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Four lectures on formal nonassociative Lie theory

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Abstract. This survey corresponds to four lectures on nonassociative Lie theory that will be held at Workshops Loops'11, Třešt', July 21 – 23, 2011. In the first lecture we focus on the importance of the space of distributions in the algebraic treatment of local loops. Formal loops replace local loops in the second lecture, where the classification of formal loops in terms of Sabinin algebras is presented. The geometrical meaning of this classification is the topic of the third lecture. The non-existence of quantum loops is discussed in the final lecture.

1. The bialgebra of distributions of a local loop

1.1. Basic definitions and structures

1.1.1. Distributions with support at a point. Let Q be an n-dimensional smooth manifold, $e \in Q$ and $(U, (x^1, \dots, x^n))$ a coordinate neighborhood of e. Let $\partial_i = \partial/\partial x^i$ and for any $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ define elements $\partial_I|_e$ in the dual space $\mathcal{C}^\infty(Q)^*$ of $\mathcal{C}^\infty(Q)$ by $\partial_I|_e = \partial_1^{i_1} \cdots \partial_n^{i_n}|_e$ if $|I| = i_1 + \cdots + i_n \geqslant 1$, and $\partial_I|_e \colon f \mapsto f(e)$ the Dirac delta δ_e in case that $|I| = 0$. Linear combinations of $\{\partial_I|_e \mid I \subseteq \mathbb{N}^n\}$ are called *distributions on Q with support at e*, and they form a vector space that we will denote by $\mathcal{D}'_e(Q)$.

1.1.2. Exercises.

(1) Prove that $\mathcal{D}'_e(Q)$ does not depend on the particular choice of the coordinate neighborhood of e.
(2) Prove that $\{\partial_I|_e \mid I \subseteq \mathbb{N}^n\}$ is a basis of $\mathcal{D}'_e(Q)$.
(3) Let $(U, (x^1, ..., x^n))$, $(V, (y^1, ..., y^m))$ and $(U \times V, (x^1, ..., x^n, y^1, ..., y^m))$ be coordinate neighborhoods of $e_1 \in Q_1$, $e_2 \in Q_2$ and $(e_1, e_2) \in$

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$Q_1 \times Q_2$ respectively. Prove that the map

$$\partial_I|_{x=e_1} \otimes \partial_J|_{y=e_2} \mapsto \partial_I \partial_J|_{(x,y)=(e_1,e_2)}$$

induces a linear isomorphism $\mathcal{D}'_{e_1}(Q_1) \otimes \mathcal{D}'_{e_2}(Q_2) \cong \mathcal{D}'_{(e_1,e_2)}(Q_1 \times Q_2)$.

(4) Prove that $\mathcal{D}'_e(e) = \mathbb{R}\delta_e$ for the zero-dimensional manifold $\{e\}$.

1.1.3. The linearization functor. By the Chain rule, any smooth map $\varphi\colon Q_1 \to Q_2$ induces a corresponding linear map

$$\begin{array}{rcl} \varphi'\colon \mathcal{D}'_e(Q_1) & \to & \mathcal{D}'_{\varphi(e)}(Q_2)\,, \\ \mu & \mapsto & \varphi'(\mu)\colon f \mapsto \mu(f \circ \varphi)\,. \end{array}$$

The assignment

$$(Q, e) \mapsto \mathcal{D}'_e(Q), \quad \varphi \mapsto \varphi'$$

defines a covariant functor from the category of smooth manifolds with base point to the category of vector spaces.

1.1.4. Examples.

(1) (*Twist map*) With the identification in Exercise 1.1.2 (3), the map

$$\begin{array}{rcl} \sigma\colon Q_1 \times Q_2 & \to & Q_2 \times Q_1 \\ (x, y) & \mapsto & (y, x) \end{array}$$

induces a corresponding map

$$\begin{array}{rcl} \sigma'\colon \mathcal{D}'_{e_1}(Q_1) \otimes \mathcal{D}'_{e_2}(Q_2) & \to & \mathcal{D}'_{e_2}(Q_2) \otimes \mathcal{D}'_{e_1}(Q_1) \\ \mu \otimes \nu & \mapsto & \nu \otimes \mu \end{array}$$

on distributions.

(2) (*Inclusion map*) The inclusion $\iota\colon e \to Q$ induces

$$\begin{array}{rcl} \iota'\colon \mathbb{R}\delta_e & \to & \mathcal{D}'_e(Q)\,, \\ \delta_e & \mapsto & \delta_e. \end{array}$$

(3) (*Constant map*) The constant map $\kappa\colon Q \to e$ induces

$$\begin{array}{rcl} \kappa'\colon \mathcal{D}'_e(Q) & \to & \mathbb{R}\delta_e \\ \partial_I|_e & \mapsto & \left\{ \begin{array}{ll} 0 & \text{if} \quad |I| \neq 0 \\ \delta_e & \text{if} \quad |I| = 0. \end{array} \right. \end{array}$$

The map $\epsilon = \kappa'$ will be called the *counit* of $\mathcal{D}'_e(Q)$.

(4) (*Projection*) With the natural identifications, the projection

$$\begin{aligned}\pi_1\colon Q_1\times Q_2 &\to Q_1\\ (x,y)&\mapsto x\end{aligned}$$

induces a corresponding map

$$\begin{aligned}\pi_1'\colon \mathcal{D}'_{e_1}(Q_1)\otimes\mathcal{D}'_{e_2}(Q_2) &\to \mathcal{D}'_{e_1}(Q_1)\\ (\mu\otimes\nu)&\mapsto \epsilon(\nu)\mu,\end{aligned}$$

where ϵ denotes the counit of $\mathcal{D}'_{e_2}(Q_2)$.

(5) (*Diagonal map*) The diagonal map $\delta\colon Q\mapsto Q\times Q\ x\mapsto(x,x)$ induces

$$\begin{aligned}\delta'\colon \mathcal{D}'_e(Q) &\to \mathcal{D}'_e(Q)\otimes\mathcal{D}'_e(Q)\\ \partial_I|_e &\mapsto \sum_{I=I'+I''}\frac{I!}{I'!I''!}\partial_{I'}|_e\otimes\partial_{I''}|_e,\end{aligned}\tag{1.1}$$

where $(i_1,\dots,i_n)! = i_1!\cdots i_n!$ The map $\Delta=\delta'$ will be called the *comultiplication* of $\mathcal{D}'_e(Q)$.

1.1.5. The coalgebra of distributions with support at a point. A *coassociative coalgebra* (or simply *coalgebra*) is a $\boldsymbol{k}$-vector space C endowed with two linear maps $\Delta\colon C\to C\otimes C$ (*comultiplication*) and $\epsilon\colon C\to\boldsymbol{k}$ (*counit*) such that the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "I\otimes\Delta"] \\ C\otimes C \arrow[r, "\Delta\otimes I"] & C\otimes C\otimes C \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} & C \arrow[dl, "I"'] \arrow[d, "\Delta"] \arrow[dr, "I"] & \\ C & C\otimes C \arrow[l, "\epsilon\otimes I"'] \arrow[r, "I\otimes\epsilon"] & C \end{tikzcd}$$

Following *Sweedler sigma notation*, the linear combination of homogeneous tensors that represents $\Delta(\mu)$ will be written as

$$\sum\mu_{(1)}\otimes\mu_{(2)}.$$

The coalgebra (C,Δ,ϵ) is called *cocommutative* if $\Delta=\Delta^{\mathrm{op}}$ where

$$\Delta^{\mathrm{op}}(\mu)=\sum\mu_{(2)}\otimes\mu_{(1)}.$$

Given two coalgebras $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$, a *coalgebra morphism* is a linear map $\psi\colon C_1 \to C_2$ that verifies $\Delta_2 \circ \psi = (\psi \otimes \psi) \circ \Delta_1$ and $\epsilon_2 \circ \psi = \epsilon_1$.

The linearizations of the the different commutative diagrams in the first column of Table 1 show that $(\mathcal{D}'_e(Q), \Delta, \epsilon)$ is a cocommutative coalgebra.

Manifolds	Distributions
$\begin{tikzcd} Q \arrow[r, "\delta"] \arrow[d, "\delta"'] & Q \times Q \arrow[d, "I \times \delta"] \\ Q \times Q \arrow[r, "\delta \times I"] & Q \times Q \times Q \end{tikzcd}$	$\begin{tikzcd} \mathcal{D}'_e(Q) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \arrow[d, "I \otimes \Delta"] \\ \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \arrow[r, "\Delta \otimes I"] & \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \end{tikzcd}$
$\begin{tikzcd} & Q \arrow[dl, "I"'] \arrow[d, "\delta"] \arrow[dr, "I"] & \\ Q & Q \times Q \arrow[l, "\pi_1 \times I"'] \arrow[r, "I \times \pi_2"] & Q \end{tikzcd}$	$\begin{tikzcd} & \mathcal{D}'_e(Q) \arrow[dl, "I"'] \arrow[d, "\Delta"] \arrow[dr, "I"] & \\ \mathcal{D}'_e(Q) & \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \arrow[l, "\epsilon \otimes I"'] \arrow[r, "I \otimes \epsilon"] & \mathcal{D}'_e(Q) \end{tikzcd}$
$\begin{tikzcd} & Q \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ Q \times Q \arrow[rr, "\sigma"] & & Q \times Q \end{tikzcd}$	$\begin{tikzcd} & \mathcal{D}'_e(Q) \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \arrow[rr, "\sigma'"] & & \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \end{tikzcd}$

Table 1.

1.1.6. Exercises.

(1) Given a coassociative coalgebra (C, Δ, ϵ) let $\Delta_i\colon C \otimes C \otimes \cdots \otimes C \to C \otimes C \otimes \cdots \otimes C$ be the map that acts as Δ on the ith slot and as the identity on the others. Prove that for a fixed r, the map $\Delta^r = \Delta_{i_r} \cdots \Delta_{i_2} \Delta$ does not depend on the particular values $i_2, \ldots, i_r$. The image of μ under Δ^r is denoted by $\sum \mu_{(1)} \otimes \cdots \otimes \mu_{(r+1)}$.

(2) Given a coassociative coalgebra (C, Δ, ϵ) prove that

(*a*) the dual space C^* is a unital associative algebra with the *convolution product*

$$f * g(\mu) = \sum f(\mu_{(1)}) g(\mu_{(2)})$$

and identity element ϵ;

(*b*) any coalgebra morphism $\psi\colon C_1 \to C_2$ induces an algebra homomorphism $\psi^*\colon C_2^* \to C_1^*$ $f \mapsto f \circ \psi$.

(3) Prove that in the coalgebra $(\mathcal{D}'_e(Q), \Delta, \epsilon)$ we have

$$\Delta(\delta_e) = \delta_e \otimes \delta_e \quad \text{and} \quad \epsilon(\delta_e) = 1.$$

(4) Prove that the *tangent space* T_eQ of Q at e is

$$T_eQ = \mathbb{R}\langle \partial_1|_e, \dots, \partial_n|_e\rangle = \{\mu \in \mathcal{D}'_e(Q) \mid \Delta(\mu) = \mu \otimes \delta_e + \delta_e \otimes \mu\}.$$

1.1.7. Local loops. A *local loop* (Q, xy, e) is a smooth manifold Q with a distinguished point e such that on a neighborhood U of e there is defined a smooth binary product $U \times U \to Q$ $(x, y) \mapsto xy$ with $xe = x = ex$ for all $x \in U$. The Inverse Function Theorem ensures that around e there are defined a *left* and a *right division*, denoted by $x\backslash y$ and x/y respectively, such that

$$x\backslash(xy) = y = x(x\backslash y) \quad \text{and} \quad (xy)/y = x = (x/y)y$$

for any x, y in a neighborhood of e.

