab^5ab^3ab^2 = 1\rangle$	S_3
		$ab^2ab^2ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^2ab^5 = 1\rangle$	$GL(2,3)$
		$abab^2ab^4ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^4 = 1\rangle$	C_2
		$abab^4ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^4 = 1\rangle$	C_2
		$abab^4ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^4ab^2 = 1\rangle$	C_2
		$abab^3ab^3ab^4$	$\langle a,b : a^2 = b^6 = abab^3ab^3ab^4 = 1\rangle$	S_3
		$abab^3ab^4ab^3$	$\langle a,b : a^2 = b^6 = abab^3ab^4ab^3 = 1\rangle$	C_2
		$abab^4ab^3ab^3$	$\langle a,b : a^2 = b^6 = abab^4ab^3ab^3 = 1\rangle$	S_3
		$ab^2ab^2ab^3ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3ab^4 = 1\rangle$	C_2
		$ab^2ab^2ab^4ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^4ab^3 = 1\rangle$	C_2
		$ab^2ab^3ab^2ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^2ab^4 = 1\rangle$	S_3
		$ab^2ab^3ab^3ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^3ab^3 = 1\rangle$	C_2
4	12	$ababab^5ab^5$	$\langle a,b : a^2 = b^6 = ababab^5ab^5 = 1\rangle$	$GAP4(72,20)$
		$abab^5abab^5$	$\langle a,b : a^2 = b^6 = abab^5abab^5 = 1\rangle$	$infinite\ group$
		$abab^2ab^4ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^4ab^5 = 1\rangle$	$infinite\ group$
		$abab^2ab^5ab^4$	$\langle a,b : a^2 = b^6 = abab^2ab^5ab^4 = 1\rangle$	$infinite\ group$
		$abab^4ab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^4ab^2ab^5 = 1\rangle$	$GAP4(252,26)$
		$abab^4ab^5ab^2$	$\langle a,b : a^2 = b^6 = abab^4ab^5ab^2 = 1\rangle$	$infinite\ group$
		$abab^5ab^2ab^4$	$\langle a,b : a^2 = b^6 = abab^5ab^2ab^4 = 1\rangle$	$GAP4(252,26)$
		$abab^5ab^4ab^2$	$\langle a,b : a^2 = b^6 = abab^5ab^4ab^2 = 1\rangle$	$infinite\ group$
		$ab^2ab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^3ab^5 = 1\rangle$	$(C_7 \sim C_6) \times C_2$
		$ab^2ab^2ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5ab^3 = 1\rangle$	$(C_7 \sim C_6) \times C_2$
		$ab^2ab^3ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^2ab^5 = 1\rangle$	$infinite\ group$
4	13	$abab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^2ab^5ab^5 = 1\rangle$	C_2
		$ab^2abab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2abab^5ab^5 = 1\rangle$	C_2
		$abab^5ab^2ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^2ab^5 = 1\rangle$	C_2
		$ab^2ab^2ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^4ab^5 = 1\rangle$	C_2
		$ab^2ab^2ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5ab^4 = 1\rangle$	C_2
		$ab^2ab^4ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^2ab^5 = 1\rangle$	C_2
		$ab^3ab^3ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^2ab^5 = 1\rangle$	S_3
		$ab^3ab^3ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5ab^2 = 1\rangle$	S_3
		$ab^3ab^2ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^2ab^3ab^5 = 1\rangle$	C_2

