Quasigroups and Related Systems **19** (2011), 169 – 172

A note on M-hypersystems and N-hypersystems in Γ-semihypergroups

Saleem Abdullah, Muhammad Aslam and Tariq Anwar

Abstract. In this paper, we have introduced the notions of M-hypersystem and N-hypersystem in Γ-semihypergroups, and some related properties are investigated. We have also proved that left Γ-hyperideal P of a Γ-semihypergroup S is quasi-prime if and only if $S\backslash P$ is an M-hypersystem.

1. Introduction

In 1986, Sen and Saha [3] defined the notion of a Γ-semigroup as a generalization of a semigroup. Recently, Davvaz, Hila and et. al. [1, 2] introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. The notion of a Γ-hyperideal of a Γ-semihypergroup was introduced in [1].

Let S and Γ be two non-empty sets. Then S is called a Γ-*semihypergroup* if every $\gamma \in \Gamma$ is a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have $x\alpha(y\beta z) = (x\alpha y)\beta z$. Let S be a Γ-semihypergroup and $\gamma \in \Gamma$. A non-empty subset A of S is called a *sub*-Γ-*semihypergroup* of S if $x\gamma y \subseteq A$ for every $x, y \in A$. A Γ-semihypergroup S is called *commutative* if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$.

Example 1.1. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$, we define $\gamma : S^2 \to P^*(S)$ by $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$. Then γ is hyperoperation. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have $(x\alpha y)\beta z = \left[0, \frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z)$. Thus S is a Γ-semihypergroup. □

Example 1.2. Let $(S, \circ)$ be a semihypergroup and Γ be a non-empty subset of S. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Thus S is a Γ-semihypergroup. □

2010 Mathematics Subject Classification: 20N20, 20M17
Keywords: Γ-semihypergroups, quasi-semiprime, M-hypersystem, N-hypersystem.

Example 1.3. Let $S = (0,1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation γ_n on S as follows

$$x\gamma_n y = \left\{\frac{xy}{2^k} \mid 0 \leqslant k \leqslant n\right\}.$$

Then $x\gamma_n y \subset S$ and for every $m, n \in \mathbb{N}$ and $x, y, z \in S$

$$(x\gamma_n y)\gamma_m z = \left\{\frac{xyz}{2^k} \mid 0 \leqslant k \leqslant n+m\right\} = x\gamma_n\,(y\gamma_m z)\,.$$

So, S is a Γ-semihypergroup. □

A Γ-semihypergroup S is called *regular* if for all $a \in S$ and $\alpha, \beta \in \Gamma$ there exists $x \in S$ such that $a \in a\alpha x\beta a$.

A non-empty subset A of S is a *left* (*right*) Γ-*hyperideal* of S if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$). A Γ-*hyperideal* is both a left and right Γ-hyperideal.

A left Γ-hyperideal P is *quasi-prime* if for any left Γ-hyperideals A and B such that $A\Gamma B \subseteq P$ it follows $A \subseteq P$ or $B \subseteq P$.

A left Γ-hyperideal P is *quasi-prime* P is *quasi-semiprime* if any left Γ-hyperideal A from $A\Gamma A \subseteq P$ it follows $A \subseteq P$.

2. M-hypersystem and N-hypersystem

A Γ-semihypergroup S is called *fully* Γ-*hyperidempotent* if every Γ-hyperideal is idempotent.

Proposition 2.1. *If S is Γ-semihypergroup and A, B are Γ-hyperideal of S, then the following are equivalent:*

(a) *S is fully Γ-hyperidempotent,*

(b) $A \cap B = \langle A\Gamma B\rangle$,

(c) *the set of all Γ-hyperideals of S form a semilattice $(L_S, \wedge)$, where $A \wedge B = \langle A\Gamma B\rangle$.*

Proof. $(a) \Rightarrow (b)$ Always hold $A\Gamma B \subseteq A \cap B$, for any Γ-hyperideals A and B of S. Hence $\langle A\Gamma B\rangle \subseteq A \cap B$.