1.1.8. The bialgebra of distributions with support at the identity. A (unital) *bialgebra* is a coalgebra (B, Δ, ϵ) endowed with two extra linear maps $m\colon B \otimes B \to B$ (*product*) and $u\colon \boldsymbol{k} \to B$ (*unit*) that make commutative the following diagrams:

$$\begin{tikzcd}
B \otimes \boldsymbol{k} \arrow[r, "\mathrm{I}\otimes u"] \arrow[d, "\cong"'] & B \otimes B \arrow[dl, "m"'] \arrow[d, "u\otimes \mathrm{I}"] \\
B \arrow[r, "\cong"] & \boldsymbol{k} \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "(\mathrm{I}\otimes\sigma\otimes\mathrm{I})(\Delta\otimes\Delta)"] \arrow[d, "m"'] & B \otimes B \otimes B \otimes B \arrow[d, "m\otimes m"] \\
b \arrow[r, "\Delta"] & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & \boldsymbol{k} \otimes \boldsymbol{k} \arrow[d, "\text{product}"] \\
B \arrow[r, "\epsilon"] & \boldsymbol{k}
\end{tikzcd}$$

The identity element of B is $1_B = u(1_{\boldsymbol{k}})$. Elements $x \in B$ with $\Delta(x) = x \otimes 1_B + 1_B \otimes x$ are called *primitive* and they form a subspace $\mathrm{Prim}(B)$.

Given a local loop $(Q, xy, e, \backslash, /)$, the maps $m\colon (x,y) \to xy$ and $\iota\colon e \to Q$ induce corresponding maps on distributions, that we will denote by m and ι respectively. The commutativity of the diagrams

$$\begin{tikzcd}
Q \times e \arrow[r, "\mathrm{I} \times \iota"] \arrow[d, "\pi_1"'] & Q \times Q \arrow[d, "\iota \times \mathrm{I}"] \arrow[dl, "m"'] \\
Q \arrow[r, "\pi_2"] & e \times Q
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \times Q \arrow[r, "(\mathrm{I} \times \sigma \times \mathrm{I})(\delta \times \delta)"] \arrow[d, "m"'] & Q \times Q \times Q \times Q \arrow[d, "m \times m"] \\
Q \arrow[r, "\delta"] & Q \times Q
\end{tikzcd}$$

$$\begin{tikzcd}
Q \times Q \arrow[r, "\kappa \times \kappa"] \arrow[d, "m"'] & e \times e \arrow[d] \\
Q \arrow[r, "\kappa"] & e
\end{tikzcd}$$

shows that $(\mathcal{D}'_e(Q), \Delta, \epsilon, m, \iota)$ is a bialgebra with identity element $1_{\mathcal{D}'_e(Q)} = \delta_e$. The left an right divisions $\backslash$ and $/$ also induce *left* and *right division maps* $\backslash$ and $/$ on distributions. The linearization of the identities

$$x\backslash(xy) = y = x(x\backslash y) \quad \text{and} \quad (xy)/y = x = (x/y)y \quad \forall x, y \in Q$$

leads to

$$\sum \mu_{(1)}\backslash(\mu_{(2)}\nu) = \epsilon(\mu)\nu = \sum \mu_{(1)}(\mu_{(2)}\backslash\nu) \tag{1.2}$$

$$\sum(\mu\nu_{(1)})/\nu_{(2)} = \epsilon(\nu)\mu = \sum(\mu/\nu_{(1)})\nu_{(2)} \tag{1.3}$$

for all $\mu, \nu \in \mathcal{D}'_e(Q)$, where $\mu\nu = m(\mu, \nu)$. By Exercise 1.1.6 (4) the tangent space of Q is recovered as

$$T_eQ = \mathrm{Prim}(\mathcal{D}'_e(Q))$$

1.1.9. Exercises.

(1) Prove that in $\mathcal{D}'_e(Q)$

$$\Delta(\mu\backslash\nu) = \sum \mu_{(1)}\backslash\nu_{(1)} \otimes \mu_{(2)}\backslash\nu_{(2)} \quad \text{and}$$
$$\Delta(\mu/\nu) = \sum \mu_{(1)}/\nu_{(1)} \otimes \mu_{(2)}/\nu_{(2)}.$$

(2) Given a unital bialgebra B and $a, b \in \mathrm{Prim}(B)$ prove that $[a,b] = ab - ba \in \mathrm{Prim}(B)$.

1.1.10. Poincaré-Birkhoff-Witt type bases for $\mathcal{D}'_e(Q)$. Given a local loop (Q, xy, e) and $\partial_{i_1}|_e \cdots \partial_{i_r}|_e$ the product in $\mathcal{D}'_e(Q)$ of $\partial_{i_1}|_e, \dots, \partial_{i_r}|_e$ with an unspecified order of parentheses then the Chain rule implies that

$$\partial_{i_1}|_e \cdots \partial_{i_r}|_e = \partial_{i_1} \cdots \partial_{i_r}|_e + \text{ linear combination of } \partial_I|_e \text{ with } |I| < r. \tag{1.4}$$

Recall that a *filtration* of an algebra A is an increasing chain of subspaces $A_0 \subseteq A_1 \subseteq \cdots$ such that $A = \cup_{i=0}^{\infty} A_n$ and $A_p A_q \subseteq A_{p+q}$ for any $p, q \geqslant 0$. Any filtration of A induces a *graded algebra* $\mathrm{Gr}(A) = \oplus_{i=0}^{\infty} A_i/A_{i-1}$ (where $A_{-1} = 0$) with the product

$$(x_p + A_{p-1})(y_q + A_{q-1}) = x_p y_q + A_{p+q-1}.$$

The subspaces $\mathcal{D}'_e(Q)_r = \mathbb{R}\langle \partial_I|_e \mid |I| \leqslant r\rangle$ $(r \geqslant 0)$ define a filtration of $\mathcal{D}'_e(Q)$ and (1.4) is equivalent to fact that the associated graded algebra is isomorphic to the symmetric algebra $\boldsymbol{k}[T_e Q]$. The set of ordered right normed monomials

$$\{((\partial_{i_1}|_e \partial_{i_2}|_e) \cdots)\partial_{i_r}|_e \mid r \geqslant 0 \quad \text{and} \quad i_1 \leqslant \cdots \leqslant i_r\}$$

is a basis of $\mathcal{D}'_e(Q)$.

1.2. Examples of algebraic structures induced on the tangent spaces of local loops

1.2.1. Lie groups. A local *Lie group* (G, xy, e) is a local analytic loop that satisfies the *associative* identity

$$(xy)z = x(yz).$$

In terms of diagrams this identity is written as

$$\begin{tikzcd} G \times G \times G \arrow[r, "I \times m"] \arrow[d, "m \times I"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

where $m(x,y) = xy$. The linearization of this diagram shows that the product on $\mathcal{D}'_e(G)$ is associative. The inverse map $x \mapsto x^{-1}$ on G induces a map

$$S\colon \mathcal{D}'_e(G) \to \mathcal{D}'_e(G)$$

with

$$\sum S(\mu_{(1)})\mu_{(2)} = \epsilon(\mu)1_{\mathcal{D}'_e(G)} = \sum \mu_{(1)}S(\mu_{(2)}) \tag{1.5}$$

so $(\mathcal{D}'_e(G), \Delta, \epsilon, m, \iota, S)$ is a unital *Hopf algebra* (i.e., an associative bialgebra with a map S, the *antipode*, satisfying (1.5)).

The *commutator product* $[x,y] = xy - yx$ in any associative algebra satisfies

$$\begin{aligned} (\textit{Skew-commutativity}) \quad & [x,y] = -[y,x] \quad \text{and} \\ (\textit{Jacobi identity}) \quad & [[x,y],z] + [[y,z],x] + [[z,x],y] = 0. \end{aligned}$$

Algebras with a skew-commutative product $[\,,\,]$ that satisfies the Jacobi identity are called *Lie algebras*. Exercise 1.1.9 (2) shows that the tangent space of a local Lie group at the identity element is a Lie algebra.

1.2.2. Exercises.

(1) Prove that any associative algebra with the commutator product is a Lie algebra.

(2) Prove that a local loop (Q, xy, e) is a local Lie group if and only if $\mathcal{D}'_e(Q)$ is associative.

(3) Prove that a local Lie group (G, xy, e) is abelian if and only if $T_e(G)$ is an *abelian Lie algebra* (i.e., $[\alpha, \beta] = 0$ for all $\alpha, \beta \in T_eG$).

1.2.3. Moufang loops. A local *Moufang loop* is a local loop (Q, xy, e) that satisfies any of the following equivalent identities

$$x(y(xz)) = ((xy)x)z, \quad (xy)(zx) = x((yz)x) \quad \text{and} \quad ((xy)z)y = x(y(zy)).$$

The linearization of these identities shows that in $\mathcal{D}'_e(Q)$

$$\begin{aligned} \sum \mu_{(1)}(\nu(\mu_{(2)}\eta)) &= \sum((\mu_{(1)}\nu)\mu_{(2)})\eta, && (1.6) \\ \sum(\mu_{(1)}\nu)(\eta\mu_{(2)}) &= \sum \mu_{(1)}((\nu\eta)\mu_{(2)}), && (1.7) \\ \sum((\mu\nu_{(1)})\eta)\nu_{(2)} &= \sum \mu(\nu_{(1)}(\eta\nu_{(2)})). && (1.8) \end{aligned}$$

Unital cocommutative bialgebras with left and right division satisfying (1.2) and (1.3) that also satisfy any of the equivalent identities (1.6), (1.7) or (1.8) are called unital cocommutative *Moufang-Hopf algebras.*

For any algebra A the (generalized) *alternative nucleus* of A is defined as

$$\mathrm{N}_{\mathrm{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a)\}$$

where $(x, y, z) = (xy)z - x(yz)$ denotes the *associator*. A *Malcev algebra* is a vector space M endowed with a skew-symmetric bilinear product $[\,,]$ such that

$$\mathrm{J}(x, y, [x, z]) = [\mathrm{J}(x, y, z), x]$$

where $\mathrm{J}(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$ denotes the *jacobian* of x, y and z.

Proposition 1.1. *For any algebra A, $\mathrm{N}_{\mathrm{alt}}(A)$ is a Malcev algebra with the commutator product.* □

Identities (1.7) and (1.8) imply that $\mathrm{Prim}(\mathcal{D}'_e(Q)) \subseteq \mathrm{N}_{\mathrm{alt}}(\mathcal{D}'_e(Q))$. Proposition 1.1 and Exercise 1.1.9 (2) then show that the tangent space at the identity element of any local Moufang loop is a Malcev algebra.

1.2.4. Exercises. L_a, R_a will denote the *left* and *right multiplication operators* by a. Recall that a *derivation* of an algebra A is a linear map $d\colon A \to A$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$.

(1) Prove that any commutative local Moufang loop is a commutative local Lie group. It is essential that the characteristic of the ground field, the real numbers, is different from 3 since over fields of characteristic 3 there exist (formal) commutative Moufang loops that are not groups [1].

(2) Given an algebra A, triples $(d_1, d_2, d_3) \in \mathrm{End}(A)^3$ such that

$$d_1(xy) = d_2(x)y + xd_3(y) \quad \forall x, y \in A$$

are called *ternary derivations* of A. Prove that the set $\mathrm{Tder}(A)$ of all ternary derivations of A is a Lie algebra with the componentwise commutator product and that

$$a \in \mathrm{N}_{\mathrm{alt}}(A) \Leftrightarrow (L_a, L_a + R_a, -L_a), (R_a, -R_a, L_a + R_a) \in \mathrm{Tder}(A).$$

(3) Prove that in any Moufang loop there exists a map $x \mapsto x^{-1}$ such that $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Conclude that for any local Moufang loop (Q, xy, e) there exists a map $S\colon \mathcal{D}'_e(Q) \to \mathcal{D}'_e(Q)$, the *antipode*, with $S^2 = \mathrm{I}$ and such that

$$\sum S(\mu_{(1)})(\mu_{(2)}\nu) = \epsilon(\mu)\nu = \sum(\nu\mu_{(1)})S(\mu_{(2)})\,.$$

(4) Prove that for any algebra A, $a, b \in \mathrm{N}_{\mathrm{alt}}(A)$ and $x \in A$ we have that

(i) $L_{ax} = L_aL_x + [R_a, L_x]$, $L_{xa} = L_xL_a + [L_x, R_a]$,

(ii) $R_{ax} = R_xR_a + [R_x, L_a]$, $R_{xa} = R_aR_x + [L_a, R_x]$,

(iii) $[L_a, R_b] = [R_a, L_b]$,

(iv) $[L_a, L_b] = L_{[a,b]} - 2[R_a, L_b]$, $[R_a, R_b] = -R_{[a,b]} - 2[L_a, R_b]$,

(v) The map $D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ is a derivation of A, $D_{a,b} = \mathrm{ad}_{[a,b]} - 3[L_a, R_b]$ and $2D_{a,b} = \mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b]$, where ad_a denotes the map $x \mapsto [a, x]$.