k	l	word	quotient group	structure
4	14	$abab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^3ab^5ab^5 = 1\rangle$	D_4
		$ab^3abab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3abab^5ab^5 = 1\rangle$	D_4
		$abab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^3ab^5 = 1\rangle$	$GAP4(24,8)$
		$ab^2ab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^2ab^5ab^5 = 1\rangle$	$GAP4(96,190)$
		$ab^2ab^5ab^2ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^2ab^5 = 1\rangle$	$infinite\ group$
		$ab^4ab^4abab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4abab^5 = 1\rangle$	D_2
		$ab^4ab^4ab^5ab$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5ab = 1\rangle$	D_2
		$ab^4abab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4abab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^2ab^3ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^2ab^3ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^5ab^4 = 1\rangle$	D_6
		$ab^2ab^4ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^3ab^5 = 1\rangle$	D_2
		$ab^2ab^4ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^5ab^3 = 1\rangle$	D_6
		$ab^2ab^5ab^4ab^3$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^4ab^3 = 1\rangle$	$infinite\ group$
		$ab^2ab^5ab^3ab^4$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^3ab^4 = 1\rangle$	D_2
		$ab^3ab^3ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^4ab^4 = 1\rangle$	D_6
		$ab^3ab^4ab^3ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^3ab^4 = 1\rangle$	$infinite\ group$
4	15	$ab^3ab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^2ab^5ab^5 = 1\rangle$	C_6
		$ab^2ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^3ab^5ab^5 = 1\rangle$	C_6
		$ab^2ab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^3ab^5 = 1\rangle$	C_6
		$ab^3ab^3ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^4ab^5 = 1\rangle$	C_6
		$ab^3ab^3ab^5ab^4$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5ab^4 = 1\rangle$	C_6
		$ab^3ab^4ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^3ab^5 = 1\rangle$	$infinite\ group$
4	16	$abab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = abab^5ab^5ab^5 = 1\rangle$	D_4
		$ab^2ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^4ab^5ab^5 = 1\rangle$	D_2
		$ab^4ab^2ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^2ab^5ab^5 = 1\rangle$	D_2
		$ab^2ab^5ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^3ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^3ab^5ab^5 = 1\rangle$	$GAP4(24,8)$
		$ab^3ab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^3ab^5 = 1\rangle$	$infinite\ group$
		$ab^4ab^4ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^3ab^5 = 1\rangle$	D_2
		$ab^4ab^4ab^5ab^3$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5ab^3 = 1\rangle$	D_2
		$ab^4ab^3ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^3ab^4ab^5 = 1\rangle$	$infinite\ group$
		$ab^4ab^4ab^4ab^4$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^4ab^4 = 1\rangle$	$infinite\ group$

k	l	word	quotient group	structure
4	17	$ab^2ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^2ab^5ab^5ab^5 = 1\rangle$	$GL(2,3)$
		$ab^3ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^4ab^5ab^5 = 1\rangle$	C_2
		$ab^4ab^3ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^3ab^5ab^5 = 1\rangle$	C_2
		$ab^4ab^5ab^3ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^3ab^5 = 1\rangle$	S_3
		$ab^4ab^4ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^4ab^5 = 1\rangle$	C_2
4	18	$ab^3ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^3ab^5ab^5ab^5 = 1\rangle$	$D_4 \times C_3$
		$ab^4ab^4ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^4ab^5ab^5 = 1\rangle$	$C_6 \times S_3$
		$ab^4ab^5ab^4ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^4ab^5 = 1\rangle$	*infinite group*
4	19	$ab^4ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^4ab^5ab^5ab^5 = 1\rangle$	C_2
4	20	$ab^5ab^5ab^5ab^5$	$\langle a,b : a^2 = b^6 = ab^5ab^5ab^5ab^5 = 1\rangle$	$\Delta(2,6,4)$

References

[1] **M. Aslam, Q. Mushtaq, T. Maqsood and M. Ashiq,** *Real quadratic irrational numbers and the group* $\langle x, y : x^2 = y^6 = 1\rangle$, Southeast Asian Bull. Math. **27** (2003), 409 – 415.

[2] **İ. N. Cangül and D. Singerman,** *Normal subgroups of Hecke groups and regular maps,* Math. Proc. Camb. Phil. Soc. **123** (1998), 59 – 74.

[3] **M. D. E. Conder,** *Three relator quotients of the modular group,* Quart. J. Math. Oxford **38** (1987), 427 – 447.

[4] **M. Demirci and İ. N. Cangül,** *A class of congruence subgroups of Hecke group* $H(\lambda_5)$*,* Bull. Inst. Math. Acad. Sinicia **1** (2004), 549 – 556.

[5] **G. Havas, M.D.E Conder and M. Newman,** *One relator quotients of the modular group,* Group St Andrews, Bath. **11** August 2009.

[6] **Q. Mushtaq,** *On the word structure of the modular group over finite and real quadratic fields,* Discrete Math. **178** (1998), 155 – 164.

[7] **Q. Mushtaq and M. Aslam,** *Group generated by two elements of order two and six acting on R and* $Q(\sqrt{n})$, Discrete Math. **179** (1998), 145 – 154.

[8] **L. A. Parson,** *Normal congruence subgroups of the Hecke groups* $G(2^{(1/2)})$ *and* $G(3^{(1/2)})$, Pacific. J. Math. **70** (1977), 481 – 487.

[9] **Y. T. Ulutas and I. N. Cangül,** *One relator quotients of the Hecke group* $H(\frac{1+\sqrt{5}}{2})$, Bull. Inst. Math. Acad. Sinicia **31** (2003), 59 – 74.

[10] **Y. T. Ulutas and I. N. Cangül,** *One relator quotients of the modular group*, Bull. Inst. Math. Acad. Sinicia **32** (2004), 291 – 296.

[11] **N. Yilmaz and İ. N. Cangül,** *Power subgroups of Hecke groups* $H(\sqrt{n})$, International J. Math. and Math. Sci. **11** (2001), 703 − 708.

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