Converse let $x \in A \cap B$. If $\langle x\rangle$ denote the principle left Γ-hyperideal generated by x, then $x \in \langle x\rangle = \langle x\rangle\Gamma\langle x\rangle \subseteq \langle A\Gamma B\rangle$. Thus $x \in \langle A\Gamma B\rangle$. Therefore $A \cap B \subseteq \langle A\Gamma B\rangle$, which proves (b).

$(b) \Rightarrow (c)$ $A \wedge B = \langle A\Gamma B\rangle = A \cap B = B \cap A = \langle B\Gamma A\rangle = B \wedge A$.

$(c) \Rightarrow (b)$ Let $(L_S, \wedge)$ be a semilattice. Then $A = A \wedge A = \langle A\Gamma A\rangle = A\Gamma A$. Hence S is fully Γ-hyperidempotent. □

Corollary 2.2. *If* Γ*-semihypergroup* S *is regular, then* $S = S\Gamma S$. $\square$

A subset M of Γ-semihypergroup S is called an M*-hypersystem* if for all $a, b \in M$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \subseteq M$.

A subset N of Γ-semihypergroup S is called an N*-hypersystem* if for all $a \in N$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \subseteq N$.

Obviously, each M-hypersystem is an N-hypersystem.

Example 2.3. The set $S_i = \left(0, 2^{-i}\right)$, where $i \in \mathbb{N}$, is an M-hypersystem of a Γ-semihypergroup S defined in Example 1.3. The set $T_i = \left(0, 4^{-i}\right)$, where $i \in \mathbb{N}$, is its an N-hypersystem of S. $\square$

Example 2.4. The set $T = [0, t]$, where $t \in [0, 1]$, is an M-hypersystem and an N-hypersystem of a Γ-semihypergroup defined in Example 1.1. $\square$

Theorem 2.5. *Let* P *be a left* Γ*-hyperideal of* Γ*-semihypergroup* S. *Then the following are equivalent:*

(1) P *is a quasi-prime,*

(2) $A\Gamma B = \langle A\Gamma B\rangle \subseteq P \Rightarrow A \subseteq P$ *or* $B \subseteq P$ *for all left* Γ*-hyperideals,*

(3) $A \nsubseteq P$ *or* $B \nsubseteq P \Rightarrow A\Gamma B \nsubseteq P$ *for all left* Γ*-hyperideals,*

(4) $a \notin P$ *or* $b \notin P \Rightarrow a\Gamma b \nsubseteq P$ *for all* $a, b \in S$,

(5) $a\Gamma b \subseteq P \Rightarrow a \in P$ *or* $b \in P$ *for all* $a, b \in S$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is straightforward.

(1) $\Leftrightarrow$ (4) Let $\langle a\rangle\Gamma\langle b\rangle \subseteq P$. Then by (1) either $\langle a\rangle \subseteq P$ or $\langle b\rangle \subseteq P$, which implies that either $a \in P$ or $b \in P$.

(4) $\Rightarrow$ (2) Let $A\Gamma B \subseteq P$. If $a \in A$ and $b \in B$, then $\langle a\rangle\Gamma\langle b\rangle \subseteq P$, now by (4) either $a \in P$ or $b \in P$, which implies that either $A \subseteq P$ *or* $B \subseteq P$.

(1) $\Rightarrow$ (5) Let P be a left Γ-hyperideal of Γ-semihypergroup S and $a\Gamma S\Gamma b \subseteq P$. Then, by $(2), (3)$ and (1), we get $S\Gamma(a\Gamma S\Gamma b) \subseteq S\Gamma P \subseteq P$, that is, $S\Gamma(a\Gamma S\Gamma b) = (S\Gamma a)\Gamma(S\Gamma b)$. Thus, $(S\Gamma a)\Gamma(S\Gamma b) \subseteq P$ implies either $S\Gamma a \subseteq P$ or $S\Gamma b \subseteq P$.