1.2.5. Bol loops. A local *right Bol loop* is a local loop (Q, xy, e) that satisfies the *right Bol identity*

$$((xy)z)y = x((yz)y).$$

The linearization of this identity shows that $\mathcal{D}'_e(Q)$ satisfies

$$\sum((\mu\nu_{(1)})\eta)\nu_{(2)} = \sum \mu((\nu_{(1)}\eta)\nu_{(2)}). \tag{1.9}$$

A vector space V equipped with a trilinear operation $[a, b, c]$ is called a *Lie triple system* if

$$[a, b, b] = 0\,,$$
$$[a, b, c] + [b, c, a] + [c, a, b] = 0\,,$$
$$[[a, b, c], x, y] = [[a, x, y], b, c] + [a, [b, x, y], c] + [a, b, [c, x, y]]$$

for all $x, y, a, b, c \in V$. A *right Bol algebra* $(V, [\ ,\ ,\ ], [\ ,\ ])$ is a Lie triple system $(V, [\ ,\ ,\ ])$ with an additional bilinear skew-symmetric operation $[a, b]$ satisfying

$$[[a, b], c, d] = [[a, c, d], b] + [a, [b, c, d]] + [[c, d], a, b] + [[a, b], [c, d]]. \tag{1.10}$$

Given an arbitrary algebra A, the *right alternative nucleus* is defined as

$$\mathrm{RN}_{\mathrm{alt}}(A) = \{a \in A \mid (x, y, a) = -(x, a, y) \quad \forall x, y \in A\}\,.$$

Proposition 1.2. *For any algebra A, $\mathrm{RN}_{\mathrm{alt}}(A)$ is a Lie triple system with the triple product*

$$[a,b,c] = (ab)c - (ac)b - (bc - cb)a.$$

Any subspace V of $\mathrm{RN}_{\mathrm{alt}}(A)$ closed under the triple product $[\ ,\ ,\ ]$ and the opposite $\langle a,b\rangle = ba - ab$ of the commutator product is a right Bol algebra with these operations. $\square$

Identity (1.9) implies that $\mathrm{Prim}(\mathcal{D}'_e(Q)) \subseteq \mathrm{RN}_{\mathrm{alt}}(\mathcal{D}'_e(Q))$ so the tangent space to any local right Bol loop is a right Bol algebra.

1.2.6. Exercises.

(1) Prove that in any right Bol loop $R_x^{-1} = R_{e/x}$ where e denotes the identity element.

(2) A local *right Bruck loop* is a local right Bol loop with the *automorphic inverse property*

$$(xy)^{-1} = x^{-1}y^{-1}$$

where $x^{-1} = e/x$. Prove that the binary product $[\ ,\ ]$ of the right Bol algebra T_eQ vanishes for any local right Bruck loop (Q, xy, e).

1.3. Applications

1.3.1. Linear local loops. Any finite-dimensional unital algebra $(A, *, 1_A)$ over the real numbers defines a local loop in a neighborhood of the identity element 1_A. By translation $x \mapsto x - 1_A$ we obtain a local loop around 0. The product xy of this local loop is related with the product $x * y$ of A by

$$xy = x + y + x * y.$$

Even in case that A is nonunital this formula still defines a local loop $(A, xy, 0)$. We say that a local loop (Q, xy, e) is *linear* if there exists a finite-dimensional algebra A and homomorphism of local loops $\varphi\colon (Q, xy, e) \to (A, xy, 0)$ such that the differential of φ at e is nonsingular.

Let us fix a basis $\{x_1, \dots, x_m\}$ of A and the corresponding dual basis $\{x^1, \dots, x^m\}$ that defines local coordinates on A around 0. Let $(A^{\#}, *)$ be the algebra obtained by adding a formal identity element to A, that we identify as a vector space with the subspace $\mathbb{R}\langle \delta_0, \partial_1|_0, \dots, \partial_m|_0\rangle$ of $\mathcal{D}'_0(A)$ in the natural way. Define the map

$$\pi_{A^{\#}} : \mathcal{D}'_e(Q) \to A^{\#}$$

which assigns to a distribution μ the component of $\varphi'(\mu)$ of degree $\leqslant 1$ in $\mathcal{D}'_0(A)$, i.e., its projection on $\mathbb{R}\langle \delta_0, \partial_1|_0, \ldots, \partial_m|_0\rangle$ parallel to $\mathbb{R}\langle \partial_I|_0 \mid |I| \geqslant 2\rangle$, and finally fix the scalars c^i_{kl} determined by

$$x_k * x_l = c^i_{kl} x_i$$

where Einstein summation convention is assumed. Since

$$\begin{aligned} \varphi'(\mu\nu)(x^i) &= \mu\nu(\varphi^i) = (\mu \otimes \nu)(\varphi^i(xy)) \\ &= (\mu \otimes \nu)(\varphi^i(x) + \varphi^i(y) + c^i_{kl}\varphi^k(x)\varphi^l(y)) \end{aligned}$$

and $\epsilon(\varphi'(\mu\nu)) = \epsilon(\mu)\epsilon(\nu)$ then

$$\pi_{A\#}(\mu\nu) = \pi_{A\#}(\mu) * \pi_{A\#}(\nu).$$

This proves the "only if" part of

Theorem 1.3. *A local loop (Q, xy, e) is linear if and only if there exists a finite-codimensional ideal I of the algebra $\mathcal{D}'_e(Q)$ with $I \cap \mathrm{Prim}(\mathcal{D}'_e(Q)) = 0$.*

In [13] it was proved that for any simple local Bruck loop (Q, xy, e) of $\dim \geqslant 2$, $\mathcal{D}'_e(Q)$ has no finite-codimensional proper ideals different from $\ker(\epsilon)$ so those local loops are not linear. By contrast, in [14] it was proved using Ado's theorem for Lie algebras that any local Moufang loop is linear.

2. Formal loops and Sabinin algebras

2.1. Formal loops

2.1.1. Formal maps. Let V be a vector space over a field $\boldsymbol{k}$ of characteristic zero, $\boldsymbol{k}[V]_i$ the *ith symmetric power* of V and $\boldsymbol{k}[V]$ the *symmetric algebra* on V. By the universal property of the symmetric algebra, the assignments $x \mapsto x \otimes 1 + 1 \otimes x$ and $x \mapsto 0$ extend to homomorphisms of algebras

$$\textit{comultiplication} \quad \Delta\colon \boldsymbol{k}[V] \to \boldsymbol{k}[V] \otimes \boldsymbol{k}[V] \quad \text{and} \quad \textit{counit} \quad \epsilon\colon \boldsymbol{k}[V] \to \boldsymbol{k}.$$

Unital bialgebras with underlying coalgebra structure isomorphic to $\boldsymbol{k}[V]$ for some V are called *connected*. By (1.1), for any local loop (Q, xy, e) the bialgebra $\mathcal{D}'_e(Q)$ is a connected bialgebra with coalgebra structure isomorphic to $\boldsymbol{k}[T_eQ]$.

By analogy, elements of the dual space $\boldsymbol{k}[V]^*$ will be referred to as *formal functions* on V, and those of $\boldsymbol{k}[V]$ as *formal distributions* on V. Recall that by Exercise 1.1.6 (2) $\boldsymbol{k}[V]^*$ is a unital associative and commutative algebra with the convolution product $*$. A *formal map* from V to W is a linear map

$$\theta\colon \boldsymbol{k}[V] \to W$$

with $\theta(1) = 0$. The projection of $\boldsymbol{k}[V]$ onto its primitive part $\boldsymbol{k}[V]_1 = V$ will be denoted by π_V.

2.1.2. Exercises.

(1) Prove that $\mathrm{Prim}(\boldsymbol{k}[V]) = V$.

(2) (*Taylor series*) Prove that if $\{x_1, \dots, x_n\}$ is a basis of V, $\{x_1^*, \dots, x_n^*\}$ is the corresponding dual basis, $x_I = x_1^{i_1} \cdots x_n^{i_n}$, $x_I^* = (x_1^*)^{i_1} * \cdots * (x_n^*)^{i_n}$ if $I = (i_1, \dots, i_n)$, then the algebra $\boldsymbol{k}[V]^*$ is isomorphic to the algebra of formal power series $\boldsymbol{k}[[x_1^*, \dots, x_n^*]]$ by

$$f \mapsto \sum_{I \in \mathbb{N}^n} \frac{f(x_I)}{I!} x_I^*.$$

2.1.3. Coalgebra morphisms induced by formal maps. The following identification will be useful

$$\begin{array}{rcl} \{f \in \boldsymbol{k}[W]^* \mid f(\boldsymbol{k}[W]_i) = 0 \quad \forall i \neq 1\} & \cong & W^* \\ f & \mapsto & f|_W \end{array}$$

By Exercise 1.1.6 (2), any coalgebra morphism $\psi\colon \boldsymbol{k}[V] \to \boldsymbol{k}[W]$ induces an algebra homomorphism $\psi^*\colon \boldsymbol{k}[W]^* \to \boldsymbol{k}[V]^*$. This homomorphism is determined by its restriction to W^*. Hence, ψ itself is determined by the formal map $\pi_W \circ \psi$. Conversely, any formal map $\theta\colon \boldsymbol{k}[V] \to W$ induces a (unique) coalgebra morphism $\theta'\colon \boldsymbol{k}[V] \to \boldsymbol{k}[W]$ with $\pi_W \theta' = \theta$, namely

$$\theta'(\mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \theta(\mu_{(1)}) \cdots \theta(\mu_{(n)}) = \epsilon(\mu)1 + \theta(\mu) + \cdots \qquad (2.1)$$

2.1.4. Exercises.

(1) Prove that for any formal map $\theta\colon \boldsymbol{k}[V] \to W$ the map θ' is a coalgebra morphism.

(2) Prove that the coalgebra morphism induced by the projection $\pi_V\colon \boldsymbol{k}[V] \to V$ is the identity map on $\boldsymbol{k}[V]$.

(3) Prove that the coalgebra morphism induced by the null map $\boldsymbol{k}[V] \to V$ is $\mu \mapsto \epsilon(\mu)1$.

2.1.5. Notation. The algebra $\boldsymbol{k}[V_1 \times \cdots \times V_n]$ is canonically isomorphic to $\boldsymbol{k}[V_1] \otimes \cdots \otimes \boldsymbol{k}[V_n]$. The formal map $\pi_{V_i} \colon \boldsymbol{k}[V_i] \to V_i$ will be denoted by $\boldsymbol{x}_i$ and the null map $\boldsymbol{k}[V_i] \to V_i$ will be denoted by $\mathbf{0}$. The induced coalgebra morphism $\boldsymbol{x}_i'$ is the identity map on $\boldsymbol{k}[V_i]$, and $\mathbf{0}'(\mu) = \epsilon(\mu)1$. Given a formal map

$$G \colon \boldsymbol{k}[V_1 \times \cdots \times V_n] \to W$$

and formal maps $\theta_i \colon \boldsymbol{k}[U_i] \to V_i$ for $1 \leqslant i \leq n$ we write $G(\theta_1, \ldots, \theta_n)$ for the map $G \circ (\theta_1' \otimes \cdots \otimes \theta_n')$.

With this notation $\boldsymbol{x}_i$ can be treated as variables. In particular, G can be also written as $G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. If

$$G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n) = G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \mathbf{0}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n)$$

we say that G *does not depend on* $\boldsymbol{x}_i$ and we omit this variable altogether. Notice that in this case

$$G(\mu_1, \ldots, \mu_n) = \epsilon(\mu_i) G(\mu_1, \ldots, \mu_{i-1}, 1, \mu_{i+1}, \ldots, \mu_n).$$

If $V_1 = \cdots = V_n = V$ the notation $G(\boldsymbol{x}, \ldots, \boldsymbol{x})$ stands for the composition of G with the map $\boldsymbol{k}[V] \to \boldsymbol{k}[V \times \cdots \times V]$ induced by the diagonal $V \to V \times \cdots \times V$:

$$\mu \mapsto \sum G(\mu_{(1)}, \ldots, \mu_{(n)}).$$

Similarly one defines $G(\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_n})$ when there are various groups of repeating indices among the i_k.

2.1.6. Formal multiplications. A *formal multiplication* on V is a formal map

$$F \colon \boldsymbol{k}[V \times V] \to V.$$

A formal multiplication on V is said to be a *formal loop* or a *formal unital multiplication* if

$$F|_{\boldsymbol{k}[V] \otimes 1} = \pi_V = F|_{1 \otimes \boldsymbol{k}[V]}.$$

Any unital formal multiplication $F(\boldsymbol{x}, \boldsymbol{y})$ induces a coalgebra morphism

$$F' \colon \boldsymbol{k}[V] \otimes \boldsymbol{k}[V] \to \boldsymbol{k}[V].$$

Moreover,

$$F'(\mu, 1) = (\pi_V)'(\mu) = \mu = F'(1, \mu) \quad \text{for any } \mu \in \boldsymbol{k}[V].$$

The unital connected bialgebra $(\boldsymbol{k}[V], \Delta, \epsilon, F', u)$ with $u\colon \boldsymbol{k} \to \boldsymbol{k}[V]$ $1 \mapsto 1$ will be denoted by $\boldsymbol{k}[F]$ and will be called *the connected bialgebra of formal distributions of* F.

Since

$$\operatorname{Hom}(\boldsymbol{k}[V] \otimes \boldsymbol{k}[V], V) \cong \prod_{p,q=0}^{\infty} \operatorname{Hom}(\boldsymbol{k}[V]_p \otimes \boldsymbol{k}[V]_q, V)$$

we can write any formal unital multiplication F as an infinite formal sum

$$F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} + \sum_{p,q \geqslant 1} F_{p,q}(\boldsymbol{x}, \boldsymbol{y}) \tag{2.2}$$

with $F_{p,q}(\boldsymbol{x}, \boldsymbol{y}) \in \operatorname{Hom}(\boldsymbol{k}[V]_p \otimes \boldsymbol{k}[V]_q, V)$, or equivalently

$$F(\mu_1 \otimes \mu_2) = \pi_V(\mu_1)\epsilon(\mu_2) + \epsilon(\mu_1)\pi_V(\mu_2) + \sum_{p,q \geqslant 1} F_{p,q}(\mu_1 \otimes \mu_2).$$

We will write $\boldsymbol{xy}$ for a unital formal multiplication $F(\boldsymbol{x}, \boldsymbol{y})$. Recall that the product of any local analytic loop around $0 \in \mathbb{R}^n$ can be expressed by a Taylor expansion of the form (2.2).

2.1.7. Poincaré-Birkhoff-Witt type bases for $\boldsymbol{k}[F]$. Given a unital formal multiplication F on V, by (2.1) the subspaces $\sum_{i=0}^{n} \boldsymbol{k}[F]_i$ $(n \geqslant 0)$ define a filtration of $\boldsymbol{k}[F]$ whose associated graded algebra $\operatorname{Gr}(\boldsymbol{k}[F])$ is isomorphic to the symmetric algebra $\boldsymbol{k}[V]$. Therefore, for any totally ordered basis of V the ordered right normed monomials on elements of that basis is a basis of $\boldsymbol{k}[F]$ (compare with Section 1.1.10).

2.1.8. The equivalence of categories. Let F and H be unital formal multiplications on V and W respectively. A formal map θ from V to W is called a *homomorphism* from F to H if

$$H(\theta(\boldsymbol{x}), \theta(\boldsymbol{y})) = \theta(F(\boldsymbol{x}, \boldsymbol{y}))$$

or, equivalently,

$$H(\theta'(\mu_1) \otimes \theta'(\mu_2)) = \theta(F'(\mu_1 \otimes \mu_2))$$

for any $\mu_1, \mu_2 \in \boldsymbol{k}[V]$. Hence, $\theta \mapsto \theta'$ gives a correspondence between homomorphisms of unital formal multiplications and homomorphisms of their connected bialgebras of formal distributions.

Theorem 2.1. [11] *The category of unital formal multiplications and the category of connected unital bialgebras are equivalent.*

Proof. The product of any connected unital bialgebra $\boldsymbol{k}[V]$ induces a unital formal multiplication $\boldsymbol{k}[V] \otimes \boldsymbol{k}[V] \to \boldsymbol{k}[V] \stackrel{\pi_V}{\to} V$ and conversely, any formal unital multiplication F defines a connected unital bialgebra $\boldsymbol{k}[F]$. □

2.1.9. Exercises.

(1) Prove that the category of formal Moufang loops, i.e. formal loops $\boldsymbol{xy}$ with $((\boldsymbol{xy})\boldsymbol{z})\boldsymbol{y} = \boldsymbol{x}(\boldsymbol{y}(\boldsymbol{zy}))$ is equivalent to the category of unital connected bialgebras that satisfy the (right) *Moufang-Hopf identity*

$$\sum((\mu\nu_{(1)})\eta)\nu_{(2)} = \sum \mu(\nu_{(1)}(\eta\nu_{(2)})).$$

(2) A formal loop is called *right alternative* if $(\boldsymbol{xy})\boldsymbol{y} = \boldsymbol{x}(\boldsymbol{yy})$. Prove that for any formal right alternative loop $(\boldsymbol{xy}^i)\boldsymbol{y}^j = \boldsymbol{x}(\boldsymbol{y}^i\boldsymbol{y}^j)$ holds where $\boldsymbol{y}^i = ((\boldsymbol{yy})\cdots)\boldsymbol{y}$ denotes the ith power of $\boldsymbol{y}$.

2.2. Sabinin algebras

2.2.1. Primitive operations and Shestakov-Umirbaev's functor. Let $\mathcal{S}$ be a set. Denote by $\boldsymbol{k}\{\mathcal{S}\}$ the unital *free non-associative algebra* generated by the elements of $\mathcal{S}$ over the field $\boldsymbol{k}$ of characteristic zero. The algebra $\boldsymbol{k}\{\mathcal{S}\}$ can be given a structure of a bialgebra: the comultiplication is defined by the condition that all elements of $\mathcal{S}$ are primitive, i.e., $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathcal{S}$.

Define by induction on the degree $|u|$ of the *nonassociative monomial* u bilinear maps $\backslash, /$ such that

$$\sum u_{(1)}\backslash(u_{(2)}v) = \epsilon(u)v = \sum(vu_{(1)})/u_{(2)}\,.$$

They also satisfy $\sum u_{(1)}(u_{(2)}\backslash v) = \epsilon(u)v = \sum(v/u_{(1)})u_{(2)}$.

Let $x_1, \dots, x_m, y_1, \dots, y_n, z \in \mathcal{S}$ different elements, $u = ((x_1x_2)\cdots)x_m$, $v = ((y_1y_2)\cdots)y_n$ and define the *primitive operations*

$$p(x_1, \dots, x_m; y_1, \dots, y_n; z) = \sum(u_{(1)}v_{(1)})\backslash(u_{(2)}, v_{(2)}, z)$$

where $(x, y, z) = (xy)z - x(yz)$ [17]. These are nonassociative polynomials that define multilinear operations when evaluated on any algebra. If the evaluation takes place on primitive elements of a unital cocommutative bialgebra then the result is again a primitive element.

Consider

$$\begin{aligned}
\langle 1; y, z\rangle &= \langle y, z\rangle = -[y,z] = -yz + zy\,,\\
\langle x_1,\dots,x_m; y, z\rangle &= \langle \underline{u}; y, z\rangle = -p(\underline{u}; y; z) + p(\underline{u}, z, y)\,,\\
\Phi^{SU}(x_1,\dots,x_m; y_1,\dots,y_n) &= \\
&\frac{1}{m!}\frac{1}{n!}\sum_{\tau\in S_m,\sigma\in S_n} p(x_{\tau(1)},\dots,x_{\tau(m)}; y_{\sigma(1)},\dots; y_{\sigma(n)})
\end{aligned}$$

with $u = ((x_1x_2)\cdots)x_m$, $\underline{u} = (x_1,\dots,x_m)$, S_m the symmetric group on m letters and $m \geqslant 1$, $n \geqslant 2$.

Any nonassociative algebra A with these operations turns out to be a *Sabinin algebra*, an algebraic structure that we will define in Section 2.2.3. Thus we have the *Shestakov-Umirbaev functor* from non-associative algebras to Sabinin algebras

$$A \mapsto \text{УШ}(A).$$

The tangent space $T_eQ = \mathrm{Prim}(\mathcal{D}'_e(Q))$ of any local analytic loop (Q, xy, e) is a Sabinin subalgebra of $\text{УШ}(\mathcal{D}'_e(Q))$. Similarly, for any unital formal multiplication F on V the vector space V of primitive elements of $\boldsymbol{k}[F]$ is a Sabinin subalgebra of $\text{УШ}(\boldsymbol{k}[F])$.

Given any unital formal multiplication F on V, consider the pair $(U(V),\iota)$ formed by a unital algebra $U(V)$ and a homomorphism of Sabinin algebras $\iota\colon V \to \text{УШ}(U(V))$ with the following universal property: any homomorphism of Sabinin algebras $\varphi\colon V \to \text{УШ}(A)$ from V to a unital algebra A extends to a unique homomorphism of unital algebras $\tilde\varphi\colon U(V) \to A$ with $\varphi = \tilde\varphi \circ \iota$. By [17], $U(V)$ is a bialgebra isomorphic to $\boldsymbol{k}[F]$, in fact the homomorphism $U(V) \to \boldsymbol{k}[F]$ induced by the inclusion $V \subseteq \boldsymbol{k}[F]$ is such an isomorphism.

Theorem 2.2. *Let F and G be two unital formal multiplications. The following statements are equivalent:*

(*i*) *The local loops F and G are isomorphic.*

(*ii*) *The connected bialgebras $\boldsymbol{k}[F]$ and $\boldsymbol{k}[G]$ are isomorphic.*

(*iii*) *The Sabinin algebras* $\mathrm{Prim}(\boldsymbol{k}[F])$ *and* $\mathrm{Prim}(\boldsymbol{k}[G])$ *are isomorphic.* □

2.2.2. Exercises.

(1) Prove that for any $x_1,\dots,x_m,y_1,\dots,y_n,z\in\mathcal{S}$

$$p(x_1,\dots,x_m;y_1,\dots,y_n;z)\in \mathrm{Prim}(\boldsymbol{k}\{\mathcal{S}\})\,.$$

(2) Prove that a formal loop F is right alternative if and only if the *multioperator* Φ^{SU} vanishes.

(3) Prove that a formal loop F is a formal group if and only if any multilinear operation on the Sabinin algebra $\mathrm{Prim}(\boldsymbol{k}[F])$ different from the binary operation $\langle\ ,\ \rangle$ vanishes. Conclude that two formal groups are isomorphic if and only if their corresponding Lie algebras are isomorphic.

(4) Prove that if A is an associative algebra then $\text{УШ}(A)$ is the Lie algebra A^-, i.e., the Shestakov-Umirbaev functor $A\mapsto \text{УШ}(A)$ generalizes the usual functor from associative algebras to Lie algebras.

2.2.3. Sabinin algebras. [16] A vector space V is called a *Sabinin algebra* if it is endowed with multilinear operations

$$\begin{aligned}&\langle x_1,x_2,\dots,x_m;y,z\rangle && m\geqslant 0\quad\text{and}\\ &\Phi(x_1,x_2,\dots,x_m;y_1,y_2,\dots,y_n), && m\geqslant 1,\ n\geqslant 2\end{aligned}$$

which satisfy the identities

$$\langle x_1,x_2,\dots,x_m;y,z\rangle=-\langle x_1,x_2,\dots,x_m;z,y\rangle,$$

$$\begin{aligned}\langle x_1,x_2,...,x_r,a,b,x_{r+1},...,x_m;y,z\rangle-\langle x_1,x_2,...,x_r,b,a,x_{r+1},...,x_m;y,z\rangle\\ +\textstyle\sum_{k=0}^r\sum_\alpha\langle x_{\alpha_1},...,x_{\alpha_k},\langle x_{\alpha_{k+1}},...,x_{\alpha_r};a,b\rangle,...,x_m;y,z\rangle=0,\end{aligned}$$

$$\begin{aligned}\sigma_{x,y,z}(\langle x_1,...,x_r,x;y,z\rangle\\ +\textstyle\sum_{k=0}^r\sum_\alpha\langle x_{\alpha_1},...,x_{\alpha_k};\langle x_{\alpha_{k+1}},...,x_{\alpha_r};y,z\rangle,x\rangle)=0\end{aligned}$$

and

$$\Phi(x_1,\dots,x_m;y_1,\dots,y_n)=\Phi(x_{\tau(1)},\dots,x_{\tau(m)};y_{\delta(1)},\dots,y_{\delta(n)}),$$

where α runs the set of all bijections of the type $\alpha\colon\{1,2,...,r\}\to\{1,2,...,r\}$, $i\mapsto\alpha_i$, $\alpha_1<\alpha_2<\cdots<\alpha_k$, $\alpha_{k+1}<\cdots<\alpha_r$, $k=0,1,\dots,r$, $r\geqslant 0$, $\sigma_{x,y,z}$ denotes the cyclic sum by x,y,z; $\tau\in S_m$, $\delta\in S_n$ and S_l is the symmetric group on l symbols. The operations $\langle\ ;\ ,\ \rangle$ and the so called *multioperator* Φ are independent and sometimes the term "Sabinin algebra" is used for a vector space equipped only with operations $\langle\ ;\ ,\ \rangle$ satisfying the corresponding properties.