Since $S\Gamma a$ and $S\Gamma b$ are left Γ-hyperideals, for $L(a) = (a \cup S\Gamma a)$ we have

$$\begin{aligned} L(a)\Gamma L(a)\Gamma L(a) &= (a \cup S\Gamma a)\,\Gamma\,(a \cup S\Gamma a)\,\Gamma\,(a \cup S\Gamma a) \\ &\subseteq a\Gamma a \cup a\Gamma S\Gamma a \cup S\Gamma a\Gamma a \cup \subseteq S\Gamma a\Gamma S\Gamma a\Gamma a \cup S\Gamma a \\ &\subseteq S\Gamma a \subseteq P. \end{aligned}$$

Hence $L(a)\Gamma L(a)\Gamma L(a) = (L(a)\Gamma L(a))\Gamma L(a) \subseteq P$. Since P is quasi-prime and $L(a)\Gamma L(a)$ is a left Γ-hyperideal of S we have $L(a)\Gamma L(a) \subseteq P$

or $L(a) \subseteq P$. If $L(a) \subseteq P$, then $a \in L(a) \subseteq P$. Let $L(a)\Gamma L(a) \subseteq P$. Since P is quasi-prime, $L(a) \subseteq P$. Thus, $a \in L(a) \subseteq P$, i.e., $a \in P$.

(5) $\Rightarrow$ (1) Assume that $A\Gamma B \subseteq P$, wher A and B are left Γ-hyperideals of S such that $A \nsubseteq P$. Then there exist $x \in A$ such that $x \notin P$. Hence $x\Gamma S\Gamma y \subseteq A\Gamma S\Gamma B \subseteq A\Gamma B \subseteq P$ for all $y \in B$. Then, by (5), $y \in P$. $\square$

Proposition 2.6. *A left Γ-hyperideal P of Γ-semihypergroup S is quasi-prime if and only if $S\backslash P$ is an M-hypersystem.*

Proof. Let $S\backslash P$ be an M-hypersystem and $a\Gamma S\Gamma b \subseteq P$ for some $a, b \in S\backslash P$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \subseteq S\backslash P$. This implies that $a\alpha x\beta b \nsubseteq P$, which is a contradiction. Hence either $a \in P$ or $b \in P$.

Conversely, if P is quasi-prime and $x, y \in S\backslash P$, then for $z \in S$ and $\alpha, \beta \in \Gamma$ such that $x\alpha z\beta y \nsubseteq S\backslash P$ we haven $x\alpha z\beta y \subseteq P$, i.e., either $x \in P$ or $y \in P$. So, $S\backslash P$ is an M-hypersystem. $\square$

Proposition 2.7. *A left Γ-hyperideal P of Γ-semihypergroup S is quasi-semiprime if and only if $S\backslash P$ is an N-hypersystem.*

Proof. Let $S\backslash P$ be an N-hypersystem and $a\Gamma S\Gamma a \subseteq P$ with $a \notin P$. Then $a\alpha x\beta b \subseteq S\backslash P$ for some $x \in S$ and $\alpha, \beta \in \Gamma$. Thus $a\alpha x\beta a \nsubseteq P$, which is a contradiction. Hence $a \in P$. The converse statement is obvious. $\square$

Theorem 2.8. *Let S be Γ-semihypergroup and P a proper left Γ-hyperideal of S. Then the following are equivalent:*

(1) *P is quasi-prime,*
(2) *$a\Gamma M\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$,*
(3) *$S\backslash P$ is an M-system,*
(4) *$S\backslash P$ is an N-system.* $\square$

References

[1] **S. M. Anvariyeh, S. Mirvakili, and B. Dawaz**, *On Γ-hyperideals in Γ-semihypergroups*, Carpathian J. Math. **26** (2010), 11 – 23.

[2] **D. Heidari, S. O. Dehkordi and B. Davvaz,** *Γ-semihypergroups and their properties,* U.P.B. Sci. Bull., Series A, **72** (2010), 197 – 210.

[3] **M. K. Sen, and N. K. Saha**, *On Γ-semigroup, I*, Bull. Cal. Math. Soc. **78** (1986), 180 – 186.

Received October 19, 2010

Department of Mathematics Quaid-i-Azam University, Islamabad 45320, Pakistan.