With the help of the bialgebra structure on $\boldsymbol{k}\{\mathcal{S}\}$ introduced in Section 2.2.1 we may write the definition of a Sabinin algebra very shortly as

$$\begin{aligned}
&\langle \underline{u}; a, b\rangle + \langle \underline{u}; b, a\rangle = 0,\\
&\langle \underline{uabv}; c, e\rangle - \langle \underline{ubav}; c, e\rangle + \sum \langle \underline{u_{(1)} \langle \underline{u_{(2)}}; a, b\rangle v}; c, e\rangle = 0,\\
&\sigma_{a,b,c}\Big(\langle \underline{uc}; a, b\rangle + \sum \langle \underline{u_{(1)}}; \langle \underline{u_{(2)}}; a, b\rangle, c\rangle\Big) = 0 \quad \text{and}\\
&\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Phi(x_{\tau(1)}, \ldots, x_{\tau(m)}; y_{\delta(1)}, \ldots, y_{\delta(n)}),
\end{aligned}$$

where $u = ((x_1x_2)\cdots)x_m)$, $v = ((y_1y_2)\cdots)y_n$ and $x_i, y_j, a, b, c, e \in \mathcal{S}$.

2.2.4. Exercises.

(1) Prove that any Lie algebra is a Sabinin algebra where all multilinear operations vanish with the possible exception of the bilinear product $\langle y, z\rangle = -[y, z]$.

(2) Prove that any Malcev algebra is a Sabinin algebra with

$$\begin{aligned}
&\langle 1; a, b\rangle = -[a, b],\\
&\langle c; a, b\rangle = -\tfrac{1}{3}J(a, b, c) \quad \text{and}\\
&\langle \underline{uc}; a, b\rangle = \textstyle\sum \langle \underline{u_{(1)}}; c, \langle \underline{u_{(2)}}; a, b\rangle\rangle \quad \text{if } |u| \geqslant 1.
\end{aligned}$$

(3) Prove that any right Bol algebra is a Sabinin algebra with

$$\begin{aligned}
&\langle 1; a, b\rangle = -[a, b],\\
&\langle c; a, b\rangle = -[c, a, b] + [c, [a, b]] \quad \text{and}\\
&\langle \underline{uc}; a, b\rangle = \textstyle\sum \langle \underline{u_{(1)}}; c, \langle \underline{u_{(2)}}; a, b\rangle\rangle \text{ if } |u| \geqslant 1.
\end{aligned}$$

(4) An *Akivis algebra* is an algebra with a skew-symmetric bilinear product $[\ ,\ ]$ and a trilinear one $\{\ ,\ ,\ \}$ related by

$$\begin{aligned}
&[[x, y], z] + [[y, z], x] + [[z, x], y]\\
&\quad = \{x, y, z\} + \{y, z, x\} + \{z, x, y\} - \{x, z, y\} - \{y, x, z\} - \{z, y, x\}.
\end{aligned}$$

Prove that any Sabinin algebra with the bilinear map $[x, y] = -\langle x, y\rangle$ and the trilinear map $\{x, y, z\} = -\frac{1}{2}\langle x; y, z\rangle$ is an Akivis algebra.

2.3. Formal integration

Formal integration of a Sabinin algebra amounts to constructing an adequate universal enveloping algebra for it.

2.3.1. Formal integration of Lie algebras. The *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ is defined as the quotient of the free unital associative algebra on $\mathfrak{g}$, i.e., noncommutative polynomials on a basis of $\mathfrak{g}$, by the ideal I generated by

$$\{xy - yx - [x, y] \mid x, y \in \mathfrak{g}\}.$$

The pair $(U(\mathfrak{g}), \iota)$ with $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$, $x \mapsto x + I$ verifies the following universal property: for any unital associative algebra A and any homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to A^-$ there exists a unique homomorphism $\tilde{\varphi}\colon U(\mathfrak{g}) \to A$ of unital algebras such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, "\tilde{\varphi}"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[r, "\varphi"] & A \end{tikzcd}$$

To integrate a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ to a formal group we need to construct a unital associative connected bialgebra $U(\mathfrak{g})$ with $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$ and such that the Sabinin structure on $\mathfrak{g}$ given by Exercise 2.2.4 (1) is the one induced on $\mathfrak{g}$ from *УШ*$(U(\mathfrak{g}))$, which is equivalent to recovering the product of $\mathfrak{g}$ as $[x, y] = xy - yx$ on $U(\mathfrak{g})$. The product on the bialgebra $U(\mathfrak{g})$ will induce a formal group

$$F\colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \stackrel{\pi_{\mathfrak{g}}}{\to} \mathfrak{g}.$$

The existence of this bialgebra $U(\mathfrak{g})$ is ensured by the Poincaré-Birkhoff-Witt Theorem for Lie algebras [4].

Theorem 2.3. *There exists an equivalence of categories between the category of formal groups and the category of Lie algebras.* □

2.3.2. Formal integration of Malcev algebras. Let $(\mathfrak{m}, [\ ,\ ])$ be a Malcev algebra. In order to integrate it to a formal Moufang loop we need to construct a connected unital bialgebra $U(\mathfrak{m})$ with $\mathrm{Prim}(U(\mathfrak{m})) = \mathfrak{m}$ satisfying the *Moufang-Hopf* identity

$$\sum((\mu\nu_{(1)})\eta)\nu_{(2)} = \sum \mu(\nu_{(1)}(\eta\nu_{(2)}))$$

and such that the Sabinin structure of $\mathfrak{m}$ given by Exercise 2.2.4 (2) agrees with the Sabinin structure induced on $\mathfrak{m}$ by *УШ*$(U(\mathfrak{m}))$. In fact, it is enough to recover the product on $\mathfrak{m}$ as $[x, y] = xy - yx \in U(\mathfrak{m})$ for any $x, y \in \mathfrak{m}$ [12].

Let us sketch the method used to construct $U(\mathfrak{m})$ assumed that $U(\mathfrak{m})$ exists [14].

(1) *The Lie algebra* $\mathcal{L}(\mathfrak{m})$. The *associative multiplication algebra* of an algebra A is the associative algebra $\mathrm{Mult}(A)$ generated by the left and right multiplication operators by elements of A. Since $\mathfrak{m} \subseteq \mathrm{N}_{\mathrm{alt}}(U(\mathfrak{m}))$, Exercise 1.2.4 (4) shows that $\mathrm{Mult}(U(\mathfrak{m}))$ is generated by left and right multiplication operators by elements of $\mathfrak{m}$. Let $\mathcal{L}(\mathfrak{m})$ be the Lie algebra generated by $\{\lambda_a, \rho_a \mid a \in \mathfrak{m}\}$ with relations

$$\begin{aligned} &\lambda_{\alpha a+\beta b} = \alpha\lambda_a + \beta\lambda_b, && \rho_{\alpha a+\beta b} = \alpha\rho_a + \beta\rho_b,\\ &[\lambda_a, \lambda_b] = \lambda_{[a,b]} - 2[\lambda_a, \rho_b], && [\rho_a, \rho_b] = -\rho_{[a,b]} - 2[\lambda_a, \rho_b],\\ &[\lambda_a, \rho_b] = [\rho_a, \lambda_b], \end{aligned}$$

$a, b \in \mathfrak{m}$, $\alpha, \beta \in \boldsymbol{k}$. The maps determined by

$$\begin{aligned} \zeta(\lambda_a) &= T_a, & \eta(\lambda_a) &= -\lambda_a,\\ \zeta(\rho_a) &= -\rho_a, & \eta(\rho_a) &= T_a, \end{aligned} \tag{2.3}$$

where $T_a = \lambda_a + \rho_a$ define two automorphisms of $\mathcal{L}(\mathfrak{m})$. There is a Lie homomorphism

$$\mathcal{L}(\mathfrak{m}) \to \mathrm{Mult}(U(\mathfrak{m}))^-$$

that extends to a homomorphism $U(\mathcal{L}(\mathfrak{m})) \to \mathrm{Mult}(U(\mathfrak{m}))$ defining a left $U(\mathcal{L}(\mathfrak{m}))$-module structure on $U(\mathfrak{m})$.

(2) *The* $\mathcal{L}(\mathfrak{m})$*-module* $U(\mathfrak{m})$. $U(\mathfrak{m})$ is a cyclic $U(\mathcal{L}(\mathfrak{m}))$-module generated by the identity element 1. The annihilator of 1 in $U(\mathcal{L}(\mathfrak{m}))$ contains the left ideal K generated by $\mathcal{L}_+ = \{\lambda_a - \rho_a, [\lambda_a, \lambda_a] + [\rho_a, \rho_b] + [\lambda_a, \rho_b]\}$. The Lie algebra $\mathcal{L}(\mathfrak{m})$ admits a $\mathbb{Z}_2$-gradation $\mathcal{L}(\mathfrak{m}) = \mathcal{L}_+ \oplus \mathcal{L}_-$ with $\mathcal{L}_- = \{\lambda_a + \rho_a \mid a \in \mathfrak{m}\} \cong \mathfrak{m}$ as vector spaces. The Poincaré-Birkhoff-Witt for Lie algebras implies that if $\{a_i \mid i \in \Omega\}$ is a totally ordered basis of $\mathfrak{m}$ then $a_{i_1} \cdots a_{i_r} \mapsto T_{a_{i_1}} \cdots T_{a_{i_r}} + K$ $(a_{i_1} \leqslant \cdots \leqslant a_{i_r})$ is an isomorphism $\theta\colon \boldsymbol{k}[\mathfrak{m}] \xrightarrow{\cong} U(\mathcal{L}(\mathfrak{m}))/K$ as vector spaces (as coalgebras in fact). The $U(\mathcal{L}(\mathfrak{m}))$-module structure of $U(\mathcal{L}(\mathfrak{m}))/K$ is transported to $\boldsymbol{k}[\mathfrak{m}]$ by $\lambda \circ x = \theta^{-1}(\lambda\theta(x))$ for any $\lambda \in U(\mathcal{L}(\mathfrak{m}))$ and $x \in \boldsymbol{k}[\mathfrak{m}]$.

(3) *The product on* $\boldsymbol{k}[\mathfrak{m}]$ (*determination*). Having identified $U(\mathfrak{m})$ and $\boldsymbol{k}[\mathfrak{m}]$, we look for a product $*$ on $\boldsymbol{k}[\mathfrak{m}]$ such that $\mathfrak{m} \subseteq \mathrm{N}_{\mathrm{alt}}((\boldsymbol{k}[\mathfrak{m}], *))$ and $a * x = 2\lambda_a x$, $x * a = 2\rho_a x$ and $1 * x = x = x * 1$. Since

$$a * (x * y) = (a * x + x * a) * y + x * (-a * y)$$

and

$$(x * y) * a = (-x * a) * y + x * (a * y + y * a),$$

the product $*$ should be a homomorphism

$$*\colon \boldsymbol{k}[\mathfrak{m}]_\zeta \otimes \boldsymbol{k}[\mathfrak{m}]_\eta \to \boldsymbol{k}[\mathfrak{m}]$$

of $\mathcal{L}(\mathfrak{m})$-modules where $\boldsymbol{k}[\mathfrak{m}]_\zeta$ denotes the vector space $\boldsymbol{k}[\mathfrak{m}]$ with the twisted action $\lambda \cdot x = \zeta(\lambda) \circ x$.

(4) *The product on* $\boldsymbol{k}[\mathfrak{m}]$ (*construction*). The inductive definition of the product $*$ is quite straightforward. Fix a totally ordered basis $\{a_i\}_{i\in\Omega}$ of $\mathfrak{m}$ and consider the basis

$$\{a_I = a_{i_1} \cdots a_{i_n} \mid I = (i_1, \ldots, i_n) \in \Omega^n, \quad a_{i_1} \leqslant \cdots \leqslant a_{i_n},\ n \geqslant 0\}.$$

For $I = (i_1, \ldots, i_r)$ denote $I' = (i_2, \ldots, i_r)$ and $l(I) = r$. The element $r_I = a_I - 2\lambda_{a_{i_1}} \circ a_{I'}$ belongs to $\boldsymbol{k}[\mathfrak{m}]_{l(I)-1}$ (in particular, if $l(I) = 1$ then $r_I = 0$). We set $1 * x = x$, and assume that we have defined $a_J * x$ for any a_J with $l(J) < l(I)$. Then we define

$$a_I * x = 2T_{a_{i_1}} \circ (a_{I'} * x) - 2a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x.$$

Theorem 2.4. *There exists an equivalence of categories between the category of formal Moufang loops and the category of Malcev algebras.* □

2.3.3. Exercises.

(1) Prove that for any Malcev algebra $(\mathfrak{m}, [\,,\,])$ the algebra $U(\mathfrak{m})$ is isomorphic to the quotient of the unital free nonassociative $\boldsymbol{k}\{\mathfrak{m}\}$ algebra by the ideal generated by

$$\{ab - ba - [a, b], (a, x, y) + (x, a, y), (x, a, y) + (x, y, a)\},$$

where $a \in \mathfrak{m}$, $x, y \in \boldsymbol{k}\{\mathfrak{m}\}$.

(2) Prove that if a Malcev algebra $\mathfrak{m}$ is a Lie algebra then $U(\mathfrak{m})$ is isomorphic to the universal enveloping algebra of the Lie algebra $\mathfrak{m}$.

(3) Given a Malcev algebra $(\mathfrak{m}, [\,,\,])$ prove that $\mathfrak{m}$ with the triple product

$$[a, b, c] = \frac{1}{3}(2[[a, b], c] - [[b, c], a] - [[c, a], b])$$

is a Lie triple system.

2.3.4. Groups, Hopf algebras and Lie algebras with triality. A *group with triality* is a group (G, xy, e) with two automorphisms σ, τ that satisfy

(1) $\sigma^2 = \rho^3 = \mathrm{I}$, $\sigma\rho\sigma = \rho^2$
and
(2) $(g^{-1}g^{\sigma})(g^{-1}g^{\sigma})^{\rho}(g^{-1}g^{\sigma})^{\rho^2} = e$ for any $g \in G$.

Theorem 2.5. [3] *Given a group with triality G the set $\mathcal{M}(G) = \{g^{-1}g^{\sigma}\}$, where $g \in G$, is a Moufang loop with respect to the multiplication law*

$$m\cdot n = m^{-\rho}nm^{-\rho^2} = n^{-\rho^2}mn^{-\rho} \qquad \forall m, n \in \mathcal{M}(G) \qquad \square$$

Given two automorphisms ρ, σ of a cocommutative Hopf algebra H such that $\sigma^2 = \rho^3 = \mathrm{I}_H$ and $\sigma\rho = \rho^2\sigma$, H is called a cocommutative *Hopf algebra with triality* relative to ρ and σ if

$$\sum P(x_{(1)})\rho(P(x_{(2)}))\rho^2(P(x_{(3)})) = \epsilon(x)1, \tag{2.4}$$

for all $x \in H$, where $P(x) = \sum \sigma(x_{(1)})S(x_{(2)})$.

Theorem 2.6. *Let H be a cocommutative Hopf algebra with triality relative to ρ and σ and define $P(x) = \sum \sigma(x_{(1)})S(x_{(2)})$ for any $x \in H$. Then*

$$\mathcal{MH}(H) = \{P(x) \mid x \in H\}$$

is a unital cocommutative Moufang-Hopf algebra with the coalgebra structure and antipode inherited from H, the same unit element and product defined by

$$u * v = \sum \rho^2(S(u_{(1)}))v\rho(S(u_{(2)})) = \sum \rho(S(v_{(1)}))u\rho^2(S(v_{(2)}))$$

for any $u, v \in \mathcal{MH}(H)$. $\square$

Given a Lie algebra $(\mathfrak{g}, [\,,\,])$, two automorphisms ρ, σ of $\mathfrak{g}$ such that $\sigma^2 = \rho^3 = \mathrm{I}_{\mathfrak{g}}$, $\sigma\rho = \rho^2\sigma$, $\mathfrak{g}$ is called a *Lie algebra with triality* relative to ρ and σ in case that

$$a - \sigma(a) + \rho(a) - \rho\sigma(a) + \rho^2(a) - \rho^2\sigma(a) = 0 \tag{2.5}$$

for any $a \in \mathfrak{g}$. For any Malcev algebra $(\mathfrak{m}, [\,,\,])$ the Lie algebra $\mathcal{L}(\mathfrak{m})$ with the automorphisms defined in (2.3) is an example of Lie algebra with triality. The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra with triality $\mathfrak{g}$ is a Hopf algebra with triality.

The following result presents another approach to the formal integration of Malcev algebras.

Theorem 2.7. [2] *Let* $(\mathfrak{m}, [\,,\,])$ *be a Malcev algebra over a field of characteristic* $\neq 2, 3$. *Then the Moufang-Hopf algebra* $U(\mathfrak{m})$ *is isomorphic to* $\mathcal{MH}(U(\mathcal{L}(\mathfrak{m})))$. □

2.3.5. Exercises.

(1) Given a local group (G, xy, e) with triality prove that the bialgebra of distributions with support at the identity element is a Hopf algebra with triality.

(2) Prove that for any connected Hopf algebra with triality the primitive elements form a Lie algebra with triality.

(3) Prove that for any Malcev algebra $(\mathfrak{m}, [\,,\,])$ the Lie algebra $\mathcal{L}(\mathfrak{m})$ is a Lie algebra with triality.

2.3.6. Integration of Sabinin algebras. Let $(V, \langle\,;\,,\,\rangle, \Phi)$ be a Sabinin algebra. The formal integration of $(V, \langle\,;\,,\,\rangle, \Phi)$ to a formal loop amounts to constructing a unital connected bialgebra $U(V)$ with $\mathrm{Prim}(U(V)) = V$ and such that the Sabinin algebra structure of V is the one induced by $\mathit{УШ}(U(V))$.

A natural candidate for the underlying vector space of $U(V)$ is a quotient of the unital free associative algebra $\mathrm{T}(V)$ on a basis of V

$$\tilde{S}(V) = \mathrm{T}(V)/\boldsymbol{k}\langle xaby - xbay + \sum x_{(1)}\langle \underline{x}_{(2)}; a, b\rangle y \mid x, y \in \mathrm{T}(V),\ \ a, b \in V\rangle.$$

With this choice the second axiom of the operations $\langle\,;\,,\,\rangle$ is automatically satisfied. Even more, $\tilde{S}(V)$ is a coalgebra isomorphic to $\boldsymbol{k}[V]$. Using a Poincaré-Birkhoff-Witt basis it can be defined a (bialgebra) product on $\tilde{S}(V)$ such that the Sabinin structure on V agrees with the one induced by $\mathit{УШ}(\tilde{S}(V))$ [12].

Theorem 2.8. *There exists an equivalence of categories between the category of formal loops and the category of Sabinin algebras.* □

3. The geometry of formal loops

3.1. Geodesic loops and similarity

3.1.1. Affine connections and local loops. Given a flat affine connection on a smooth manifold Q and $e \in Q$, the parallel transportation τ^x_y from x to y along a curve does not depend on the curve itself. Around e we may define a binary operation $x \times y = \exp_x(\tau_x^e(\exp_e^{-1}(y)))$ that extends the usual sum on $\mathbb{R}^n$.

$$\begin{tikzcd}
 & y & & x \times y \\
e \arrow[ur, "\exp_e^{-1}(y)"] \arrow[rr] & & x \arrow[ur, "\tau_x^e(\exp_e^{-1}(y))"'] &
\end{tikzcd}$$

Clearly $e \times y = \exp_e(\tau_e^e(\exp_e^{-1}(y))) = y$ and $x \times e = \exp_x(\tau_x^e(0)) = \exp_x(0) = x$ so $(Q, x \times y, e)$ is a local loop, *the geodesic loop* at e. For $v \in T_eQ$ small enough, both $(x \times \exp_e(tv)) \times \exp_e(sv)$ and $x \times (\exp_e(tv) \times \exp_e(sv))$ are geodesics $\gamma_t(s)$ with $\gamma_t(0) = x \times \exp_e(tv)$ and $\dot{\gamma}_t(0) = \tau^e_{\gamma_t(0)}(v)$ so

$$(x \times y) \times y = x \times (y \times y),$$

i.e., $(Q, x \times y, e)$ is *right alternative.* Conversely, given any local loop (Q, xy, e)

$$\tau_y^x(v) = dL_y|_e(dL_x|_e)^{-1}(v)$$

defines around e the parallel transportation of the so called *canonical flat affine connection.* The geodesic loop $(Q, x \times y, e)$ obtained from this affine connection might not be isomorphic to the original loop (Q, xy, e). However, both loops are related through a certain map Φ by

$$x \times \Phi(x, y) = xy \tag{3.1}$$

that verifies

$$\Phi(e, y) = y, \quad \Phi(x, e) = e \quad \text{and} \quad d\Phi(x, y)|_{y=e} = \mathrm{I}_{T_eQ}. \tag{3.2}$$

Any map $\Phi\colon Q \times Q \to Q$ satisfying (3.2) is called a *similarity.* Two local loops (Q, xy, e) and $(Q, x \times y, e)$ that define the same canonical flat connection are *similar.* This is equivalent to the existence of a similarity that relates both products by (3.1).

Mikheev and Sabinin proved that a local loop is similar to a unique right alternative local loop. The classification of right alternative local loops is equivalent to the classification of local flat affine connections. Any such connection is determined by its torsion which is encoded in the $\langle\, ;\, , \,\rangle$ operations

$$\langle x_1, \ldots, x_n; y, z\rangle_F = \nabla_{x_1^*} \cdots \nabla_{x_n^*} T(y^*, z^*)(e).$$

The multioperator Φ encodes the similarity needed to pass from the geodesic loop to the target loop [16].

3.1.2. Right alternative loops. Given a local loop (Q, xy, e), any similarity $\Phi\colon Q \times Q \to Q$ induces a corresponding map $\Phi'\colon \mathcal{D}'_e(Q) \otimes \mathcal{D}'_e(Q) \to \mathcal{D}'_e(Q)$ on distributions with

$$\Phi'(e, \nu) = \nu, \quad \Phi'(\mu, e) = \epsilon(\mu)e \quad \text{and} \quad \Phi'(\mu, \alpha) = \epsilon(\mu)\alpha$$

for any $\mu, \nu \in \mathcal{D}'_e(Q)$ and α primitive. A formal map $\Phi\colon \boldsymbol{k}[V] \otimes \boldsymbol{k}[V] \to V$ such that

$$\Phi|_{1\otimes \boldsymbol{k}[V]} = \pi_V \quad \text{and} \quad \Phi|_{\boldsymbol{k}[V]_{\geqslant 1}\otimes(1\oplus V)} = 0$$

is called a *similarity*. Two formal loops F_1 and F_2 on V are *similar* if there exists a similarity $\Phi\colon \boldsymbol{k}[V] \otimes \boldsymbol{k}[V] \to V$ such that $F_1(\boldsymbol{x}, \boldsymbol{y}) = F_2(\boldsymbol{x}, \Phi(\boldsymbol{x}, \boldsymbol{y}))$. Notice that in this case

$$F'_1(\mu, \alpha) = F'_2(\mu, \alpha) \tag{3.3}$$

for any primitive α.

Lemma 3.9. *Each formal loop is similar to a unique formal right alternative loop.*

Proof. In Section 2.1.7 we saw that the graded algebra $\mathrm{Gr}(\boldsymbol{k}[F])$ of the the bialgebra of formal distributions $\boldsymbol{k}[F]$ of a formal unital multiplication F on V is isomorphic to the symmetric algebra $\boldsymbol{k}[V]$. Hence given a totally ordered basis $\{a_i \mid i \in \Omega\}$ of V, the elements $\mathrm{sym}(a_{i_1}, \dots, a_{i_r}) = \frac{1}{r!}\sum_{\sigma\in S_r}((a_{i_{\sigma(1)}}a_{i_{\sigma(2)}})\cdots)a_{i_{\sigma(r)}}$ $(r \geqslant 0)$ form a basis of $\boldsymbol{k}[V]$.

We may define a new product $\times$ on $\boldsymbol{k}[V]$ by

$$x \times \mathrm{sym}(a_{i_1}, \dots, a_{i_r}) = \frac{1}{r!}\sum_{\sigma\in S_r}((xa_{i_{\sigma(1)}})\cdots)a_{i_{\sigma(r)}}$$

for any $x \in \boldsymbol{k}[V]$. With this new product $\boldsymbol{k}[V]$ is a unital connected bialgebra and it induces a formal right alternative unital multiplication $\boldsymbol{x} \times \boldsymbol{y}$. Both formal loops $\boldsymbol{x} \times \boldsymbol{y}$ and the original formal loop $\boldsymbol{xy}$ are related by $\boldsymbol{x} \times \Phi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{xy}$ for certain formal map Φ. The definition of the product $\times$ on $\boldsymbol{k}[V]$ implies that Φ is a similarity. The uniqueness is a consequence of (3.3) and the right alternativity. □

Proposition 3.10. *Let $\boldsymbol{k}[V]$ be a bialgebra with respect to two similar products $\mu\nu$ and $\mu \times \nu$. Then for any $\alpha, \alpha_1, \dots, \alpha_r, \beta \in V$*

$$\langle \alpha_1, \dots, \alpha_r; \alpha, \beta\rangle = \langle \alpha_1, \dots, \alpha_r; \alpha, \beta\rangle^{\times}. \qquad \square$$

Using a basis formed by elements $\mathrm{sym}(a_{i_1}, \dots, a_{i_r})$ as in the proof of Lemma 3.9, the similarity that relates a formal loop on V with its formal right alternative loop can be expressed by a set of multilinear operations $\Phi_{m,n}(x_1, \dots, x_m; y_1, \dots, y_n)$ on V symmetric on $x_1, \dots, x_m$ and on $y_1, \dots, y_n$.

3.1.3. Formal flat affine connections. In this section the commutative and associative product of the symmetric algebra $\boldsymbol{k}[V]$ plays a crucial role. We will denote this product by $\mu \cdot \nu$. Juxtaposition is reserved for products induced by other formal unital multiplications on V. The convolution product on $\boldsymbol{k}[V]^*$ will be denoted by fg instead of $f * g$ since we adopt it as the natural product of $\boldsymbol{k}[V]^*$.

A *formal vector field* is a linear map $A : \boldsymbol{k}[V] \to V$. The product of a formal vector field A with a formal function f is given by

$$fA \colon \mu \mapsto \sum f\big(\mu_{(1)}\big) A\big(\mu_{(2)}\big).$$

This action provides the formal vector fields with the structure of a free $\boldsymbol{k}[V]^*$–module. In fact, any set $\{A_i\}_i$ of formal vector fields such that $\{A_i(1)\}$ is a basis of V gives a $\boldsymbol{k}[V]^*$–basis of $\mathrm{Hom}(\boldsymbol{k}[V], V)$.

A formal vector field A gives a derivation D_A of the algebra $\boldsymbol{k}[V]^*$ of formal functions into itself:

$$D_A(f) = A(f) \colon \mu \mapsto \sum f\big(\mu_{(1)} \cdot A(\mu_{(2)})\big).$$

We have $(fA)(g) = fA(g)$. Formal vector fields form a Lie algebra with the Lie bracket $[A, B]$ given by

$$[A, B] \colon \mu \mapsto \sum B\big(\mu_{(1)} \cdot A(\mu_{(2)})\big) - A\big(\mu_{(1)} \cdot B(\mu_{(2)})\big).$$

Clearly $[D_A, D_B] = D_{[A,B]}$. We also have that

$$[A, fB] = A(f)B + f[A, B].$$

A *formal flat affine connection* is a linear map $\boldsymbol{k}[V] \otimes V \to V$ whose restriction to $1 \otimes V$ is the identity. For a given formal connection, $\mu \in \boldsymbol{k}[V]$ and $v \in V$, we write $\mu * v$ for the image of $\mu \otimes v$. The vector field $v^* \colon \mu \mapsto \mu * v$ is said to be *adapted* to the *tangent vector* v. There always exists a unique "inverse" map $\boldsymbol{k}[V] \otimes V \to V$ sending $\mu \otimes u$ to an element that we denote by $\mu \backslash^* u$ and such that $\sum \mu_{(1)} \backslash^* (\mu_{(2)} * v) = \epsilon(\mu) v = \sum \mu_{(1)} * (\mu_{(2)} \backslash^* v)$.

The *covariant differentiation* with respect to the formal vector field A is defined as

$$\nabla_A(B)\colon \mu \mapsto \sum B\big(\mu_{(1)} \cdot A(\mu_{(2)})\big) - \big(\mu_{(1)} \cdot A(\mu_{(2)})\big) * \big(\mu_{(3)} \backslash^* B(\mu_{(4)})\big).$$

Proposition 3.11. *Let A, B be formal vector fields, f a formal function and $v, w \in V$. Then*

(1) $\nabla_{fA}(B) = f\nabla_A(B)$,

(2) $\nabla_A(fB) = A(f)B + f\nabla_A(B)$,

(3) $\nabla_{v^*}(w^*) = 0$.

Given a formal loop F on a vector space V, the *formal canonical connection* of F is the restriction of F to the subspace

$$\boldsymbol{k}[V] \otimes V \subset \boldsymbol{k}[V] \otimes \boldsymbol{k}[V].$$

In fact, this is all one needs to construct a right alternative product on $\boldsymbol{k}[V]$ as in the proof of Lemma 3.9.

Exercises.

(1) Prove that the formal vector fields form a free $\boldsymbol{k}[V]^*$-module.

(2) Prove that D_A is a derivation of the algebra $\boldsymbol{k}[V]^*$ of formal functions.

(3) Prove that the space of formal vector fields is a Lie algebra with the product

$$[A, B]\colon \mu \mapsto \sum B\big(\mu_{(1)} \cdot A(\mu_{(2)})\big) - A\big(\mu_{(1)} \cdot B(\mu_{(2)})\big).$$

(4) Prove Proposition 3.11.

(5) Let $\{v_i \mid i \in \Omega\}$ a basis of V and ∇ the covariant derivative of a formal flat affine connection. Given a formal vector field $A = \sum_i f_i v_i^*$ prove that

$$\nabla_{v^*}(A) = \sum_i v^*(f_i)v_i^*.$$

(6) Prove that two formal loops on the vector space V are similar if and only if their formal canonical connections agree.

(7) Prove that the set of formal flat affine connections is a group with the product

$$C * C'\colon \mu \otimes v \mapsto \sum C(\mu_{(1)} \otimes C'(\mu_{(2)} \otimes v)).$$

3.1.5. The torsion of a formal flat affine connection. The torsion of two formal vector fields A and B is defined in the usual way

$$T(A,B) = \nabla_A(B) - \nabla_B(A) - [A,B].$$

In the case of adapted vector fields x^*, y^* with $x, y \in V$ we get

$$T(x^*, y^*) = -[x^*, y^*].$$

Setting

$$\langle x_1, \ldots, x_n; y, z\rangle_F = \nabla_{x_1^*} \cdots \nabla_{x_n^*} T(y^*, z^*)(1)$$

we obtain an $n+2$-linear operation on V for all $n \geqslant 0$.

Proposition 3.12. *Assigning the set of operations $\langle x_1, \ldots, x_n, y, z\rangle_F$ to a formal multiplication F gives a functor from the category of formal loops to that of Sabinin algebras with trivial multioperator.* □

Given a formal unital multiplication F on V, the torsion tensor of the formal canonical connection of F admits a simple interpretation in terms of the product on $\boldsymbol{k}[F]$.

Lemma 3.13. *For any $x, y \in V$ and $\mu \in \boldsymbol{k}[V]$ we have that*

. $$T(x^*, y^*)(\mu) = \pi_V\left((\mu y)x - (\mu x)y\right).$$ □

This provides the geometrical interpretation of the multilinear operations involved in the definition of the Shestakov-Umirbaev functor.

Theorem 3.14. [11] *The multilinear operations $\langle x_1, \ldots, x_n; y, z\rangle$ of Shestakov and Umirbaev identically coincide with the operations $\langle x_1, \ldots, x_n; y, z\rangle_F$ of Mikheev and Sabinin.* □

3.1.6. Exercises.

(1) Use Exercise 3.1.4 (5) to prove that the torsion of a formal flat affine connection is determined by the multilinear operations $\langle x_1, \ldots, x_n; y, z\rangle$ $(n \geqslant 0)$.

(2) Given a local group and its canonical formal flat connection, prove that the commutator of two adapted vector fields is an adapted vector field. Conclude that any covariant derivative of the corresponding torsion with respect to any adapted vector field vanishes.

(3) [15] Prove that the adapted vector fields on a right Bol loop (Q, xy, e) form a Lie triple system with the product

$$[x^*, y^*, z^*] = [x^*, [y^*, z^*]].$$

Take a basis $\{x_1, \ldots, x_n\}$ of T_eQ and define $R^l_{i,jk}$ and a^l_{ij} by

$$[x^*_i, [x^*_j, x^*_k]] = R^l_{i,jk} x^*_l \quad \text{and} \quad [x^*_i, x^*_j](1) = a^l_{ij} x_l.$$

Prove that the tangent space T_eQ with the operations determined by

$$[x_i, x_j, x_k] = R^l_{i,jk} x_l \quad \text{and} \quad [x_i, x_j] = a^l_{ij} x_l$$

is a right Bol algebra. This Bol algebra structure is the same as the one provided by Proposition 1.2.

(4) [15] Let (Q, xy, e) be a right Bol algebra, $\{x_1, \ldots, x_n\}$ a basis of T_eQ and denote $x^*_i(f)$ by $\nabla_i(f)$ for any formal function f. Define formal functions C^l_{jk} such that $[x^*_j, x^*_k] = C^l_{jk} x^*_l$ and $T^l_{jk} = -C^l_{jk}$ so that $T(x^*_j, x^*_k) = T^l_{jk} x^*_l$. Use Exercise 3.1.6 (3) to prove that

$$\nabla_r(\nabla_l T^i_{jk} + T^s_{jk} T^i_{sl}) = 0. \tag{3.4}$$

(5) [15] Let F be a formal geodesic loop on V. Prove that if the torsion satisfies (3.4) then the right multiplication operators by primitive elements of $\boldsymbol{k}[F]$ form a Lie triple system. Use [12, Proof of Proposition 32] to conclude that the Sabinin structure inherited by V corresponds to that of a right Bol algebra (Exercise 2.2.4 (3)). Conclude that F is a formal right Bol loop.

4. Beyond Lie's theorems

4.1. Quantum loops

4.1.1. Hopf algebra deformations. A *topologically free Hopf algebra H* over the ring of formal power series $K = \boldsymbol{k}[[h]]$ with coefficients in the base field $\boldsymbol{k}$ is a topologically free K-module equipped with a product, coproduct, unit, counit and antipode which satisfy the axioms of a Hopf algebra over K with tensor products understood in a complete sense. If $H/hH \cong U(\mathfrak{g})$ as Hopf algebras over $\boldsymbol{k}$ for some Lie algebra $(\mathfrak{g}, [\,,\,])$ then H is called a *quantized universal enveloping algebra* or *Hopf algebra deformation* of $U(\mathfrak{g})$.

4.1.2. Rigidity of universal enveloping algebra of the traceless octonions. Given a Malcev algebra $\mathfrak{m}$ over a field $\boldsymbol{k}$, a coassociative *bialgebra deformation* of $U(\mathfrak{m})$ over $K = \boldsymbol{k}[[h]]$ is a topologically free K-module B endowed with four maps of K-modules

(*unit*)	$\iota_h\colon K \to B$, $1 \mapsto 1_B$,	(*product*)	$m_h\colon B\tilde{\otimes}B \to B$,
(*counit*)	$\epsilon_h\colon B \to K$,	(*coproduct*)	$\Delta_h\colon B \to B\tilde{\otimes}B$,

where $\tilde{\otimes}$ stands for the completed tensor product in the h-adic topology, such that

(1) $(B, \Delta_h, \epsilon_h, m_h, \iota_h)$ satisfies the axioms of bialgebra over the commutative ring K but with the algebraic tensor products replaced by their completions,
(2) $B/hB \cong U(\mathfrak{m})$ as a $\boldsymbol{k}$-vector space and, with this identification,
(3) $m_h \equiv m (\operatorname{mod} h)$ and $\Delta_h \equiv \Delta (\operatorname{mod} h)$

with μ and Δ the multiplication and comultiplication of $U(\mathfrak{m})$ respectively. Since B is topologically free and $B/hB \cong U(\mathfrak{m})$, we can identify B with $U(\mathfrak{m})[[h]]$ as a K-module. The product μ_h and the comultiplication Δ_h are uniquely determined by their restrictions to $U(\mathfrak{m}) \otimes U(\mathfrak{m})$ and $U(\mathfrak{m})$ respectively. We can write $m_h|_{U(\mathfrak{m})\otimes U(\mathfrak{m})} = m + hm_1 + h^2 m_2 + \cdots$ and $\Delta_h|_{U(\mathfrak{m})} = \Delta + h\Delta_1 + h^2\Delta_2 + \cdots$ for some $\boldsymbol{k}$-linear maps $m_i\colon U(\mathfrak{m})\otimes U(\mathfrak{m}) \to U(\mathfrak{m})$ and $\Delta_i\colon U(\mathfrak{m}) \to U(\mathfrak{m}) \otimes U(\mathfrak{m})$ $(i \geqslant 1)$. The *null deformation* of $U(\mathfrak{m})$ is obtained by extending K-linearly the structure maps of $U(\mathfrak{m})$. *Trivial deformations* are those isomorphic to the null deformation under a K-linear bialgebra isomorphism which is the identity modulo h.

Proposition 4.1. *Define* $\delta\colon U(\mathfrak{m}) \to U(\mathfrak{m}) \otimes U(\mathfrak{m})$ *by*

$$\delta(x) = \Delta_1(x) - \Delta_1^{\mathrm{op}}(x) = \frac{\Delta_h(a) - \Delta_h^{\mathrm{op}}(a)}{h} \pmod h$$

for any $a \in B$ *that reduces to* $x (\operatorname{mod} h)$*. Then*

i) δ *is skew-symmetric and* $\sum_{\mathrm{cyclic}}(\delta \otimes \mathrm{I})\delta = 0$*;*

ii) $(\Delta \otimes \mathrm{I})\delta = (\mathrm{I} \otimes \delta) + \sigma_{23}(\delta \otimes \mathrm{I})\Delta$*;*

iii) $\delta(x_1 x_2) = \delta(x_1)\Delta(x_2) + \Delta(x_1)\delta(x_2)$ *for all* $x_1, x_2 \in U(\mathfrak{m})$*.* □

Proposition 4.2. *Let* $\delta = \Delta_1 - \Delta_1^{\mathrm{op}}$*. Then* $\delta(\mathfrak{m}) \subseteq \mathfrak{m} \otimes \mathfrak{m}$ *and* $\delta_{\mathfrak{m}} = \delta|_{\mathfrak{m}}$ *satisfies:*

i) $\delta^*_{\mathfrak{m}}\colon \mathfrak{m}^* \otimes \mathfrak{m}^* \to \mathfrak{m}^*$ *is a Lie bracket on* $\mathfrak{m}^*$ *and*

ii) $\delta_{\mathfrak{m}}([x,y]) = (\mathrm{ad}_x \otimes \mathrm{I}_{\mathfrak{m}} + \mathrm{I}_{\mathfrak{m}} \otimes \mathrm{ad}_x)\delta_{\mathfrak{m}}(y) - (\mathrm{ad}_y \otimes \mathrm{I}_{\mathfrak{m}} + \mathrm{I}_{\mathfrak{m}} \otimes \mathrm{ad}_y)\delta_{\mathfrak{m}}(x)$ *for all* $x, y \in \mathfrak{m}$. □

The traceless octonions [18] $\mathbb{M}(\alpha,\beta,\gamma) = \{x \in \mathbb{O}(\alpha,\beta,\gamma) \mid t(x) = 0\}$ $(\alpha,\beta,\gamma \neq 0)$ with the commutator product are up to isomorphism the only central simple Malcev algebras which are not Lie algebras. In contrast with finite-dimensional semisimple Lie algebras (or symmetrizable Kac-Moody algebras) for which non cocommutative quantized universal enveloping algebras exist, the simple Malcev algebras $\mathbb{M}(\alpha,\beta,\gamma)$ show an exceptional behavior.

Theorem 4.3. [6] *We have that* $\delta_{\mathbb{M}(\alpha,\beta,\gamma)} = 0$. □

Corollary 4.4. *Any coassociative bialgebra deformation of* $U(\mathbb{M}(\alpha,\beta,\gamma))$ *is cocommutative.* □

Corollary 4.5. *Any coassociative bialgebra deformation of* $U(\mathbb{M}(\alpha,\beta,\gamma))$ *satisfying*

$$\sum((xy_{(1)})z)y_{(2)} = \sum x(y_{(1)}(zy_{(2)}))$$

is trivial. □

4.2. Nilpotent loops

4.2.1. The dimension filtration. Let $\boldsymbol{k}$ be a field of characteristic zero, (Q, xy, e) a loop and $\boldsymbol{k}Q$ the loop algebra of Q over $\boldsymbol{k}$, which is a unital cocommutative bialgebra with the comultiplication and counit determined by $\Delta\colon x \mapsto x \otimes x$ and $\epsilon\colon x \mapsto 1$ for any $x \in Q$. Let $I = \ker \epsilon$ be the ideal of $\boldsymbol{k}Q$ spanned by elements of the form $x - e$. The bialgebra structure of $\boldsymbol{k}Q$ induces a bialgebra structure on the graded space $\mathcal{I}(Q,\boldsymbol{k}) = \oplus_{n=0}^{\infty} I^n/I^{n+1}$ $(I^0 = \boldsymbol{k}Q$ by convention).

The unital bialgebra $\mathcal{I}(Q,\boldsymbol{k})$ is connected so it determines a formal loop on the space of primitive elements. This space admits a beautiful description in terms of the so called *dimension subloops*. The nth dimension subloop of Q over $\boldsymbol{k}$ is the intersection

$$D_n(Q,\boldsymbol{k}) = Q \cap (e + I^n).$$

The filtration $D_1(Q,\boldsymbol{k}) \supseteq D_2(Q,\boldsymbol{k}) \supseteq \cdots$ is called the *dimension filtration* of Q over $\boldsymbol{k}$. The quotient $D_n(Q,\boldsymbol{k})/D_{n+1}(Q,\boldsymbol{k})$ is an abelian group so we can extend scalars to get a vector space

$$\mathcal{D} = \bigoplus_{n=1}^{\infty} \boldsymbol{k} \otimes (D_n(Q, \boldsymbol{k})/D_{n+1}(Q, \boldsymbol{k})).$$

The map

$$\begin{array}{rcl} \mathcal{D} & \to & \mathcal{I}(Q, \boldsymbol{k}) \\ xD_{n+1}(Q, \boldsymbol{k}) & \mapsto & x - e + I^{n+1} \end{array}$$

is an inclusion of $\mathcal{D}$ into $\mathcal{I}(Q, \boldsymbol{k})$.

Theorem 4.6 ([10]). *$\mathcal{I}(Q, \boldsymbol{k})$ is a unital connected bialgebra and the image of $\mathcal{D}$ in $\mathcal{I}(Q, \boldsymbol{k})$ coincides with the subspace of primitive elements.* □

4.2.2. The commutator-associator filtration. Given a group (G, xy, e), the ith term G_i of its *lower central series*

$$G = G_1 \supseteq G_2 \supseteq \cdots$$

is the subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in G_p$ and $y \in G_q$, $p + q \geqslant i$. In case that $G_{n+1} = \{e\}$ for some n then G is called *nilpotent*.

Since $[x_p x_q, x_r] \in G_{p+q+r}$ for any $x_p \in G_p, x_q \in G_q$ and $x_r \in G_r$ then the commutator of G induces a bilinear product on the abelian group $\oplus_i G_i/G_{i+1}$ by

$$[\bar{x}_p, \bar{x}_q] = [x_q, x_q]G_{p+q+1}.$$

This product defines a *graded Lie ring structure* on $\oplus_i G_i/G_{i+1}$.

Given a loop (Q, xy, e), the *commutator* of x, y is defined as

$$[x, y] = (yx)\backslash(xy)$$

and the *associator* of x, y and z is

$$(x, y, z) = (x(yz))\backslash((xy)z).$$

To define a series with abelian group factors so that the commutator and the associator induce multilinear product one introduces the *associator deviations* of level one

$$\begin{array}{rcl} (x, y, z, t)_1 & = & ((x, z, t)(y, z, t))\backslash(xy, z, t), \\ (x, y, z, t)_2 & = & ((x, y, t)(x, z, t))\backslash(x, yz, t), \\ (x, y, z, t)_3 & = & ((x, y, z)(x, y, t))\backslash(x, y, zt). \end{array}$$

Given a deviation $A\colon Q^{l+2} \to Q$ of level $l-1$ and $1 \leqslant \alpha_l \leqslant l+2$, an associator deviation of level l is defined by

$$A_{\alpha_l}(\dots, x_{\alpha_l}, x_{\alpha_{l+1}}, \dots) = (A(\dots, x_{\alpha_l}, \dots,)A(\dots, x_{\alpha_{l+1}}, \dots))\backslash A(\dots, x_{\alpha_l}x_{\alpha_{l+1}}, \dots).$$

Hence, for any $\alpha_1, \dots, \alpha_l$ with $1 \leqslant \alpha_i \leqslant i+2$ there exist associator deviations

$$(x_1, \dots, x_{l+3})_{\alpha_1,\dots,\alpha_l}$$

of level l. The associator is thought of as the associator deviation of level zero.

Define $\gamma_1 Q = Q$ and for $n > 1$ define $\gamma_n Q$ to be the minimal normal subloop of Q containing

- $[\gamma_p Q, \gamma_q Q]$ with $p + q \geqslant n$;
- $(\gamma_p Q, \gamma_q Q, \gamma_r Q)$ with $p + q + r \geqslant n$;
- $(\gamma_{p_1} Q, \dots, \gamma_{p_{l+3}} Q)_{\alpha_1,\dots,\alpha_l}$ with $p_1 + \dots + p_{l+3} \geqslant n$.

The subloop $\gamma_n Q$ is called the nth *commutator-associator subloop* of Q and $\gamma_1 Q \supseteq \gamma_2 Q \supseteq \cdots$ is the *commutator-associator filtration* of Q [7]. We say that Q is *nilpotent* if there exists n such that $\gamma_{n+1} Q = \{e\}$.

4.2.3. Jennings Theorem. Let F be the free loop on a single generator x and $\delta\colon F \to \mathbb{Z}$ the homomorphism that sends x to 1. The *degree* of an element $w(x) \in F$ is defined to be the integer $\delta(w(x))$. Given a loop (Q, xy, e), the *isolator* $\sqrt{K}$ of a normal subloop K is the minimal normal subloop of Q containing all $x \in Q$ such that $w(x) \in K$ for some word w of non-zero degree.

Theorem 4.7 ([8]). *For any field $\boldsymbol{k}$ of characteristic* 0 *and for any loop (Q, xy, e), the isolator $\sqrt{\gamma_n Q}$ in Q coincides with the dimension subloop $D_n(Q, \boldsymbol{k})$.* □

The associative version of this theorem is due to Jennings [5].

4.2.4. Coquecigrues. A *Leibniz algebra* is a vector space equipped with a bilinear product $[\,,\,]$ that satisfies the identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. In case that $[\,,\,]$ is skew-symmetric we recover the definition of a Lie algebra.

In the same way that Lie algebras are the tangent spaces of Lie groups at the identity element, it has been suggested that Leibniz algebras could be integrated to some hypothetic objects called *coquecigrues*. One of the several attempts of finding these coquecigrues is based on formal integration of Lie algebra to formal group in the so called Loday-Pirashvili category [9].

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